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The Hausdorff Lower Semicontinuous Envelope of the Length in the Plane

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Abstract. We study the Hausdorff lower semicontinuous envelope of the length in the plane. This envelope is taken with respect to the Hausdorff metric on the space of the continua. The resulting quantity appeared naturally as the rate function of a large deviation principle in a statistical mechanics context and seems to deserve further analysis. We provide basic simple results which parallel those available for the perimeter of Caccioppoli and De Giorgi.

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1. – Introduction

The results reported here come as a side product of our endeavour to provide a rigorous mathematical analysis of the phase coexistence phenomena in models of statistical mechanics in dimension higher than 3 [3], [4], [5]. The main obstacle that prevented the 2D proofs to be extended to dimensions higher than 3 was to find a higher dimensional analog of the skeleton technique, an intrinsically 2D tool relying on a combinatorial bound which is at the heart of the probabilistic proof [1], [7]. Our strategy to go around this obstacle was to first rewrite the 2D result in a weaker yet more robust form through a large deviation principle, the proof of which still relied on skeletons [4]. Then we proved a 3D version of the large deviation principle by replacing the skeleton argument by a compactness argument with the help of the theory of Caccioppoli sets [5]. The topology used to express the 3D large deviation principle, namely the Lebesgue measure of the symmetric difference, is weaker than the Hausdorff distance which was employed in 2D. In fact, in our preliminary 2D attempt [4], we proved two different large deviation principles with both topologies. One of the rate functions was an anisotropic version of the classical perimeter of Caccioppoli and De Giorgi, the other was an anisotropic version of the following

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quantity: for K a continuum, we define

$$\mathcal{S}(K) = \inf\left\{\liminf_{n\to\infty}\mathcal{H}^1(\partial K_n)\right\} \tag{1}$$

where the infimum is taken over all sequences $(K_n)_{n\in\mathbb{N}}$ of non-degenerate polyhedra (that is, connected polyhedra whose boundary is a finite union of disjoint Jordan curves) converging towards K with respect to the Hausdorff metric, and $\mathcal{H}^1$ is the standard one dimensional Hausdorff measure. The difference between $\mathcal{S}$ and the classical perimeter lies in the topology used in the definition, but both are lower semicontinuous envelopes of the usual length for regular sets. Our aim here is to provide the beginning of the analysis of this quantity and to prove results similar in flavor to the ones available for the classical perimeter developed by Caccioppoli and De Giorgi [2], [6]. The interest is twofold. First it will enable to reprove the 2D large deviation principle of [4] without using skeletons and it might help to analyze further 2D models in statistical mechanics. Second we think that $\mathcal{S}(K)$ is a geometrically interesting quantity on its own which deserves a thorough study.

Let us sum up briefly our main results. We start with an alternative definition of $\mathcal{S}$: for any continuum K, let

$$\mathcal{S}(K) = \sup_{\mathcal{U}}\sum_{U\in\mathcal{U}}\ \sum_{O\in\mathcal{C}(K,U)}\mathcal{H}^1(\partial O\setminus\partial U) \tag{2}$$

where $\mathcal{C}(K,U)$ is the collection of all residual domains of K in U and the supremum is taken over all families $\mathcal{U}$ of pairwise disjoint domains of $\mathbb{R}^2$. To some extent, this definition is the analog to the distributional definition of the perimeter (see [9]). We prove that $\mathcal{S}$ is lower semicontinuous with respect to the Hausdorff metric (restricted to continua). We single out a specific subset $\partial^\circ K$ of the topological boundary of a continuum K and we analyze its structure whenever $\mathcal{S}(K)$ is finite. At $\mathcal{H}^1$ almost all the points of $\partial^\circ K$ there is a true tangent, in a sense even stronger than the classical measure theoretic definition. Up to a set of $\mathcal{H}^1$ measure zero, the subset of $\partial^\circ K$ where there is a true tangent can be further partitioned into two sets: a set $\partial^*_I K$ consisting of the points where K locally looks like a half-plane and a set $\partial^*_{II} K$ where K locally looks like a line. We rewrite $\mathcal{S}(K)$ as

$$\mathcal{S}(K) = \mathcal{H}^1(\partial^*_I K) + 2\mathcal{H}^1(\partial^*_{II} K)\,.$$

These results parallel the corresponding ones for the reduced boundary of sets having finite perimeter. We finally prove that both definitions (1) and (2) of $\mathcal{S}$ agree. An interesting question, which is not handled at all here, is to compare $\mathcal{S}(K)$ with other classical quantities, like for instance the perimeter or the Minkowski content.

The proofs rely on a few classical results on 1-sets in the plane and on the Vitali covering theorem on one hand, and on arguments from planar geometry and topology on the other hand.

The paper is organized as follows. In Section 2, we give the notation and basic definitions. In Section 3, we state some useful topological lemmas. In Section 4, we recall several standard results concerning 1-sets in the plane. In Section 5, we define the subset $\partial^\circ K$ of the boundary of a continuum K. The notion of true tangents is introduced in Section 6. In Section 7, we analyze the local structure of $\partial^\circ K$ at the points where there is a tangent. In Sections 8 and 9, we consider the case of continua such that $\mathcal{H}^1(\partial^\circ K) < \infty$. In Section 10, we define and we study the quantity $\mathcal{S}(K)$ with the help of the previous results.

2. – Notation and basic definitions

In this section we fix the notation and we recall some standard definitions.

2.1. – Topology

Let E be a subset of $\mathbb{R}^2$. We denote its interior by $\mathring{E}$, its closure by $\overline{E}$, its boundary by ∂E. The collection of all compact subsets of $\mathbb{R}^2$ is denoted by $\mathcal{K}$. A continuum is a compact connected set with at least two points. The collection of all compact connected sets is denoted by $\mathcal{K}_c$. Our usual notation for a set which is either a continuum or is reduced to a single point is K. If E is a connected set, then any set F such that $E \subset F \subset \overline{E}$ is also connected.

A domain is a non-empty open connected set. Our usual notation for a domain is O or U.

Let K be an element of $\mathcal{K}_c$ and let U be a domain. A residual domain of K in U is a connected component of $U \setminus K$ (i.e. a maximal connected set included in $U \setminus K$). The collection of all residual domains of K in U is denoted by $\mathcal{C}(K, U)$. The collection of all residual domains of K in $\mathbb{R}^2$ is denoted by $\mathcal{C}(K)$. A compact set K is said to disconnect two sets A_1 and A_2 inside a domain U if there is no residual domain of K in U intersecting both A_1 and A_2. We will make use of the following facts. Every residual domain of a continuum in $\mathbb{R}^2$ is simply connected and has a connected boundary ([12, Chapter VI, Paragraph 4.3 and Theorem 4.4]).

2.2. – Metric

For x a point of $\mathbb{R}^2$, we denote by $|x|_2$ its Euclidean norm. The associated distance is denoted by d. The diameter of a set E is $\operatorname{diam} E = \sup\{|x - y|_2 : x, y \in E\}$. A set E is bounded if its diameter is finite. The distance between two sets E_1 and E_2 is

$$d(E_1, E_2) = \inf\{|x_1 - x_2|_2 : x_1 \in E_1,\ x_2 \in E_2\}\,.$$

The r-neighbourhood of a set E is the set

$$\mathcal{V}(E, r) = \{x \in \mathbb{R}^2 : d(x, E) < r\}\,.$$

Let E_1, E_2 be two bounded subsets of $\mathbb{R}^2$. We define successively

$$e(E_1, E_2) = \inf\{r > 0 : E_2 \subset \mathcal{V}(E_1, r)\}$$

and the Hausdorff distance between E_1 and E_2

$$D(E_1, E_2) = \max\{e(E_1, E_2), e(E_2, E_1)\}\,.$$

The restriction of D to $\mathcal{K}$ is a metric and the metric space $(\mathcal{K}, D)$ is complete. We claim that $\mathcal{K}_c$ is a closed subspace of $(\mathcal{K}, D)$. Indeed, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of connected compact sets converging to K. Suppose K is not connected, so that there exist two open disjoint sets U, V such that $K \subset U \cup V$ and $K \cap U \neq \varnothing$, $K \cap V \neq \varnothing$. For n sufficiently large, we will also have $K_n \subset U \cup V$, $K_n \cap U \neq \varnothing$, $K_n \cap V \neq \varnothing$, which is absurd since K_n is connected.

2.3. – Measure

We denote by $\mathcal{L}^2$ the planar Lebesgue measure and by $\mathcal{H}^1$ the standard one dimensional Hausdorff measure in $\mathbb{R}^2$. We recall that for any subset E of $\mathbb{R}^2$,

$$\mathcal{H}^1(E) = \sup_{\delta>0} \inf \left\{ \sum_{i\in I} \operatorname{diam} E_i : \sup_{i\in I} \operatorname{diam} E_i \leq \delta,\, E \subset \bigcup_{i\in I} E_i \right\}.$$

2.4. – Geometry

Let x be a point of $\mathbb{R}^2$ and let r be positive. The closed ball of center x and Euclidean radius r is denoted by $B(x, r)$. The sphere of center x and radius r is $\partial B(x, r)$. Let E be a set in $\mathbb{R}^2$. We define $E(x, r) = E \cap B(x, r)$. Let θ be an angle. We denote by $(u(\theta), v(\theta))$ the orthonormal basis whose angle with the canonical basis is θ, that is $u(\theta) = (\cos\theta, \sin\theta)$, $v(\theta) = (-\sin\theta, \cos\theta)$.

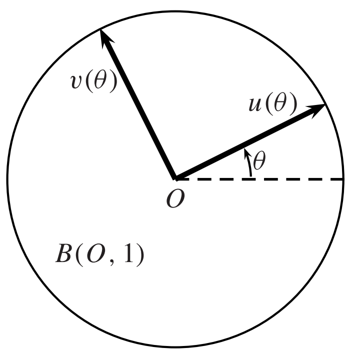

Fig. 1.

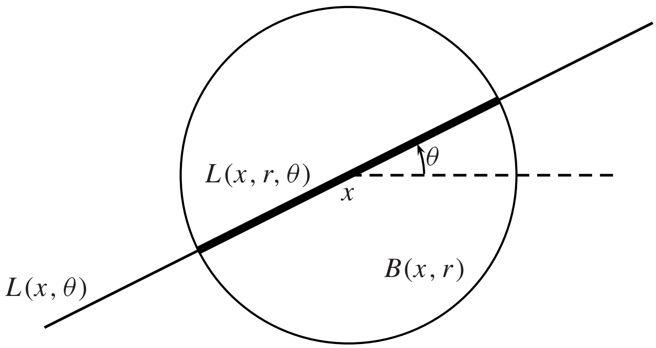

Fig. 2.

We denote by $L(x, \theta)$ the line passing through x parallel to $u(\theta)$ (here θ is defined modulo π), and by $L(x, r, \theta)$ its intersection with $B(x, r)$, that is

$$L(x, \theta) = \{x + tu(\theta) : t \in \mathbb{R}\}, \qquad L(x, r, \theta) = L(x, \theta) \cap B(x, r).$$

We denote by $HL(x, \theta)$ the half-line passing through x oriented by $u(\theta)$ (here θ is defined modulo 2π), and by $HL(x, r, \theta)$ its intersection with $B(x, r)$, that is

$$HL(x, \theta) = \{x + tu(\theta) : t \in \mathbb{R}^+\}, \qquad HL(x, r, \theta) = HL(x, \theta) \cap B(x, r).$$

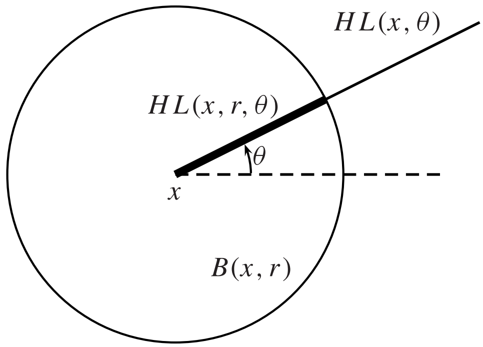

Fig. 3.

The closed angular sector of vertex x and angles ϕ_1, ϕ_2 is the set

$$S(x, \phi_1, \phi_2) = \{x + ru(\theta) : r \geq 0, \ \phi_1 \leq \theta \leq \phi_2\}.$$

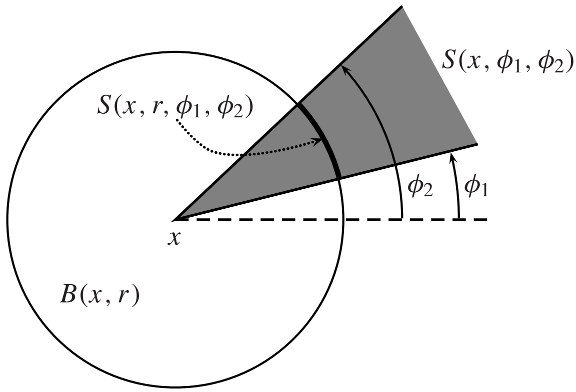

Fig. 4.

We set also $S(x,r,\phi_1,\phi_2) = S(x,\phi_1,\phi_2) \cap \partial B(x,r)$.
Let ϕ belong to $[0,\pi/2]$. We define

$$U_-(x,r,\theta,\phi) = S(x,\pi+\theta+\phi,\theta-\phi) \cap B(x,r)\,,\quad U_-(x,r,\theta) = U_-(x,r,\theta,0)\,,$$
$$U_+(x,r,\theta,\phi) = S(x,\theta+\phi,\pi+\theta-\phi) \cap B(x,r)\,,\quad U_+(x,r,\theta) = U_+(x,r,\theta,0)\,,$$

and

$$U(x,r,\theta,\phi) = U_-(x,r,\theta,\phi) \cup U_+(x,r,\theta,\phi)\,.$$

Let ε be positive. We set also

$$V_-(x,r,\varepsilon,\theta,\phi) = \{\, y \in U_-(x,r,\theta,\phi) : d(y,\mathbb{R}^2 \setminus U_-(x,r,\theta,\phi)) > \varepsilon r\}\,,$$
$$V_+(x,r,\varepsilon,\theta,\phi) = \{\, y \in U_+(x,r,\theta,\phi) : d(y,\mathbb{R}^2 \setminus U_+(x,r,\theta,\phi)) > \varepsilon r\}\,.$$

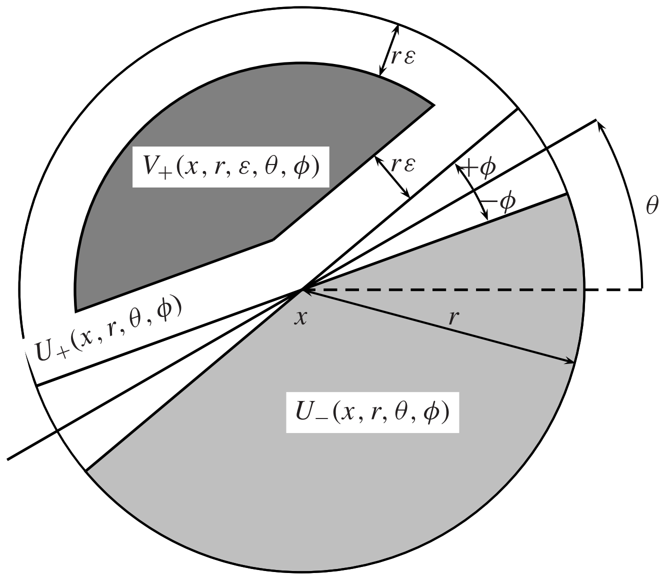

Fig. 5.

3. – Topological lemmas

This section is devoted to the statement of some basic topological results.

LEMMA 3.1. *Let O be a domain with compact closure. There exists a sequence $(O_n)_{n\in\mathbb{N}}$ of increasing domains included in O such that:*

$$\forall n \in \mathbb{N} \quad \forall x \in O_n \quad d(x, \partial O) > 1/n \quad \textit{and} \quad \bigcup_{n\in\mathbb{N}} O_n = O\,.$$

PROOF. Let n belong to $\mathbb{N}$. We define a relation $\mathcal{R}_n$ on the points of O by: $x\mathcal{R}_n y$ if and only if there exists a continuous path $\gamma : [0,1] \to O$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(t), \partial O) > 1/n$ for all t in $[0,1]$. For any pair x, y of points of O there exists n_0 such that $x\mathcal{R}_n y$ for all n larger than n_0. In fact, O is an open connected subset of $\mathbb{R}^2$ and is therefore arcwise connected. Thus there exists a continuous path $\gamma : [0,1] \to O$ such that $\gamma(0) = x$, $\gamma(1) = y$. Since $\gamma([0,1])$ does not intersect ∂O and is compact, the distance $d(\gamma([0,1]), \partial O)$ is positive. It follows that $x\mathcal{R}_n y$ as soon as $d(\gamma([0,1]), \partial O) > 1/n$. Let us fix a point x_0 in O and let $C(x_0, n)$ be its equivalence class for the relation $\mathcal{R}_n$. Then $(C(x_0, n))_{n\in\mathbb{N}}$ is an increasing sequence of open connected sets satisfying the requirements of the lemma. □

COROLLARY 3.2. *Let O be a domain with compact closure. Let ε be positive. There exists a domain U included in O such that $e(U, \overline{O}) < \varepsilon$ and $d(U, \partial O) > 0$.*

LEMMA 3.3. *Let K be a continuum and let δ be positive. There is a finite number of residual domains of K in $\mathbb{R}^2$ of Lebesgue measure larger than δ.*

PROOF. Let B be a closed ball containing K in its interior. Let O_0 be the residual domain of K containing $\mathbb{R}^2 \setminus B$ and let $O_1, \dots, O_n$ be other residual domains of K of Lebesgue measure larger than δ. We have then $\mathcal{L}^2(O_1 \cup \cdots \cup O_n) \geq n\delta$ and $O_1 \cup \cdots \cup O_n \subset B$ whence $n\delta \leq \mathcal{L}^2(B)$. Thus there exist at most $\lceil \mathcal{L}^2(B)/\delta \rceil$ residual domains of K of Lebesgue measure larger than δ. □

COROLLARY 3.4. *A continuum K has a finite or countable number of residual domains.*

4. – The 1-sets in the plane

A subset E of $\mathbb{R}^2$ is a 1-set if E is $\mathcal{H}^1$-measurable and $0 < \mathcal{H}^1(E) < \infty$. We recall here without proofs some definitions and facts concerning 1-sets in the plane. Everything is extracted from [8, Chapter 3].

A collection of sets $\mathcal{U}$ is called a Vitali class for E if for each x in E and δ positive there exists a set U in $\mathcal{U}$ containing x such that $0 < \operatorname{diam} U < \delta$. We will use extensively the following result [8, Theorem 1.10].

THEOREM 4.1 (Vitali covering theorem). *Let E be an $\mathcal{H}^1$-measurable subset of $\mathbb{R}^2$ and let $\mathcal{U}$ be a Vitali class of closed sets for E. Then we may select a finite or countable disjoint sequence $(U_i)_{i\in I}$ from $\mathcal{U}$ such that either $\sum_{i\in I} \operatorname{diam} U_i = \infty$ or $\mathcal{H}^1(E \setminus \bigcup_{i\in I} U_i) = 0$. If $\mathcal{H}^1(E) < \infty$ then, given $\varepsilon > 0$, we may also require that $\mathcal{H}^1(E) \le \sum_{i\in I} \operatorname{diam} U_i + \varepsilon$.*

We recall next an important result, first proved by Golab [8, Theorem 3.18].

THEOREM 4.2 (Golab theorem). *If $(E_n)_{n\in\mathbb{N}}$ is a sequence of continua in $\mathbb{R}^2$ converging in the Hausdorff metric to a compact connected set E then $\mathcal{H}^1(E) \le \liminf_{n\to\infty} \mathcal{H}^1(E_n)$.*

For any 1-set E in the plane, we have

$$1/2 \le \limsup_{r\to 0} \frac{1}{2r}\mathcal{H}^1(E \cap B(x,r)) \le 1 \quad \mathcal{H}^1 \text{ a.e. on } E\,.$$

COROLLARY 4.3. *Let E be 1-set of $\mathbb{R}^2$ and let $\mathcal{U}$ be a Vitali class of closed balls for E. Then for any positive ε we may select a finite disjoint sequence $(U_i)_{i\in I}$ from $\mathcal{U}$ such that $\mathcal{H}^1(E \setminus \bigcup_{i\in I} U_i) \le \varepsilon \sum_{i\in I} \operatorname{diam} U_i$ and $\mathcal{H}^1(E) \le (1+\varepsilon) \sum_{i\in I} \operatorname{diam} U_i$.*

PROOF. Let E^* be the subset of E defined by

$$E^* = \left\{ x \in E : 1/2 \le \limsup_{r\to 0} \frac{1}{2r}\mathcal{H}^1(E \cap B(x,r)) \le 1 \right\}.$$

We know that $\mathcal{H}^1(E \setminus E^*) = 0$. The collection of closed balls

$$B(x,r), \quad x \in E^*, \quad r \text{ such that } r/3 < \mathcal{H}^1(E \cap B(x,r)) < 4r/3$$

is a Vitali class for E^*. We apply the Vitali covering Theorem 4.1 to E^* and this Vitali class; let $(U_i)_{i\in I}$ be the resulting collection of balls. Since

$$\frac{3}{4}\mathcal{H}^1\left(E \cap \bigcup_{i\in I} U_i\right) \le \sum_{i\in I} r_i < 3\mathcal{H}^1(E^*) = 3\mathcal{H}^1(E) < \infty$$

we do not have $\sum_{i\in I} \operatorname{diam} U_i = \infty$ and therefore $\mathcal{H}^1(E \setminus \bigcup_{i\in I} U_i) = 0$. By Theorem 4.1, given ε in $]0,1/3[$, we may further impose that

$$\mathcal{H}^1(E) \le \sum_{i\in I} \operatorname{diam} U_i + \varepsilon\frac{3}{8}\mathcal{H}^1(E)\,.$$

Let J be a finite subset of I such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i\in J} U_i\right) \le \varepsilon\frac{3}{4}\mathcal{H}^1(E), \qquad \sum_{i\in I} \operatorname{diam} U_i \le \sum_{i\in J} \operatorname{diam} U_i + \varepsilon\frac{3}{8}\mathcal{H}^1(E)\,.$$

We have then $\mathcal{H}^1(E \setminus \bigcup_{i\in J} U_i) \le \varepsilon \sum_{i\in J} \operatorname{diam} U_i$ and $\mathcal{H}^1(E) \le (1+\varepsilon) \sum_{i\in J} \operatorname{diam} U_i$. $\square$

A set E is said to be regular if

$$\lim_{r\to 0}\frac{1}{2r}\mathcal{H}^1(E\cap B(x,r)) = 1 \quad \mathcal{H}^1 \text{ a.e. on } E\,.$$

Definition 4.4 (tangent of a set E at a point x). A 1-set E has a tangent at x in the direction θ if $\limsup_{r\to 0}\ \mathcal{H}^1(E\cap B(x,r))/r > 0$ and

$$\forall\,\phi\in]0,\pi/2] \qquad \lim_{r\to 0}\frac{1}{r}\mathcal{H}^1(E\cap U(x,r,\theta,\phi)) = 0\,.$$

Remark. Clearly the direction θ is defined modulo π. Moreover we obtain an equivalent definition if we impose that the angle ϕ belongs to an arbitrarily small interval $]0,\eta]$, $\eta > 0$.

A curve γ is a continuous injection $\gamma : [a,b]\mapsto \mathbb{R}^2$ where $[a,b]$ is a non-degenerate closed interval. Sometimes we do not distinguish between γ and its range $\gamma([a,b])$. The length of a curve γ coincides with its $\mathcal{H}^1$-measure, that is

$$\mathcal{H}^1(\gamma) = \mathcal{H}^1(\gamma([a,b])) = \sup_{a<t_1<\ldots<t_l<b}\sum_j |\gamma(t_{j+1})-\gamma(t_j)|_2\,,$$

the supremum being taken over all finite subdivisions of $[a,b]$. The curve γ is said to be rectifiable if it has finite length or equivalently if $\mathcal{H}^1(\gamma)<\infty$. In this case, we may parametrize γ by arc length, that is, we may suppose that the map γ is defined on the interval $[0,\mathcal{H}^1(\gamma)]$ and is Lipschitz with constant 1: $\forall\,t_1,t_2 \quad |\gamma(t_1)-\gamma(t_2)|_2 \le |t_1-t_2|$.

Any 1-set contained in a countable union of rectifiable curves is a regular set and has a tangent at $\mathcal{H}^1$ almost all of its points. We next consider the case of continua. Any continuum E satisfies $\mathcal{H}^1(E)\ge \operatorname{diam}(E)$.

Theorem 4.5. *A continuum having a finite $\mathcal{H}^1$-measure consists of a countable union of rectifiable curves, together with a set of $\mathcal{H}^1$-measure zero.*

Corollary 4.6. *Any continuum E such that $\mathcal{H}^1(E)<\infty$ is a regular 1-set and has a tangent at $\mathcal{H}^1$ almost all of its points.*

5. – Lower semicontinuity of $\mathcal{H}^1(\partial^\circ K)$

In this section, we define a special subset $\partial^\circ K$ of the boundary of a continuum K and we prove that the map $K\in\mathcal{K}_c\mapsto \mathcal{H}^1(\partial^\circ K)$ is lower semicontinuous with respect to the Hausdorff metric.

DEFINITION 5.1. Let K be a continuum. Let $(O_i,\ i \in I)$ be the residual domains of K. We define $\partial^\circ K = \bigcup_{i\in I} \partial O_i$.

REMARK. The sets ∂O_i are compact because they are closed subsets of K and they are connected because the residual domains O_i, $i \in I$, are simply connected (since K is connected). Hence the set $\partial^\circ K$ is a finite or countable union of continua. However it is not necessarily closed; in general, it is a strict subset of ∂K.

LEMMA 5.2. *For any K_1, K_2 in $\mathcal{K}_c$, we have $\partial^\circ(K_1 \cup K_2) \subset \partial^\circ K_1 \cup \partial^\circ K_2$.*

PROOF. Let x belong to $\partial^\circ(K_1 \cup K_2)$. There exists a residual domain O of $K_1 \cup K_2$ such that x belongs to ∂O. Moreover x belongs to $K_1 \cup K_2$. Suppose for instance that x is in K_1. Let O_1 be the residual domain of K_1 containing O. Then x belongs to ∂O_1 so that x is in $\partial^\circ K_1$. □

COROLLARY 5.3. *For any K_1, K_2 in $\mathcal{K}_c$, we have*

$$\mathcal{H}^1(\partial^\circ(K_1 \cup K_2)) \le \mathcal{H}^1(\partial^\circ K_1) + \mathcal{H}^1(\partial^\circ K_2)\,.$$

LEMMA 5.4. *Let K belong to $\mathcal{K}_c$ and let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_c$ converging to K for the Hausdorff distance. Let O be a residual domain of K in $\mathbb{R}^2$. Let $(O_i^n,\ i \in I_n)$ be the residual domains of K_n in $\mathbb{R}^2$. We have*

$$\lim_{n\to\infty} \inf_{m\in I_n} \sup_{x\in\partial O} d(x, \partial O_m^n) = 0\,.$$

PROOF. Let ε be positive. By Corollary 3.2, there exist a positive δ and a domain U included in O such that $e(U, \overline{O}) < \varepsilon$ and $d(U, \partial O) \ge \delta$. Let n_0 be such that $D(K_n, K) < \delta$ for $n \ge n_0$. Let n be larger than n_0. Clearly the set K_n does not intersect U so that U is included in a residual domain of K_n: there exists m in I_n such that $U \subset O_m^n$. Let x belong to ∂O. There exists y in K_n such that $d(x, y) < \delta \le \varepsilon$ and z in U such that $d(x, z) < \varepsilon$. In particular the point z belongs to O_m^n, therefore the segment $[z\,y]$ intersects ∂O_m^n. It follows that $d(x, \partial O_m^n) < \varepsilon$. We have thus proved that $\inf_{m\in I_n} \sup\{\, d(x, \partial O_m^n) : x \in \partial O\,\} < \varepsilon$. □

PROPOSITION 5.5. *The map $K \in \mathcal{K}_c \mapsto \mathcal{H}^1(\partial^\circ K)$ is lower semicontinuous with respect to the Hausdorff metric i.e. for any sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_c$ such that $D(K_n, K)$ converges to 0 as n goes to ∞, we have $\liminf_{n\to\infty} \mathcal{H}^1(\partial^\circ K_n) \ge \mathcal{H}^1(\partial^\circ K)$.*

PROOF. Let $(O_i,\ i \in I)$ be a finite family of residual domains of K. For each i in I, there exists by Lemma 5.4 a sequence of domains $(O_i^n)_{n\in\mathbb{N}}$ such that: for any n in $\mathbb{N}$, O_i^n is a residual domain of K_n, and $\sup\{\, d(x, \partial O_i^n) : x \in \partial O_i\,\}$ goes to 0 as n goes to ∞. Since we deal with a finite number of sequences of domains $(O_i^n)_{n\in\mathbb{N}}$, $i \in I$, up to the extraction of a subsequence, we may assume that:

$$\forall\, i, j \in I \quad \text{either} \quad [\forall\, n \in \mathbb{N} \quad \partial O_i^n \cap \partial O_j^n = \varnothing] \quad \text{or} \quad [\forall\, n \in \mathbb{N} \quad \partial O_i^n \cap \partial O_j^n \ne \varnothing]\,.$$

We define a relation $\mathcal{R}$ on the set I by: $i\,\mathcal{R}\,j \iff \forall n\in\mathbb{N}\quad \partial O_i^n\cap\partial O_j^n\neq\varnothing$. Let $\sim$ be the transitive closure of the relation $\mathcal{R}$: $i\sim j \iff \exists\, i_1,\dots,i_r\in I\quad i\,\mathcal{R}\,i_1\,\mathcal{R}\cdots\mathcal{R}\,i_r\,\mathcal{R}\,j$. The relation $\sim$ is an equivalence relation on I. Let $I/\sim$ be the quotient set of the equivalence classes. By construction, the sets $\big(\bigcup_{i\in\pi}\partial O_i^n,\ \pi\in I/\sim\big)$ are pairwise disjoint continua included in $\partial^\circ K_n$. Therefore

$$\mathcal{H}^1(\partial^\circ K_n)\geq\mathcal{H}^1\left(\bigcup_{\pi\in I/\sim}\bigcup_{i\in\pi}\partial O_i^n\right)=\sum_{\pi\in I/\sim}\mathcal{H}^1\left(\bigcup_{i\in\pi}\partial O_i^n\right).$$

Since the sequence $(K_n)_{n\in\mathbb{N}}$ converges for the Hausdorff metric, it is contained in a bounded set, and up to the extraction of another subsequence, we may assume that for each π in $I/\sim$, the sequence $\big(\bigcup_{i\in\pi}\partial O_i^n\big)_{n\in\mathbb{N}}$ converges to some element F_π of $\mathcal{K}_c$. Necessarily the set F_π contains $\bigcup_{i\in\pi}\partial O_i$. Applying Golab Theorem 4.2, we get for any π in $I/\sim$

$$\liminf_{n\to\infty}\ \mathcal{H}^1\left(\bigcup_{i\in\pi}\partial O_i^n\right)\geq\mathcal{H}^1(F_\pi)\geq\mathcal{H}^1\left(\bigcup_{i\in\pi}\partial O_i\right).$$

Coming back to the preceding inequality, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\ \mathcal{H}^1(\partial^\circ K_n)&\geq\sum_{\pi\in I/\sim}\ \liminf_{n\to\infty}\ \mathcal{H}^1\left(\bigcup_{i\in\pi}\partial O_i^n\right)\geq\sum_{\pi\in I/\sim}\mathcal{H}^1\left(\bigcup_{i\in\pi}\partial O_i\right)\\&\geq\mathcal{H}^1\left(\bigcup_{\pi\in I/\sim}\bigcup_{i\in\pi}\partial O_i\right)=\mathcal{H}^1\left(\bigcup_{i\in I}\partial O_i\right).\end{aligned}$$

This inequality is valid for any finite family $(O_i,\ i\in I)$ of residual domains of K. The monotone continuity of $\mathcal{H}^1$ implies that $\liminf_{n\to\infty}\mathcal{H}^1(\partial^\circ K_n)\geq\mathcal{H}^1(\partial^\circ K)$. □

6. – True tangents

In this section, we introduce a stronger definition of tangency.

DEFINITION 6.1 (true tangent of a set E at a point x). A 1-set E has a true tangent at x in the direction θ if it has a tangent at x in the direction θ (in the sense of Definition 4.4) and in addition

$$\lim_{r\to 0} r^{-1}e(E\cap B(x,r),L(x,r,\theta))=0\,.$$

REMARK. A segment has a tangent at its endpoints but not a true tangent.

Proposition 6.2. *Let* $\gamma : [0,1] \mapsto \mathbb{R}^2$ *be a rectifiable curve and let* t_0 *belong to* $]0,1[$. *If* γ *is differentiable at* t_0 *and* $\gamma'(t_0) \neq 0$, *then the curve* γ *has a true tangent at* $\gamma(t_0)$.

Proof. Set $x = \gamma(t_0)$. The density of γ at x is at least $1/2$ because γ is a continuum. Let θ be the angle such that $L(x,\theta) = x + \gamma'(t_0)(\mathbb{R})$. The derivative $\gamma'(t_0)$ maps linearly $\mathbb{R}$ onto $L(x,\theta)$; it can be written $\gamma'(t_0)(s) = x + \alpha s u(\theta)$ for some $\alpha \neq 0$. Yet, by definition of the derivative,

$$\forall \varepsilon > 0 \quad \exists \eta > 0 \quad |t - t_0| < \eta \quad \Rightarrow \quad |\gamma(t) - x - \alpha(t - t_0)u(\theta)| \leq \varepsilon |t - t_0| .$$

Let $\varepsilon > 0$ and let $\eta > 0$ be associated to ε as in the above formula. Let ϕ be the angle such that $\tan\phi = \varepsilon/\alpha$. The preceding inequality implies that for t in $[t_0 - \eta, t_0 + \eta]$, $\gamma(t)$ belongs to the cone $S(x, \pi+\theta-\phi, \pi+\theta+\phi) \cup S(x, \theta-\phi, \theta+\phi)$. Since γ is one to one and continuous, the set $\gamma([0, t_0 - \eta] \cup [t_0 + \eta, 1])$ is compact and does not contain x. Hence there exists r_0 such that $0 < r_0 < d(x, \gamma([0, t_0 - \eta] \cup [t_0 + \eta, 1]))$. Therefore for r smaller than r_0, the set $\gamma \cap B(x,r) \setminus (S(x, \pi + \theta - \phi, \pi + \theta + \phi) \cup S(x, \theta - \phi, \theta + \phi))$ is empty. This proves that γ has a tangent at x in the direction θ in the sense of Definition 4.4. We finally prove that this tangent is a true tangent. Let r be smaller than $\alpha\eta$ and set $r' = r(1 - \varepsilon/\alpha)$. For s in $[-r', r']$, we have $|\gamma(t_0 + s/\alpha) - x - su(\theta)| \leq s\varepsilon/\alpha \leq r'\varepsilon/\alpha$ and also $|\gamma(t_0 + s/\alpha) - x| \leq s + r'\varepsilon/\alpha \leq r'(1 + \varepsilon/\alpha) \leq r$, whence $\gamma(t_0 + s/\alpha)$ belongs to $B(x,r)$. Consequently,

$$\begin{aligned} e(\gamma \cap B(x,r), L(x,r,\theta)) &\leq e(\gamma \cap B(x,r), L(x,r',\theta)) + e(L(x,r',\theta), L(x,r,\theta)) \\ &\leq r'\varepsilon/\alpha + r - r' \ \leq \ 2r\varepsilon/\alpha , \end{aligned}$$

so that $r^{-1} e(\gamma \cap B(x,r), L(x,r,\theta))$ goes to zero as r goes to zero. □

By [8, Theorem 3.8], we know that a rectifiable curve has a tangent at $\mathcal{H}^1$ almost all of its points. We have a slightly stronger result.

Corollary 6.3. *A rectifiable curve has a true tangent at* $\mathcal{H}^1$ *almost all of its points.*

Proof. Let $\gamma : [0,1] \mapsto \mathbb{R}^2$ be a Lipschitz parametrization of the curve γ. By the Rademacher Theorem [11, Theorem 7.3], the map γ is differentiable $\mathcal{H}^1$ almost everywhere in $[0,1]$; we denote by γ' its derivative when it is defined. Since γ is a Lipschitz map,

$$\mathcal{H}^1(\{\gamma(t) : t \text{ such that } \gamma \text{ is not differentiable at } t\}) = 0 .$$

By Proposition 6.2, the curve γ has a true tangent at the point $\gamma(t)$, $0 < t < 1$, whenever $\gamma'(t) \neq 0$. However, by the Sard-type theorem for Lipschitz maps [11, Theorem 7.6], we have $\mathcal{H}^1(\{\gamma(t) : \gamma'(t) = 0\}) = 0$. □

Corollary 6.4. *A continuum* E *such that* $\mathcal{H}^1(E)$ *is finite has a true tangent at* $\mathcal{H}^1$ *almost all of its points.*

Proof. This result is an easy consequence of Theorem 4.5 and Corollaries 4.6, 6.3. □

7. – Structure of $\partial^\circ K$

In this section we analyze the local behavior of $\partial^\circ K$.

7.1. – Preparatory lemmas

LEMMA 7.1. *Let U be a domain, let K be a continuum. If $\mathcal{H}^1(\partial^\circ K \cap U) = 0$ then either $U \subset K$ or $U \subset \mathbb{R}^2 \setminus K$.*

PROOF. Suppose that neither $U \subset K$ nor $U \subset \mathbb{R}^2 \setminus K$. Then there exists a pair (x, y) in $U \cap K \times U \cap (\mathbb{R}^2 \setminus K)$. Let O be the residual domain of K containing y. Clearly $\partial O \subset \partial^\circ K$ and $\mathcal{H}^1(\partial O \cap U) \leq \mathcal{H}^1(\partial^\circ K \cap U)$ whence by hypothesis $\mathcal{H}^1(\partial O \cap U) = 0$. If $\mathcal{H}^1(\partial O) = 0$ then $\operatorname{diam} O = 0$, which is impossible. Therefore $\partial O \cap (\mathbb{R}^2 \setminus U) \neq \varnothing$. Let γ be a curve in U joining x to y (U is arcwise connected). This curve intersects ∂O at some point z. Yet ∂O is connected and contains z and some point in $\mathbb{R}^2 \setminus U$. Thus $\mathcal{H}^1(\partial O \cap U) \geq d(z, \mathbb{R}^2 \setminus U) > 0$, which is absurd. □

The next lemma is a technical result which will be used repeatedly in the proofs.

LEMMA 7.2. *Let K be a continuum and let A be a closed set such that both A and $\mathbb{R}^2 \setminus A$ are connected. We suppose that K is not included in A and that $\mathcal{H}^1(\partial^\circ K \cap A) \leq \delta$. Let V be a domain included in A such that $d(V, \mathbb{R}^2 \setminus A) > \delta$. Then either $K \subset \mathbb{R}^2 \setminus V$ or $V \subset \mathcal{V}(K, \delta)$. If $K \cap V \neq \varnothing$, no residual domain of K intersects both V and $\mathbb{R}^2 \setminus A$.*

REMARK. The final conclusion of Lemma 7.2 is still valid for residual domains of K in a domain W.

PROOF. Suppose we have not $K \subset \mathbb{R}^2 \setminus V$ i.e. there exists x in $K \cap V$. Suppose there exists a residual domain O of K intersecting both V and $\mathbb{R}^2 \setminus A$. Let y belong to $O \cap V$ and let γ be a curve in V joining x to y. This curve intersects ∂O at some point z. Similarly, considering x' in $K \cap (\mathbb{R}^2 \setminus A)$ and y' in $O \cap (\mathbb{R}^2 \setminus A)$, we see that ∂O contains some point z' of $\mathbb{R}^2 \setminus A$. Yet ∂O is connected and contains z and z'. Thus $\mathcal{H}^1(\partial O \cap A) \geq d(z, \mathbb{R}^2 \setminus A) \geq d(V, \mathbb{R}^2 \setminus A) > \delta$ which is absurd. Suppose now that there exists y in V such that $d(y, K) \geq \delta$. Let O be the residual domain of K containing y. The previous argument shows that $O \cap (\mathbb{R}^2 \setminus A) = \varnothing$ whence $\overline{O} \subset A$ and $\mathcal{H}^1(\partial O) \leq \mathcal{H}^1(\partial^\circ K \cap A) \leq \delta$, implying $\operatorname{diam} O \leq \delta$, which is absurd since O contains the interior of the ball $B(y, \delta)$. □

LEMMA 7.3. *For any continuum K, any point x, any angles θ, ϕ and any $r > 0$, we have*

$$e(\mathbb{R}^2 \setminus U_-(x, r, \theta, \phi), K) \,+\, e(K, U_-(x, r, \theta, \phi)) \geq r \cos\phi (1 + \cos\phi)^{-1} .$$

PROOF. Let $x(r) = x + r(1+\cos\phi)^{-1}u(\theta - \pi/2)$. We have

$$\begin{aligned} d(\mathbb{R}^2 \setminus U_-(x,r,\theta,\phi), x(r)) &= r\cos\phi(1+\cos\phi)^{-1} \\ &\le e(\mathbb{R}^2 \setminus U_-(x,r,\theta,\phi), U_-(x,r,\theta,\phi)) \\ &\le e(\mathbb{R}^2 \setminus U_-(x,r,\theta,\phi), K) + e(K, U_-(x,r,\theta,\phi))\,. \end{aligned}$$

LEMMA 7.4. *Let x belong to $\mathbb{R}^2$ and let θ be an arbitrary angle. For ϕ in $]0,\pi/4[$, ε in $]0,1/4[$, r positive, the set $\bigcup_{0<s<r} V_-(x,s,\varepsilon,\theta,\phi)$ is a domain containing x in its boundary.*

PROOF. Indeed, for ϕ in $]0,\pi/4[$, ε in $]0,1/4[$, s in $]0,r[$, the point $x + (s/2)u(\theta-\pi/2)$ belongs to $V_-(x,s,\varepsilon,\theta,\phi)$. Therefore the open segment $]x, x+(r/2)u(\theta-\pi/2)[$ is in the union $\bigcup_{0<s<r} V_-(x,s,\varepsilon,\theta,\phi)$, which implies the claims of the lemma. □

7.2. – Classification of the points in $\partial^\circ K$

We classify now the points of $\partial^\circ K$.

PROPOSITION 7.5. *Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x in the direction of θ. One and only one of the two following cases occurs:*

$$\begin{array}{lll} \textit{either} & \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(x,r,\theta),K)=0, & \liminf_{r\to 0} r^{-1}e(K,U_-(x,r,\theta))\ge 1/6\,, \\ \textit{or} & \lim_{r\to 0} r^{-1}e(K,U_-(x,r,\theta))=0, & \liminf_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(x,r,\theta),K)\ge 1/6\,. \end{array}$$

The same result holds for $U_+(x,r,\theta)$.

PROOF. Since the point x and the direction θ are fixed for the whole proof, we will omit them in the notation. For instance $U(r,\phi)$ stands for $U(x,r,\theta,\phi)$. By the definition of a tangent, we have

$$\forall\phi>0 \quad \forall\varepsilon>0 \quad \exists r_0 \quad \forall r<r_0 \qquad \mathcal{H}^1(\partial^\circ K \cap U(x,r,\theta,\phi)) \le r\varepsilon\,.$$

We work with ε,ϕ small, r_0 smaller than $\operatorname{diam} K/2$ and $r<r_0$. More specifically, we require that $\cos\phi\,(1+\cos\phi)^{-1} > 1/4$ (for instance $\phi < \pi/4$) and $\varepsilon < 1/48$. Let us consider the set $V_-(r,2\varepsilon,\phi)$. Clearly this set is included in $U_-(r,\phi)$. Moreover, for ε small enough, $U_-(r,\phi)$ is included in $\mathcal{V}(V_-(r,2\varepsilon,\phi),3\varepsilon r)$. We apply Lemma 7.2 to the sets $K, U_-(r,\phi), V_-(r,2\varepsilon,\phi)$. Since K is not included in $U_-(r,\phi)$ (because $r < \operatorname{diam} K/2$), $\mathcal{H}^1(\partial^\circ K \cap U_-(r,\phi)) \le r\varepsilon$ and $d(V_-(r,2\varepsilon,\phi), \mathbb{R}^2\setminus U_-(r,\phi)) > r\varepsilon$ then either $K \subset \mathbb{R}^2\setminus V_-(r,2\varepsilon,\phi)$ or $V_-(r,2\varepsilon,\phi)\subset\mathcal{V}(K,r\varepsilon)$. Therefore, for any r smaller than r_0,

$$\text{either}\quad K\subset\mathcal{V}(\mathbb{R}^2\setminus U_-(r,\phi),4r\varepsilon)\quad\text{or}\quad U_-(r,\phi)\subset\mathcal{V}(K,4r\varepsilon)\,.$$

Fix some $r < r_0$.

- Suppose that $K \subset \mathcal{V}(\mathbb{R}^2 \setminus U_-(r,\phi), 4r\varepsilon)$. For $s < r$, we have $K \subset \mathcal{V}(\mathbb{R}^2 \setminus U_-(s,\phi), 4r\varepsilon)$ and $e(\mathbb{R}^2 \setminus U_-(s,\phi), K) \leq 4r\varepsilon$. Suppose that $U_-(s,\phi) \subset \mathcal{V}(K, 4s\varepsilon)$. Then $e(K, U_-(s,\phi)) \leq 4s\varepsilon$ and Lemma 7.3 implies that $s\cos\phi\,(1+\cos\phi)^{-1} \leq 4r\varepsilon + 4s\varepsilon$. Because of the conditions imposed on ϕ, ε, this inequality implies that $s < r/2$. Thus for s in $[r/2, r]$ we have $K \subset \mathcal{V}(\mathbb{R}^2 \setminus U_-(s,\phi), 4s\varepsilon)$.
- Suppose that $U_-(r,\phi) \subset \mathcal{V}(K, 4r\varepsilon)$. For $s < r$, we have $U_-(s,\phi) \subset \mathcal{V}(K, 4r\varepsilon)$ and $e(K, U_-(s,\phi)) \leq 4r\varepsilon$. Suppose that $K \subset \mathcal{V}(\mathbb{R}^2 \setminus U_-(s,\phi), 4s\varepsilon)$. Then $e(\mathbb{R}^2 \setminus U_-(s,\phi), K) \leq 4s\varepsilon$ and Lemma 7.3 implies that $s\cos\phi(1+\cos\phi)^{-1} \leq 4r\varepsilon + 4s\varepsilon$. Because of the conditions imposed on ϕ, ε, this inequality implies that $s < r/2$. Thus for s in $[r/2, r]$ we have $U_-(s,\phi) \subset \mathcal{V}(K, 4s\varepsilon)$.

  Since $]0, r] = \bigcup_{n\in\mathbb{N}}]2^{-n-1}r, 2^{-n}r]$, we see that

$$\begin{aligned} &\text{either} \quad \forall r < r_0 \quad K \subset \mathcal{V}(\mathbb{R}^2 \setminus U_-(r,\phi), 4r\varepsilon) \\ &\text{or} \quad \forall r < r_0 \quad U_-(r,\phi) \subset \mathcal{V}(K, 4r\varepsilon)\,. \end{aligned}$$

  Because of Lemma 7.3, we have the two exclusive cases:

$$\begin{aligned} &\text{either } \forall r < r_0 \;\; r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) \leq 4\varepsilon \text{ and } r^{-1}e(K, U_-(r,\phi)) \geq 1/6 \\ &\text{or} \quad \forall r < r_0 \; r^{-1}e(K, U_-(r,\phi)) \leq 4\varepsilon \quad \text{and } r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) \geq 1/6\,. \end{aligned}$$

  For $\varepsilon < 1/48$, we have $4\varepsilon < 1/6$, so that the case which occurs does not depend on ε. Therefore, for any ϕ in $]0, \pi/4[$, we have

$$\begin{aligned} \text{either} \quad & \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) = 0 \qquad \text{and} \\ & \liminf_{r\to 0} r^{-1}e(K, U_-(r,\phi)) \geq 1/6 \\ \text{or} \quad & \lim_{r\to 0} r^{-1}e(K, U_-(r,\phi)) = 0 \qquad \text{and} \\ & \liminf_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) \geq 1/6\,. \end{aligned}$$

  Moreover, for $0 < \phi_1 < \phi_2 < \pi/4$ and $r > 0$, we have $U_-(r,\phi_2) \subset U_-(r,\phi_1)$, so that

$$\begin{aligned} e(K, U_-(r,\phi_2)) &\leq e(K, U_-(r,\phi_1)), \\ e(\mathbb{R}^2 \setminus U_-(r,\phi_2), K) &\leq e(\mathbb{R}^2 \setminus U_-(r,\phi_1), K)\,. \end{aligned}$$

  Consequently if one of the two cases occurs for some ϕ in $]0, \pi/4[$, it occurs for all ϕ in $]0, \pi/4[$. Therefore

$$\begin{aligned} \text{either} \quad & \forall \phi \in]0, \pi/4[\;\; \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) = 0, \\ & \liminf_{r\to 0} r^{-1}e(K, U_-(r,\phi)) \geq 1/6 \\ \text{or} \quad & \forall \phi \in]0, \pi/4[\;\; \lim_{r\to 0} r^{-1}e(K, U_-(r,\phi)) = 0, \\ & \liminf_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(r,\phi), K) \geq 1/6\,. \end{aligned}$$

Finally, for any $\phi > 0$, we have

$$e(U_-(x,r,\theta,\phi), U_-(x,r,\theta)) = e(\mathbb{R}^2 \backslash U_-(x,r,\theta), \mathbb{R}^2 \backslash U_-(x,r,\theta,\phi)) \le r\phi\,.$$

Suppose for instance that the first case occurs. Then for any ϕ in $]0,\pi/4[$, we have

$$\liminf_{r\to 0} r^{-1} e(K, U_-(x,r,\theta)) \ge \liminf_{r\to 0} r^{-1} e(K, U_-(x,r,\theta,\phi)) \ge 1/6$$

and also

$$\limsup_{r\to 0} r^{-1} e(\mathbb{R}^2 \setminus U_-(x,r,\theta), K) \le \phi\,.$$

Letting ϕ go to zero, we get

$$\lim_{r\to 0} r^{-1} e(\mathbb{R}^2 \setminus U_-(x,r,\theta), K) = 0\,.$$

The second case can be handled analogously. □

PROPOSITION 7.6. *Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x in the direction of θ. Then*

$$\lim_{r\to 0} \min\left\{ r^{-1} e(K, HL(x,r,\theta)), r^{-1} e(K, HL(x,r,\pi+\theta)) \right\} = 0\,.$$

PROOF. If $r^{-1}e(K, U_-(x,r,\theta))$ or $r^{-1}e(K, U_+(x,r,\theta))$ converges to 0 as r goes to 0, then clearly so does $r^{-1}e(K, L(x,r,\theta))$. According to Proposition 7.5, the only remaining possibility is that

$$\lim_{r\to 0} r^{-1} e(\mathbb{R}^2 \backslash U_-(x,r,\theta), K) = 0 \text{ and } \lim_{r\to 0} r^{-1} e(\mathbb{R}^2 \backslash U_+(x,r,\theta), K) = 0\,.$$

By the definition of $\partial^\circ K$, the point x belongs to the boundary ∂O of some residual domain O of K. Yet ∂O is a continuum. Let r be smaller than $\operatorname{diam} O/2$ and let $F(r)$ be the connected component of $\partial O \cap B(x,r)$ containing x. Because of the particular shapes of the sets $U_-(x,r,\theta)$, $U_+(x,r,\theta)$, we have for any positive s

$$\mathcal{V}(\mathbb{R}^2 \backslash U_-(x,r,\theta), s) \cap \mathcal{V}(\mathbb{R}^2 \backslash U_+(x,r,\theta), s) = \mathcal{V}(L(x,r,\theta) \cup (\mathbb{R}^2 \setminus B(x,r)), s)$$

whence

$$\begin{aligned} &e(L(x,r,\theta) \cup (\mathbb{R}^2 \setminus B(x,r)), K) \\ &\quad \le \max\left\{ e(\mathbb{R}^2 \setminus U_-(x,r,\theta), K), e(\mathbb{R}^2 \setminus U_+(x,r,\theta), K) \right\} \end{aligned}$$

and

$$\lim_{r\to 0} r^{-1} e(L(x,r,\theta) \cup (\mathbb{R}^2 \setminus B(x,r)), K) = 0\,.$$

Let ε be positive. There exists r_0 such that $e(L(x,r,\theta)\cup(\mathbb{R}^2\setminus B(x,r)),K) < r\varepsilon$ for $r < r_0$. Then for $r < r_0$ the set $\mathcal{V}(L(x,r,\theta)\cup(\mathbb{R}^2\setminus B(x,r)),r\varepsilon)\cap B(x,r(1-2\varepsilon))$ is included in $\mathcal{V}(L(x,r,\theta),r\varepsilon)$ whence $F(r(1-2\varepsilon))\subset\mathcal{V}(L(x,r,\theta),r\varepsilon)$. Moreover $F(r(1-2\varepsilon))$ intersects the sphere $\partial B(x,r(1-2\varepsilon))$. Let ϕ be the angle in $]0,\pi/2[$ such that $\sin\phi = 2\varepsilon/(1-2\varepsilon)$. With these choices, the set $\mathcal{V}(L(x,r,\theta),r\varepsilon)\cap\partial B(x,r(1-2\varepsilon))$ is included in $S(x,r(1-2\varepsilon),\pi+\theta-\phi,\pi+\theta+\phi)\cup S(x,r(1-2\varepsilon),\theta-\phi,\theta+\phi)$. Suppose for instance that

$$F(r(1-2\varepsilon))\cap S(x,r(1-2\varepsilon),\theta-\phi,\theta+\phi)\neq\varnothing\,.$$

Let y be a point of the above set. The continuum $F(r(1-2\varepsilon))$ contains x and y and is included in $\mathcal{V}(L(x,r,\theta),r\varepsilon)$. Yet for any s positive smaller than $r(1-4\varepsilon)$, the segment $[x+su(\theta)-r\varepsilon v(\theta),x+su(\theta)+r\varepsilon v(\theta)]$ disconnects x from y inside $\mathcal{V}(L(x,r,\theta),r\varepsilon)$. Therefore $F(r(1-2\varepsilon))$ intersects this segment and $e(F(r(1-2\varepsilon)),HL(x,r(1-4\varepsilon),\theta))\leq r\varepsilon$. Since $e(HL(x,r(1-4\varepsilon),\theta),HL(x,r,\theta))\leq 4r\varepsilon$, it follows that $e(F(r),HL(x,r,\theta))\leq 5r\varepsilon$. We handle similarly the case where $F(r(1-2\varepsilon))\cap S(x,r(1-2\varepsilon),\pi+\theta-\phi,\pi+\theta+\phi)\neq\varnothing$ to get that for $r<r_0$, either $e(F(r),HL(x,r,\theta))\leq 5r\varepsilon$ or $e(F(r),HL(x,r,\pi+\theta))\leq 5r\varepsilon$. $\square$

Propositions 7.5, 7.6 allow to introduce the following classification of tangent points.

DEFINITION 7.7 (classification of tangent points). Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x in the direction of θ. The point x is of exactly one of the following types.

- type O: $\lim_{r\to 0} r^{-1}e(K,U_-(x,r,\theta))=0,\quad \lim_{r\to 0} r^{-1}e(K,U_+(x,r,\theta))=0.$
- type 1/2: $\lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(x,r,\theta),K)=0,\quad \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_+(x,r,\theta),K)=0,$
  $\liminf_{r\to 0} r^{-1}e(K,L(x,r,\theta))>0.$
- type I: either $\lim_{r\to 0} r^{-1}e(K,U_-(x,r,\theta))=0,\quad \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_+(x,r,\theta),K)=0$
  or $\lim_{r\to 0} r^{-1}e(K,U_+(x,r,\theta))=0,\quad \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(x,r,\theta),K)=0.$
- type II: $\lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_-(x,r,\theta),K)=0,\quad \lim_{r\to 0} r^{-1}e(\mathbb{R}^2\setminus U_+(x,r,\theta),K)=0,$
  $\lim_{r\to 0} r^{-1}e(K,L(x,r,\theta))=0.$

We denote respectively by $\partial_O K$, $\partial_{1/2}K$, $\partial_I K$, $\partial_{II}K$ the points of $\partial^\circ K$ where there is a tangent and which are respectively of type O, type 1/2, type I, type II.

REMARK. Because the maps $(x,r,\theta)\in\mathbb{R}^2\times\mathbb{R}^+\times\mathbb{R}\mapsto U_-(x,r,\theta)$, $L(x,r,\theta)$, $U_+(x,r,\theta)$ are continuous with respect to the Hausdorff distance D, for any continuum K, the sets $\partial_O K$, $\partial_{1/2}K$, $\partial_I K$, $\partial_{II}K$ are all $\mathcal{H}^1$-measurable.

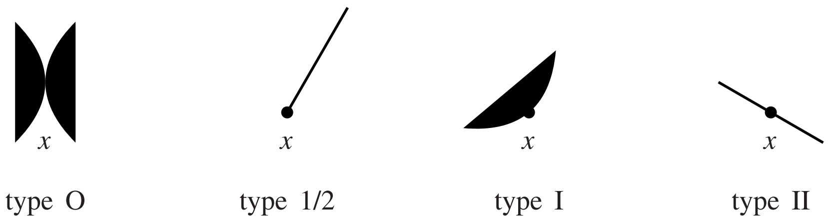

Fig. 6.

NOTATION 7.8. Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x. From now onwards, we denote by $\theta(x)$ the direction of the tangent to $\partial^\circ K$ at x. As a line direction, this angle $\theta(x)$ is defined modulo π. But whenever x is of type I, we choose $\theta(x)$ modulo 2π so that

$$\lim_{r\to 0} r^{-1} e(K, U_-(x,r,\theta(x))) = 0, \qquad \lim_{r\to 0} r^{-1} e(\mathbb{R}^2 \setminus U_+(x,r,\theta(x)), K) = 0\,.$$

PROPOSITION 7.9. *Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x. We have the following characterization of the type of x (recall that $K(x,r) = K \cap B(x,r)$):*

- *x is of type O* $\iff$ $\lim_{r\to 0}\ r^{-1} D(K(x,r), B(x,r)) = 0\,.$
- *x is of type 1/2* $\iff$ $\liminf_{r\to 0}\ r^{-1} e(K(x,r), L(x,r,\theta(x))) > 0\,.$
- *x is of type I* $\iff$ $\lim_{r\to 0}\ r^{-1} D(K(x,r), U_-(x,r,\theta(x))) = 0\,.$
- *x is of type II* $\iff$ $\lim_{r\to 0}\ r^{-1} D(K(x,r), L(x,r,\theta(x))) = 0\,.$

PROOF. It is clear that the four conditions on the right are mutually exclusive. Hence it is enough to check the implications from the left to the right for each type. From Definition 7.7, the cases of the points of type O and type 1/2 are immediate. Let us consider a point x of type I. Let $\theta = \theta(x)$ be the direction of the tangent at x. By Definition 7.7 and Notation 7.8, we have

$$\lim_{r\to 0} r^{-1} e(K, U_-(x,r,\theta(x))) = 0, \qquad \lim_{r\to 0} r^{-1} e(\mathbb{R}^2 \setminus U_+(x,r,\theta(x)), K) = 0\,.$$

Thus for any positive ε, there exists $r_0 > 0$ such that for $r < r_0$, $e(K, U_-(x,r,\theta)) \le r\varepsilon$, $e(\mathbb{R}^2 \setminus U_+(x,r,\theta), K) \le r\varepsilon$. We have then for $r < r_0$

$$\begin{aligned} e(K(x,r), U_-(x,r,\theta)) &\le e(K(x,r), U_-(x, r(1-2\varepsilon), \theta)) \\ &\qquad + e(U_-(x, r(1-2\varepsilon), \theta), U_-(x,r,\theta)) \\ &\le r(1-2\varepsilon)\varepsilon + 2r\varepsilon \ \le\ 3r\varepsilon\,. \end{aligned}$$

Therefore $r^{-1} e(K(x,r), U_-(x,r,\theta(x)))$ goes to 0 as r goes to 0.

Similarly, for ε in $]0, 1/2[$ and for $r < r_1 = r_0/(1+2\varepsilon)$,

$$e(U_-(x,r,\theta), K(x,r)) \le 2\varepsilon r + e(U_-(x, r(1+2\varepsilon), \theta), K(x,r)) .$$

Because $r(1+2\varepsilon) < r_0$, we have $e(\mathbb{R}^2 \setminus U_+(x, r(1+2\varepsilon), \theta), K) \le r(1+2\varepsilon)\varepsilon$. Moreover $d(K(x,r), \partial B(x, r(1+2\varepsilon))) \ge 2\varepsilon r > r(1+2\varepsilon)\varepsilon$. Therefore

$$\begin{aligned} e(U_-(x, r(1+2\varepsilon), \theta), K(x,r)) &\le e(L(x, r(1+2\varepsilon), \theta), K(x,r) \cap U_+(x,r,\theta)) \\ &= e(\mathbb{R}^2 \setminus U_+(x, r(1+2\varepsilon), \theta), K(x,r)) \\ &\le r(1+2\varepsilon)\varepsilon \end{aligned}$$

and we obtain $e(U_-(x,r,\theta), K(x,r)) \le 2r\varepsilon + r(1+2\varepsilon)\varepsilon$. Thus $r^{-1}e(U_-(x,r,\theta(x)), K(x,r))$ goes to 0 as r goes to 0.

Let us consider finally a point x of type II. Let $\theta = \theta(x)$ be the direction of the tangent at x. By Definition 7.7, the three quantities $r^{-1}e(\mathbb{R}^2 \setminus U_-(x,r,\theta), K)$, $r^{-1}e(\mathbb{R}^2 \setminus U_+(x,r,\theta), K)$, $r^{-1}e(K, L(x,r,\theta))$ go to 0 as r goes to 0. Thus for any positive ε, there exists $r_0 > 0$ such that for $r < r_0$, the three of them are smaller than ε. For $r < r_0$ we have then

$$\begin{aligned} e(K(x,r), L(x,r,\theta)) &\le e(K(x,r), L(x, r(1-2\varepsilon), \theta)) \\ &\quad + e(L(x, r(1-2\varepsilon), \theta), L(x,r,\theta)) \\ &\le r(1-2\varepsilon)\varepsilon + 2r\varepsilon . \end{aligned}$$

Hence $r^{-1}e(K(x,r), L(x,r,\theta))$ goes to 0 when r goes to 0.

Similarly, for ε in $]0, 1/2[$ and for $r < r_1 = r_0/(1+2\varepsilon)$,

$$e(L(x,r,\theta), K(x,r)) \le 2\varepsilon r + e(L(x, r(1+2\varepsilon), \theta), K(x,r)) .$$

Because $r(1+2\varepsilon) < r_0$ both $e(\mathbb{R}^2 \setminus U_-(x, r(1+2\varepsilon), \theta), K)$ and $e(\mathbb{R}^2 \setminus U_+(x, r(1+2\varepsilon), \theta), K)$ are smaller than $r(1+2\varepsilon)\varepsilon$ so that

$$e(\partial B(x, r(1+2\varepsilon)) \cup L(x, r(1+2\varepsilon), \theta), K(x,r)) \le r(1+2\varepsilon)\varepsilon .$$

Moreover $d(K(x,r), \partial B(x, r(1+2\varepsilon))) \ge 2\varepsilon r > r(1+2\varepsilon)\varepsilon$. Therefore

$$e(L(x, r(1+2\varepsilon), \theta), K(x,r)) \le r(1+2\varepsilon)\varepsilon$$

and we obtain $e(L(x,r,\theta), K(x,r)) \le 2r\varepsilon + r(1+2\varepsilon)\varepsilon$. Thus $r^{-1}e(L(x,r,\theta(x)), K(x,r))$ goes to 0 as r goes to 0. $\square$

7.3. – Local structure of $\partial^\circ K$

We next analyze successively the local structure of $\partial^\circ K$ near each type of tangent point.

LEMMA 7.10 (type 0). *Let K be a continuum. Let x be a point of $\partial^\circ K$ such that $\partial^\circ K$ has a tangent at x. Suppose x is of type O. Then there exists a positive r such that for any domain U containing x and included in $B(x,r)$, there does not exist a residual domain O of K in U such that ∂O has a true tangent at x.*

PROOF. The point x and the direction $\theta(x)$ being fixed for the whole proof, we will omit them in the notation as usual. Since x if of type O, we have:

$$\forall \phi \in]0,\pi/4[\quad \forall \varepsilon \in]0,1/8[\quad \exists r_0 \quad \forall r < r_0$$
$$\mathcal{H}^1(\partial^\circ K \cap U(r,\phi)) \leq r\varepsilon, \quad e(K, U(r,\phi)) \leq r\varepsilon\,.$$

We impose in addition that $r_0 < \operatorname{diam} K/2$. As in the proof of Proposition 7.5, we consider the set $V_-(r,2\varepsilon,\phi)$ for $r < r_0$. We check that the hypothesis of Lemma 7.2 are satisfied by the sets $K, U_-(r,\phi), V_-(r,2\varepsilon,\phi)$. The set K is not included in $U_-(r,\phi)$, $\mathcal{H}^1(\partial^\circ K \cap U_-(r,\phi)) \leq r\varepsilon$ and $d(V_-(r,2\varepsilon,\phi), \mathbb{R}^2 \setminus U_-(r,\phi)) > r\varepsilon$. If $K \cap V_-(r,2\varepsilon,\phi) = \varnothing$ then

$$e(\mathbb{R}^2 \setminus V_-(r,2\varepsilon,\phi), V_-(r,2\varepsilon,\phi)) \leq e(K, U_-(r,\phi)) \leq e(K, U(r,\phi)) \leq r\varepsilon\,.$$

A direct computation gives $e(\mathbb{R}^2 \setminus V_-(r,2\varepsilon,\phi), V_-(r,2\varepsilon,\phi)) = r(1-4\varepsilon)\cos\phi(1+\cos\phi)^{-1}$. Hence the preceding inequality cannot occur when $0 < \phi < \pi/4$, $0 < \varepsilon < 1/8$ so that we have $K \cap V_-(r,2\varepsilon,\phi) \neq \varnothing$. The last part of Lemma 7.2 then implies that no residual domain of K intersects both $V_-(r,2\varepsilon,\phi)$ and $\mathbb{R}^2 \setminus U_-(r,\phi)$. The same result holds for the sets $V_+(r,2\varepsilon,\phi)$ and $\mathbb{R}^2 \setminus U_+(r,\phi)$.

Let us consider the set F_0 defined by

$$F_0 = \bigcup_{0<r<r_0} V_-(r,2\varepsilon,\phi) \cup \{x\} \cup \bigcup_{0<r<r_0} V_+(r,2\varepsilon,\phi)\,.$$

This set is connected: Lemma 7.4 shows that it is the union of two connected sets having a common point. Moreover the set F_0 contains the segment $[x-(r_0/2)v(\theta), x+(r_0/2)v(\theta)]$ which disconnects the interior of the angular sectors $S(x,\pi+\theta-\phi,\pi+\theta+\phi)$, $S(x,\theta-\phi,\theta+\phi)$ inside $B(x,r_0/2)$.

Let U be a domain containing x and included in $B(x,r_0/2)$. Let r_1 positive be such that $B(x,r_1) \subset U$. Suppose there exists a residual domain O of K in U such that ∂O has a true tangent at x. By definition, we have then $\lim_{s\to 0} s^{-1} e(\partial O \cap B(x,s), L(x,s,\theta)) = 0$. Hence there exists s_0 smaller than $r_0/2$ and r_1 such that $e(\partial O \cap B(x,s), L(x,s,\theta)) < (s/4)\sin\phi$ for $s < s_0$. Let s be smaller than s_0. We have then

$$\begin{aligned} d(x \pm (s/2)u(\theta), \partial O) &< (s/4)\sin\phi \\ &< d\big(x \pm (s/2)u(\theta), \mathbb{R}^2 \setminus (S(x,\pi+\theta-\phi,\pi+\theta+\phi) \cup S(x,\theta-\phi,\theta+\phi))\big) \end{aligned}$$

so that O intersects both $S(x, \pi + \theta - \phi, \pi + \theta + \phi)$ and $S(x, \theta - \phi, \theta + \phi)$ inside $B(x, s)$. Thus the domain O intersects $\mathbb{R}^2 \setminus (U_-(s, \phi) \cup U_+(s, \phi))$ for $s > 0$. By Lemma 7.2, this implies that O does not intersect $V_-(s, 2\varepsilon, \phi)$ nor $V_+(s, 2\varepsilon, \phi)$ for $0 < s < r_0$. Since x does not belong to O, it follows that O does not intersect F_0, which is absurd. □

LEMMA 7.11 (type 1/2). *Let K be a continuum, let O be a residual domain of K. Let x be a point of ∂O where $\partial^\circ K$ has a tangent. If ∂O has a true tangent at x then x is not of type 1/2.*

PROOF. Since ∂O has a true tangent in the direction $\theta = \theta(x)$ (the direction of the tangent is the same for ∂O and $\partial^\circ K$), then $r^{-1} e(\partial O \cap B(x, r), L(x, r, \theta))$ goes to 0 when r goes to 0. But ∂O is a subset of K, whence $r^{-1} e(K(x, r), L(x, r, \theta))$ goes to 0 as well when r goes to 0. □

LEMMA 7.12 (type I). *Let K be a continuum and let x belong to $\partial_I K$. For any positive ε there exists a positive $r(x, \varepsilon)$ such that*

$$\forall r < r(x, \varepsilon) \ \ \forall K' \in \mathcal{K}_c \ \ D(K, K') \le r\varepsilon \Rightarrow D(K'(x, r), U_-(x, r, \theta(x))) \le 4r\varepsilon\,.$$

PROOF. By Proposition 7.9, since x is a point of type I, then $r^{-1} D(K(x, r), U_-(x, r, \theta(x)))$ goes to 0 when r goes to 0. Let ε be positive and smaller than one. There exists a positive r_0 such that $D(K(x, r), U_-(x, r, \theta(x))) \le r\varepsilon$ for $r < r_0$. We set $r_1 = r_0(1-\varepsilon)$. Let r be smaller than r_1 and let K' be a compact connected set such that $D(K, K') \le r\varepsilon$. We have then $e(K'(x, r), K(x, r(1 - \varepsilon))) \le r\varepsilon$ so that

$$\begin{aligned} e(K'(x, r), U_-(x, r, \theta(x))) &\le e(K'(x, r), K(x, r(1 - \varepsilon))) \\ &\quad + e(K(x, r(1 - \varepsilon)), U_-(x, r(1 - \varepsilon), \theta(x))) \\ &\quad + e(U_-(x, r(1 - \varepsilon), \theta(x)), U_-(x, r, \theta(x))) \\ &\le r\varepsilon + r(1 - \varepsilon)\varepsilon + r\varepsilon \le 3r\varepsilon\,. \end{aligned}$$

Since $r(1 + \varepsilon) \le r_0(1 - \varepsilon^2) < r_0$, we have $e(U_-(x, r(1 + \varepsilon), \theta(x)), K(x, r(1 + \varepsilon))) \le r(1 + \varepsilon)\varepsilon$ so that

$$\begin{aligned} e(U_-(x, r, \theta(x)), K'(x, r)) &\le e(U_-(x, r, \theta(x)), U_-(x, r(1 + \varepsilon), \theta(x))) \\ &\quad + e(U_-(x, r(1 + \varepsilon), \theta(x)), K(x, r(1 + \varepsilon))) \\ &\quad + e(K(x, r(1 + \varepsilon)), K'(x, r)) \\ &\le r\varepsilon + r\varepsilon(1 + \varepsilon) + r\varepsilon \le 4r\varepsilon\,. \end{aligned}$$

Thus $r(x, \varepsilon) = r_1$ answers the problem. □

LEMMA 7.13 (type II). *Let K be a continuum and let x belong to $\partial_{II} K$. For any positive ε there exists a positive $r(x, \varepsilon)$ such that*

$$\forall r < r(x, \varepsilon) \ \ \forall K' \in \mathcal{K}_c \ \ D(K, K') \le r\varepsilon \Rightarrow D(K'(x, r), L(x, r, \theta(x))) \le 4r\varepsilon\,.$$

PROOF. The proof is similar to the proof of Lemma 7.12. □

Lemma 7.14. *For any compact sets K_1, K_2, the sets $\partial_{II}(K_1 \cup K_2) \cap (\partial_I K_1 \cup \partial_I K_2)$ and $\partial_I(K_1 \cup K_2) \cap \partial_{II} K_1 \cap \partial_{II} K_2$ are empty.*

Proof. By $K_1 \cup K_2(x,r)$ we denote the set $(K_1 \cup K_2)(x,r)$. For any point x, any positive r and any angles θ_1, θ, we have

$$\begin{aligned} r &\le e(L(x,r,\theta), U_-(x,r,\theta_1)) \\ &\le e(L(x,r,\theta), K_1 \cup K_2(x,r)) + e(K_1(x,r), U_-(x,r,\theta_1)) \end{aligned}$$

so that $r^{-1}e(L(x,r,\theta), K_1 \cup K_2(x,r))$ and $r^{-1}e(K_1(x,r), U_-(x,r,\theta_1))$ cannot go simultaneously to 0 when r goes to 0. Therefore $\partial_{II}(K_1 \cup K_2) \cap \partial_I K_1$ is empty. Analogously, for any point x, any positive r and any angles $\theta_1, \theta_2, \theta$, we have

$$\begin{aligned} r/\sqrt{2} &\le e(L(x,r,\theta_1) \cup L(x,r,\theta_2), U_-(x,r,\theta)) \\ &\le \max\left\{e(L(x,r,\theta_1), K_1(x,r)), e(L(x,r,\theta_2), K_2(x,r))\right\} \\ &\quad + e(K_1 \cup K_2(x,r), U_-(x,r,\theta)) \end{aligned}$$

so that the three quantities

$$r^{-1}e(L(x,r,\theta_1), K_1(x,r)), \quad r^{-1}e(L(x,r,\theta_2), K_2(x,r)),$$
$$r^{-1}e(K_1 \cup K_2(x,r), U_-(x,r,\theta))$$

cannot go simultaneously to 0 when r goes to 0. Therefore $\partial_I(K_1 \cup K_2) \cap \partial_{II} K_1 \cap \partial_{II} K_2$ is empty. $\square$

8. – The continua K with $\mathcal{H}^1(\partial^\circ K)$ finite

The goal of this section is to show that if K is a continuum with $\mathcal{H}^1(\partial^\circ K) < \infty$, then $\mathcal{H}^1$ almost all points of $\partial^\circ K$ have true tangents and are of type I or II.

Notation 8.1. If O is a domain, we denote by $\partial^* O$ the set of the points of ∂O where ∂O has a true tangent.

Definition 8.2. Let K be a continuum. We set

$$\partial^* K = \left(\partial_O K \cup \partial_{1/2} K \cup \partial_I K \cup \partial_{II} K\right) \setminus \bigcup_U \bigcup_O (\partial O \setminus (\partial^* O \cup \partial U))$$

where the first union is over all the domains U of the plane and the second union is over all domains O in $\mathcal{C}(K,U)$. We set also $\partial_I^* K = \partial^* K \cap \partial_I K$ and $\partial_{II}^* K = \partial^* K \cap \partial_{II} K$.

Lemma 8.3. *Let O be a domain such that $\mathcal{H}^1(\partial O)$ is finite. Let x belong to $\mathbb{R}^2$ and let s, r be two positive real numbers with $s < r$. There is at most a finite number of connected components of $O \cap \mathring{B}(x,r)$ which intersect $B(x,s)$.*

PROOF. Let $n \geq 2$ and suppose that $O_1, \ldots, O_n$ are connected components of $O \cap \mathring{B}(x,r)$ intersecting $B(x,s)$. Let t be such that $s < t < r$. For each i in $\{1 \cdots n\}$, the domain O_i intersects both spheres $\partial B(x,s)$ and $\partial B(x,t)$ (otherwise O_i would not be connected). Since O_i is arcwise connected, there exists a simple arc $\gamma_i : [0,1] \mapsto O_i$ such that: $\gamma_i(0) \in \partial B(x,t)$, $\gamma_i(1) \in \partial B(x,s)$ and $\gamma_i(u) \in \mathring{B}(x,t) \setminus B(x,s)$ for u in $]0,1[$ (we first consider an arc in O_i joining $\partial B(x,t)$ to $\partial B(x,s)$ and we look at the portion between the last visit to $\partial B(x,t)$ and the hitting time of $\partial B(x,s)$). Clearly the arcs γ_i, $1 \leq i \leq n$, are pairwise disjoint. We may order the sequence $\gamma_1, \ldots, \gamma_n$ so that when we move counterclockwise on $\partial B(x,t)$ we observe successively $\gamma_1(0), \ldots, \gamma_n(0)$. Necessarily, if we move counterclockwise on $\partial B(x,s)$ we observe $\gamma_1(1), \ldots, \gamma_n(1)$ in the same order (otherwise two arcs would intersect). These n arcs separate the annulus $B(x,t) \setminus \mathring{B}(x,s)$ into n domains $A_1, \ldots, A_n$, where A_1 is delimited by (γ_1, γ_2), $\ldots$, A_{n-1} by (γ_{n-1}, γ_n), A_n by (γ_n, γ_{n+1}) (we make the convention that $\gamma_{n+1} = \gamma_1$). Let ψ be the map from $\mathbb{R}^2$ to $\mathbb{R}^+$ defined by $\psi(y) = |y - x|_2$. Clearly ψ is Lipschitz with constant 1. Applying [11, Theorem 7.7, p. 104], we have

$$\mathcal{H}^1(\partial O) \geq \mathcal{H}^1\big(\partial O \cap (B(x,t) \setminus \mathring{B}(x,s))\big) \geq \int_s^t \operatorname{card}\big(\partial O \cap \psi^{-1}(u)\big)\, du\,.$$

Let u belong to $]s,t[$. Each arc γ_i, $1 \leq i \leq n$, intersects the sphere $\partial B(x,u)$. For i in $\{1 \cdots n\}$, let $\overline{x_i x_{i+1}}$ be a subarc of $\partial B(x,u)$ such that $x_i \in \gamma_i$, $x_{i+1} \in \gamma_{i+1}$ and the arcs γ_j, $j \in \{1 \cdots n+1\} \setminus \{i, i+1\}$, do not intersect $\overline{x_i x_{i+1}} \setminus \{x_i, x_{i+1}\}$. Necessarily the arc $\overline{x_i x_{i+1}} \setminus \{x_i, x_{i+1}\}$ meets ∂O. Since there are n such subarcs with pairwise disjoint interiors, we see that $\partial O \cap \psi^{-1}(u)$ contains at least n points. Therefore $n(t-s) \leq \mathcal{H}^1(\partial O)$ and the number n of connected components of $O \cap \mathring{B}(x,r)$ is bounded. □

LEMMA 8.4. *Let O be a domain such that $\mathcal{H}^1(\partial O)$ is finite. Let x belong to ∂O. For any domain U containing x, there exists a connected component O' of $O \cap U$ such that x belongs to $\partial O'$.*

PROOF. Let s, r be such that $0 < s < r$ and $B(x,r) \subset U$. By Lemma 8.3, there is at most a finite number of connected components of $O \cap \mathring{B}(x,r)$ intersecting $B(x,s)$, say $O_1, \ldots, O_n$. We have then $\partial O \cap B(x,s) = (\partial O_1 \cup \cdots \cup \partial O_n) \cap B(x,s)$ so that there exists i in $\{1 \cdots n\}$ such that x belongs to ∂O_i. Let O' be the connected component of $O \cap U$ containing O_i. Then $\partial O_i \setminus \partial B(x,r) \subset \partial O'$ so that x is in $\partial O'$. □

COROLLARY 8.5. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. Let x be a point of $\partial^\circ K$. Let U be a domain containing x. There exists a residual domain O of K in U such that x belong to ∂O.*

LEMMA 8.6. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. Let x be a point of $\partial^\circ K$ where $\partial^\circ K$ has a tangent. Let U be a domain containing x and suppose that there exists a residual domain O of K in U such that x belongs to ∂O and ∂O has not a true tangent at x. Then for any domain U' containing x and included in U,* *there exists a residual domain O' of K in U' such that O' is included in O, x belongs to $\partial O'$ and $\partial O'$ has not a true tangent at x.*

Proof. Let K, U, O, U' be as in the statement of the lemma. Let s, r be such that $0 < s < r$ and $B(x, r) \subset U'$. Certainly $\mathcal{H}^1(\partial O)$ is finite, hence by Lemma 8.3, there is at most a finite number of connected components of $O \cap \overset{\circ}{B}(x, r)$ intersecting $B(x, s)$, say $O_1, \dots, O_n$. We have then $\partial O \cap B(x, s) = (\partial O_1 \cup \cdots \cup \partial O_n) \cap B(x, s)$ so that there exists i in $\{1 \cdots n\}$ such that x belongs to ∂O_i. Let O' be the connected component of $O \cap U'$ containing O_i. Then $\partial O_i \setminus \partial B(x, r) \subset \partial O'$ so that x is in $\partial O'$. Moreover $\partial O' \cap U' \subset \partial O$. Since $\partial^\circ K$ has a tangent at x, necessarily

$$\forall \phi \in]0, \pi/2] \qquad \lim_{t \to 0} \frac{1}{t} \mathcal{H}^1(\partial O \cap U(x, t, \theta, \phi)) = 0\,.$$

However ∂O has not a true tangent at x. Either $\lim_{t\to 0} \mathcal{H}^1(\partial O \cap B(x, t))/t = 0$ or

$$\liminf_{t \to 0} \frac{1}{t} e(\partial O \cap B(x, t), L(x, t, \theta)) > 0\,.$$

In both cases, the same property holds for $\partial O'$, hence $\partial O'$ has not a true tangent at x. □

Corollary 8.7. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of domains which is a basis for the topology of $\mathbb{R}^2$. Then*

$$\partial^* K = \big(\partial_O K \cup \partial_{1/2} K \cup \partial_I K \cup \partial_{II} K\big) \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{O \in \mathcal{C}(K, U_n)} (\partial O \setminus (\partial^* O \cup \partial U_n))\,.$$

Proof. Indeed, let x be a point of $\partial^\circ K$ where $\partial^\circ K$ has a tangent and suppose that for some domain U, there exists O in $\mathcal{C}(K, U)$ such that x belongs to $\partial O \setminus (\partial^* O \cup \partial U)$. Since $(U_n)_{n\in\mathbb{N}}$ is a basis for the topology of $\mathbb{R}^2$, there exists n in $\mathbb{N}$ such that x belongs to U_n and U_n is included in U. By Lemma 8.6, there exists a residual domain O_n of K in U_n such that x belongs to $\partial O_n \setminus (\partial^* O_n \cup \partial U_n)$. □

Proposition 8.8. *Let K be a continuum. If $\mathcal{H}^1(\partial^\circ K) < \infty$ then $\partial^\circ K$ is a regular 1-set and moreover $\mathcal{H}^1(\partial^\circ K \setminus \partial^* K) = 0$.*

Proof. We recall that $\partial^\circ K = \bigcup_{i\in I} \partial O_i$ where $(O_i,\, i \in I)$ are the residual domains of K (see Definition 5.1), and the set I is finite or countable. Each set ∂O_i is a continuum of finite $\mathcal{H}^1$-measure because $\mathcal{H}^1(\partial O_i) \le \mathcal{H}^1(\partial^\circ K) < \infty$. Theorem 4.5 implies that each ∂O_i, $i \in I$, as well as $\partial^\circ K$, consists of a countable union of rectifiable curves, together with a set of $\mathcal{H}^1$-measure zero. Hence $\partial^\circ K$ is a regular 1-set and has a tangent at $\mathcal{H}^1$-almost all of its points (by Corollaries 3.4, 6.3 or [8, Corollaries 3.9, 3.10]). Therefore we have

$$\mathcal{H}^1\big(\partial^\circ K \setminus (\partial_O K \cup \partial_{1/2} K \cup \partial_I K \cup \partial_{II} K)\big) = 0\,.$$

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of domains which is a basis for the topology of $\mathbb{R}^2$ and such that $\mathcal{H}^1(\partial U_n)$ is finite for any n (choose for instance a collection of open balls). Then for any n in $\mathbb{N}$ and any O in $\mathcal{C}(K, U_n)$ we have $\partial O \setminus \partial U_n \subset \partial^\circ K$ (if O' is the residual domain of K in $\mathbb{R}^2$ containing O then $\partial O \setminus \partial U_n \subset \partial O'$) and

$$\mathcal{H}^1(\partial O) \leq \mathcal{H}^1(\partial O \setminus \partial U_n) + \mathcal{H}^1(\partial U_n) \leq \mathcal{H}^1(\partial^\circ K) + \mathcal{H}^1(\partial U_n) < \infty\,.$$

By Corollary 6.4, $\mathcal{H}^1(\partial O \setminus \partial^* O) = 0$. Therefore the set

$$\bigcup_{n\in\mathbb{N}} \bigcup_{O\in\mathcal{C}(K,U_n)} (\partial O \setminus (\partial^* O \cup \partial U_n))$$

is a countable union of sets having zero $\mathcal{H}^1$-measure (by Corollary 3.4) and therefore it has $\mathcal{H}^1$-measure zero. By Corollary 8.7, this set contains $(\partial_O K \cup \partial_{1/2} K \cup \partial_I K \cup \partial_{II} K) \setminus \partial^* K$ whence $\mathcal{H}^1(\partial^\circ K \setminus \partial^* K) = 0$. □

Proposition 8.9. *Let K be a continuum. If $\mathcal{H}^1(\partial^\circ K) < \infty$ then $\mathcal{H}^1(\partial_O K \cup \partial_{1/2} K) = 0$.*

Proof. By Lemma 7.11, the set $\partial_{1/2} K$ is included in $\bigcup_{O\in\mathcal{C}(K)} (\partial O \setminus \partial^* O)$ and by Corollary 6.4, $\mathcal{H}^1(\partial O \setminus \partial^* O) = 0$ for any O in $\mathcal{C}(K)$. Hence $\mathcal{H}^1(\partial_{1/2} K) = 0$.

We finally prove that $\mathcal{H}^1(\partial_O K) = 0$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of domains which is a basis for the topology of $\mathbb{R}^2$ and such that $\mathcal{H}^1(\partial U_n)$ is finite for any n (choose for instance a collection of open balls). Let x belong to $\partial_O K$. We apply Lemma 7.10: there exists a positive r such that for any domain U containing x and included in $B(x, r)$, there does not exist a residual domain O of K in U such that ∂O has a true tangent at x. Let n in $\mathbb{N}$ be such that U_n contains x and is included in $B(x, r)$. By Lemma 8.5, there exists a residual domain O of K in U_n such that x belongs to ∂O. Since U_n is included in $B(x, r)$, ∂O has not a true tangent at x so that x belongs to $\partial O \setminus \partial^* O$. Therefore we have

$$\partial_O K \subset \bigcup_{n\in\mathbb{N}} \bigcup_{O\in\mathcal{C}(K,U_n)} (\partial O \setminus \partial^* O)\,.$$

For any n in $\mathbb{N}$ and O in $\mathcal{C}(K, U_n)$, we have $\partial O \setminus \partial U_n \subset \partial^\circ K$ (if O' is the residual domain of K in $\mathbb{R}^2$ containing O then $\partial O \setminus \partial U_n \subset \partial O'$) and

$$\mathcal{H}^1(\partial O) \leq \mathcal{H}^1(\partial O \setminus \partial U_n) + \mathcal{H}^1(\partial U_n) \leq \mathcal{H}^1(\partial^\circ K) + \mathcal{H}^1(\partial U_n) < \infty\,,$$

whence by Corollary 6.4, $\mathcal{H}^1(\partial O \setminus \partial^* O) = 0$. Hence $\partial_O K$ is included in the countable union of sets of $\mathcal{H}^1$-measure zero and $\mathcal{H}^1(\partial_O K) = 0$. □

Corollary 8.10. *Let K be a continuum. If $\mathcal{H}^1(\partial^\circ K) < \infty$ then $\mathcal{H}^1(\partial^\circ K \setminus \partial_I^* K \setminus \partial_{II}^* K) = 0$.*

9. – Local structure of $\partial_I^* K$ and $\partial_{II}^* K$

In this section, we focus further on the points of types I and II where there is a true tangent. We recall that a point x belonging to the boundary ∂O of an open set O is said to be accessible from O if there exists a continuous arc $\gamma : [0,1] \mapsto \overline{O}$ such that $\gamma([0,1[) \subset O$ and $\gamma(1) = x$.

Proposition 9.1. *Let K be a continuum and let x belong to $\partial_I^* K$. There exists r positive such that for any domain U containing x and included in $B(x,r)$, there exists a unique residual domain O of K in U such that x belongs to ∂O. Moreover x is accessible from O.*

Proof. Let $\theta = \theta(x)$ be the direction of the tangent to $\partial^\circ K$ at x. Since x is of type I, we have: $\forall \phi \in]0, \pi/4[\quad \forall \varepsilon \in]0, 1/8[\quad \exists r_0 \quad \forall r < r_0$

$$\mathcal{H}^1(\partial^\circ K \cap U(r,\phi)) \le r\varepsilon\,, \quad e(K, U_-(r,\phi)) \le r\varepsilon\,, \quad e(\mathbb{R}^2 \setminus U_+(r,\phi), K) \le r\varepsilon\,.$$

We impose that $r_0 < \operatorname{diam} K/2$. We have then $U_-(r,\phi) \subset \mathcal{V}(K, r\varepsilon)$ for $r < r_0$. Let us consider the set $V_+(r, 2\varepsilon, \phi)$. Since $d(V_+(r,2\varepsilon,\phi), \mathbb{R}^2 \setminus U_+(r,\phi)) > r\varepsilon$ we have $V_+(r,2\varepsilon,\phi) \cap K = \varnothing$ for $r < r_0$. Let F_+ be the domain $F_+ = \bigcup_{r<r_0} V_+(r,2\varepsilon,\phi)$. Then F_+ does not intersect K and contains the segment $]x, x + r_0(1-3\varepsilon)v(\theta)]$.

Let U be a domain containing x and included in $B(x, r_0/2)$. Let O be the residual domain of K in U containing $F_+ \cap B(x, r_0/2)$. Clearly x belongs to ∂O and x is accessible from O. Suppose there is another residual domain O' of K in U such that x belongs to $\partial O'$. Since $O \cap O' = \varnothing$ then $O' \cap F_+ = \varnothing$. Yet x belongs to $\partial_I^* K$, so that $\partial O'$ must have a true tangent at x. This tangent is necessarily in the direction θ (because $\partial O' \setminus \partial U \subset \partial^\circ K$). Necessarily, O' meets both $S(x, \pi+\theta-\phi, \pi+\theta+\phi)$ and $S(x, \theta-\phi, \theta+\phi)$ inside $B(x,r)$ for r sufficiently small, say $r < r_1 < r_0/2$.

We check that the hypothesis of Lemma 7.2 are satisfied by the sets $K, U_-(r,\phi), V_-(r,2\varepsilon,\phi)$ for $r < r_0$:

$$K \cap (\mathbb{R}^2 \setminus U_-(r,\phi)) \neq \varnothing, \quad \mathcal{H}^1(\partial^\circ K \cap U_-(r,\phi)) \le r\varepsilon,$$
$$d(V_-(r,2\varepsilon,\phi), \mathbb{R}^2 \setminus U_-(r,\phi)) > r\varepsilon\,.$$

If $K \cap V_-(r,2\varepsilon,\phi) = \varnothing$ then

$$e(\mathbb{R}^2 \setminus V_-(r,2\varepsilon,\phi), V_-(r,2\varepsilon,\phi)) \le e(K, U_-(r,\phi)) \le r\varepsilon\,.$$

A direct computation gives $e(\mathbb{R}^2 \setminus V_-(r,2\varepsilon,\phi), V_-(r,2\varepsilon,\phi)) = r(1-4\varepsilon)\cos\phi(1+\cos\phi)^{-1}$. Hence the preceding inequality cannot occur when $0 < \phi < \pi/4$, $0 < \varepsilon < 1/8$ so that we have $K \cap V_-(r,2\varepsilon,\phi) \neq \varnothing$. The last part of Lemma 7.2 then implies that no residual domain of K intersects both $V_-(r,2\varepsilon,\phi)$ and $\mathbb{R}^2 \setminus U_-(r,\phi)$.

The set O' intersects $B(x,r) \setminus U_-(r,\phi)$ for $r < r_1$, hence it intersects $\mathbb{R}^2 \setminus U_-(r,\phi)$ for $r < r_0$ and thus it does not intersect $V_-(r,2\varepsilon,\phi)$ for $r <$

r_0. If we set $F_- = \bigcup_{r<r_0} V_-(r, 2\varepsilon, \phi)$ then $O' \cap F_- = \varnothing$. It follows that $O' \cap (F_- \cup \{x\} \cup F_+) = \varnothing$. However $F_- \cup \{x\} \cup F_+$ disconnects the interior of the angular sectors $S(x, \pi + \theta - \phi, \pi + \theta + \phi)$, $S(x, \theta - \phi, \theta + \phi)$ inside $B(x, r_0/2)$, which is absurd. □

Proposition 9.2. *Let K be a continuum and let x belong to $\partial^*_{II}K$. There exists r positive such that for any domain U containing x and included in $B(x, r)$, there exist either one or two residual domains O of K in U such that x belongs to ∂O. Moreover x is accessible from each such domain.*

Proof. Let $\theta = \theta(x)$ be the direction of the tangent to $\partial^\circ K$ at x. Since x is of type II, we have: $\forall \phi \in]0, \pi/4[\quad \forall \varepsilon \in]0, 1/8[\quad \exists r_0 \quad \forall r < r_0$

$$\mathcal{H}^1(\partial^\circ K \cap U(r, \phi)) \le r\varepsilon\,, \quad e(\mathbb{R}^2 \setminus U_-(r, \phi), K) \le r\varepsilon\,, \quad e(\mathbb{R}^2 \setminus U_+(r, \phi), K) \le r\varepsilon\,.$$

We impose that $r_0 < \operatorname{diam} K/2$. Let us consider as usual the set $V_-(r, 2\varepsilon, \phi)$. Since $d(V_-(r, 2\varepsilon, \phi), \mathbb{R}^2 \setminus U_-(r, \phi)) > r\varepsilon$ we have $V_-(r, 2\varepsilon, \phi) \cap K = \varnothing$ for $r < r_0$. Let F_- be the domain $F_- = \bigcup_{r<r_0} V_-(r, 2\varepsilon, \phi)$. Then F_- does not intersect K and contains the segment $]x, x - r_0(1 - 3\varepsilon)v(\theta)]$. Similarly, the domain $F_+ = \bigcup_{r<r_0} V_+(r, 2\varepsilon, \phi)$ does not intersect K and contains the segment $]x, x + r_0(1 - 3\varepsilon)v(\theta)]$. Let U be a domain containing x and included in $B(x, r_0/2)$. Let O_- (respectively O_+) be the residual domain of K in U containing F_- (respectively F_+). It might happen that $O_- = O_+$. Clearly x belongs to ∂O_- and ∂O_+ and x is accessible from both O_- and O_+. Suppose there is another residual domain O' of K in U such that x belongs to $\partial O'$. Since $(O_- \cup O_+) \cap O' = \varnothing$ then $O' \cap (F_- \cup F_+) = \varnothing$. Yet x belongs to $\partial^*_I K$, so that $\partial O'$ must have a true tangent at x. This tangent is necessarily in the direction θ (because $\partial O' \setminus \partial U \subset \partial^\circ K$). Necessarily, O' meets both $S(x, \pi + \theta - \phi, \pi + \theta + \phi)$ and $S(x, \theta - \phi, \theta + \phi)$ inside $B(x, r)$ for r sufficiently small. However $F_- \cup \{x\} \cup F_+$ disconnects the interior of the angular sectors $S(x, \pi + \theta - \phi, \pi + \theta + \phi)$, $S(x, \theta - \phi, \theta + \phi)$ inside $B(x, r_0/2)$, which is absurd. □

Corollary 9.3. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$ and let U be a domain.*

- *Any x in $\partial^*_I K \cap U$ belongs to the boundary of exactly one residual domain of K in U.*
- *Any x in $\partial^*_{II} K \cap U$ belongs to the boundary of one or two residual domains of K in U.*

Proof. This result is a consequence of Lemma 8.4 and Propositions 9.1, 9.2. □

10. – The surface energy $\mathcal{S}$

We first prove a covering lemma for the sets of points of type I and type II.

LEMMA 10.1. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. Let ε be positive. Suppose that to each point of $\partial_I^* K$ (respectively $\partial_{II}^* K$) there is associated a positive number $r_1(x)$ (respectively $r_2(x)$), possibly depending on ε. There exists a finite family of disjoint balls $B(x_i, r_i)$, $i \in I_1 \cup I_2$, such that: for i in I_1, x_i belongs to $\partial_I^* K$ and $0 < r_i < r_1(x_i)$, for i in I_2, x_i belongs to $\partial_{II}^* K$ and $0 < r_i < r_2(x_i)$, and*

$$\mathcal{H}^1(\partial_I^* K) + 2\mathcal{H}^1(\partial_{II}^* K) \leq (1 + 2\varepsilon)\left(2\sum_{i\in I_1} r_i + 4\sum_{i\in I_2} r_i\right),$$

$$\mathcal{H}^1\left(\partial^\circ K \setminus \bigcup_{i\in I_1\cup I_2} B(x_i, r_i)\right) \leq 2\varepsilon \sum_{i\in I_1\cup I_2} r_i\,.$$

PROOF. Under the hypothesis that $\mathcal{H}^1(\partial^\circ K) < \infty$, the sets $\partial_I^* K$ and $\partial_{II}^* K$ are $\mathcal{H}^1$-measurable and their $\mathcal{H}^1$-measures are finite (see the remark after Definition 7.7 together with Definition 8.2 and Proposition 8.8). Moreover $\partial^\circ K$ is a regular 1-set by Proposition 8.8 and has density 1 at $\mathcal{H}^1$ almost all of its points. Hence if we define

$$\partial^{\circ *} K = \{x \in \partial^* K : \lim_{r\to 0}(2r)^{-1}\mathcal{H}^1(\partial^\circ K \cap B(x, r)) = 1\}$$

and

$$\partial_I^{**} K = \partial_I^* K \cap \partial^{\circ *} K, \quad \partial_{II}^{**} K = \partial_{II}^* K \cap \partial^{\circ *} K\,,$$

then we have $\mathcal{H}^1(\partial^\circ K \setminus \partial^{\circ *} K) = 0$ so that $\mathcal{H}^1(\partial_I^* K \setminus \partial_I^{**} K) = 0$, $\mathcal{H}^1(\partial_{II}^* K \setminus \partial_{II}^{**} K) = 0$. Now for each x in $\partial_I^{**} K \cup \partial_{II}^{**} K$, there exists $r(x, \varepsilon)$ positive such that

$$\forall r \in]0, r(x, \varepsilon)[\quad 2r(1 - \varepsilon) \leq \mathcal{H}^1(\partial^\circ K \cap B(x, r)) \leq 2r(1 + \varepsilon)\,.$$

The family of closed balls $\{B(x, r) : x \in \partial_{II}^{**} K,\ 0 < r < \min\{r_2(x),\ r(x, \varepsilon)\}\}$ is a Vitali class for $\partial_{II}^{**} K$. By the Corollary 4.3 to the Vitali covering theorem, we may select a finite disjoint sequence of balls in this class, $(B(x_i, r_i), i \in I_2)$, such that

$$\mathcal{H}^1\left(\partial_{II}^{**} K \setminus \bigcup_{i\in I_2} B(x_i, r_i)\right) \leq 2\varepsilon \sum_{i\in I_2} r_i, \quad \mathcal{H}^1(\partial_{II}^{**} K) \leq 2(1 + \varepsilon) \sum_{i\in I_2} r_i\,.$$

The family of closed balls

$$\left\{ B(x, r) : x \in \partial_I^{**} K,\ 0 < r < \min\left\{r_1(x),\ r(x, \varepsilon),\ d\left(x, \bigcup_{i\in I_2} B(x_i, r_i)\right)\right\}\right\}$$

is a Vitali class for $\partial_I^{**}K \setminus \bigcup_{i\in I_2} B(x_i, r_i)$. By the Corollary 4.3 to the Vitali covering theorem, we may select a finite disjoint sequence of balls in this class, $(B(x_i, r_i), i \in I_1)$, such that

$$\mathcal{H}^1\left(\partial_I^{**}K \setminus \bigcup_{i\in I_1\cup I_2} B(x_i, r_i)\right) \le 2\varepsilon \sum_{i\in I_1} r_i\,,$$

$$\mathcal{H}^1\left(\partial_I^{**}K \setminus \bigcup_{i\in I_2} B(x_i, r_i)\right) \le 2(1+\varepsilon) \sum_{i\in I_1} r_i\,.$$

We have then

$$\begin{aligned}\mathcal{H}^1(\partial_I^{*}K \cup \partial_{II}^{*}K) &= \mathcal{H}^1\left((\partial_I^{**}K \cup \partial_{II}^{**}K) \setminus \bigcup_{i\in I_1\cup I_2} B(x_i, r_i)\right)\\ &\quad + \mathcal{H}^1\left((\partial_I^{**}K \cup \partial_{II}^{**}K) \cap \bigcup_{i\in I_1\cup I_2} B(x_i, r_i)\right)\\ &\le 2\varepsilon \sum_{i\in I_1\cup I_2} r_i + \sum_{i\in I_1\cup I_2} \mathcal{H}^1(\partial^\circ K \cap B(x_i, r_i))\\ &\le 2(1+2\varepsilon) \sum_{i\in I_1\cup I_2} r_i\,.\end{aligned}$$

Combining this inequality with $\mathcal{H}^1(\partial_{II}^{**}K) \le 2(1+\varepsilon)\sum_{i\in I_2} r_i$, we get the desired estimation. ☐

We now define the surface energy of a continuum K.

DEFINITION 10.2. Let K be a continuum. For A a domain we define the surface energy $\mathcal{S}(K, A)$ of K in A by

$$\mathcal{S}(K, A) = \sup_{\mathcal{U}} \sum_{U\in\mathcal{U}} \sum_{O\in\mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U)$$

the supremum being taken over all families $\mathcal{U}$ of pairwise disjoint domains included in A. The surface energy of the whole set K is $\mathcal{S}(K) = \mathcal{S}(K, \mathbb{R}^2)$.

REMARK. Obviously, for any continuum K, any domains A_1, A_2 such that $A_1 \subset A_2$, we have $\mathcal{S}(K, A_1) \le \mathcal{S}(K, A_2)$.

LEMMA 10.3. *For any continuum K, we have $\mathcal{S}(K) \ge 2\operatorname{diam} K$.*

PROOF. Let x, y belong to K with $|x-y|_2 = \operatorname{diam} K$. Let θ be the angle between the horizontal axis and the vector xy. Let U be the open strip

$$U = \{x + au(\theta) + bv(\theta) : 0 < a < |x-y|_2,\ b \in \mathbb{R}\}\,.$$

For b larger than $\operatorname{diam} K$ and any a, the point $x + au(\theta) + bv(\theta)$ does not belong to K. Let O_+ (respectively O_-) be the residual domain of K in U containing the set

$$\{x + au(\theta) + bv(\theta) : 0 < a < |x - y|_2,\ b > \operatorname{diam} K\}$$

(respectively the set $\{x + au(\theta) - bv(\theta) : 0 < a < |x - y|_2,\ b > \operatorname{diam} K\}$). Suppose that $O_- = O_+$. Then there exists an arc γ in $U \setminus K$ joining $(x+y)/2 + 2(\operatorname{diam} K)v(\theta)$ to $(x+y)/2 - 2(\operatorname{diam} K)v(\theta)$; we can extend this arc in $\mathbb{R}^2 \setminus K$ to a Jordan curve γ' such that x is in the interior of γ' and y is in the exterior of γ', contradicting the fact that K is connected. Thus the domains O_- and O_+ are distinct. Clearly, for any a in $]0, |x-y|_2[$, the line $x + au(\theta) + \mathbb{R}v(\theta)$ intersects both ∂O_- and ∂O_+. Thus $\mathcal{S}(K) \geq \mathcal{H}^1(\partial O_- \setminus \partial U) + \mathcal{H}^1(\partial O_+ \setminus \partial U) \geq 2\operatorname{diam} K$. □

Lemma 10.4. *Let K be a continuum and let A_1, A_2 be two disjoint domains in $\mathbb{R}^2$. We have $\mathcal{S}(K, A_1 \cup A_2) = \mathcal{S}(K, A_1) + \mathcal{S}(K, A_2)$.*

Proof. Let $\mathcal{U}$ be a family of pairwise disjoint domains included in $A_1 \cup A_2$. Since A_1 and A_2 are disjoint, each domain U of $\mathcal{U}$ is either a subdomain of A_1 or a subdomain of A_2. Let us define

$$\mathcal{U}_1 = \{U \in \mathcal{U} : U \subset A_1\}, \quad \mathcal{U}_2 = \{U \in \mathcal{U} : U \subset A_2\}.$$

We have then

$$\begin{aligned}\sum_{U \in \mathcal{U}} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) &= \sum_{U \in \mathcal{U}_1} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \\ &\quad + \sum_{U \in \mathcal{U}_2} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \\ &\leq \mathcal{S}(K, A_1) + \mathcal{S}(K, A_2).\end{aligned}$$

Taking the supremum over $\mathcal{U}$, we get $\mathcal{S}(K, A_1 \cup A_2) \leq \mathcal{S}(K, A_1) + \mathcal{S}(K, A_2)$. To prove the converse inequality, we consider two families of pairwise disjoint domains $\mathcal{U}_1$, $\mathcal{U}_2$ included in A_1 and A_2 respectively. Let $\mathcal{U}$ be the union of $\mathcal{U}_1$ and $\mathcal{U}_2$. Then

$$\begin{aligned}&\sum_{U \in \mathcal{U}_1} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) + \sum_{U \in \mathcal{U}_2} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \\ &\quad = \sum_{U \in \mathcal{U}} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \leq \mathcal{S}(K, A_1 \cup A_2).\end{aligned}$$

Taking the supremum over $\mathcal{U}_1$ and $\mathcal{U}_2$, we get $\mathcal{S}(K, A_1) + \mathcal{S}(K, A_2) \leq \mathcal{S}(K, A_1 \cup A_2)$. □

LEMMA 10.5. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. For any domain A we have*

$$\mathcal{S}(K, A) \leq \mathcal{H}^1(\partial_I K \cap A) + 2\mathcal{H}^1(\partial_{II} K \cap A)\,.$$

REMARK. When $\mathcal{H}^1(\partial^\circ K)$ is finite, we have $\mathcal{H}^1(\partial^\circ K \setminus (\partial_I^* K \cup \partial_{II}^* K)) = 0$ by Corollary 8.10 so that $\mathcal{H}^1(\partial_I K \cap A) = \mathcal{H}^1(\partial_I^* K \cap A)$, $\mathcal{H}^1(\partial_{II} K \cap A) = \mathcal{H}^1(\partial_{II}^* K \cap A)$ for any domain A.

PROOF. Let U be a domain. By Corollary 9.3, we have

$$\forall x \in \partial_I^* K \cap U \qquad \sum_{O \in \mathcal{C}(K,U)} \chi(x \in \partial O) = 1$$

whence by integrating over $\partial_I^* K \cap U$ with respect to $\mathcal{H}^1$

$$\sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \cap \partial_I^* K \setminus \partial U) = \mathcal{H}^1(\partial_I^* K \cap U)$$

and for any x in $\partial_{II}^* K \cap U$, we have $\sum_{O \in \mathcal{C}(K,U)} \chi(x \in \partial O) \leq 2$ whence by integrating over $\partial_{II}^* K \cap U$

$$\sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \cap \partial_{II}^* K \setminus \partial U) \leq 2\mathcal{H}^1(\partial_{II}^* K \cap U)\,.$$

Adding the two previous relations yields

$$\sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \cap (\partial_I^* K \cup \partial_{II}^* K) \setminus \partial U) \leq \mathcal{H}^1(\partial_I^* K \cap U) \;+\; 2\mathcal{H}^1(\partial_{II}^* K \cap U)\,.$$

For any O in $\mathcal{C}(K, U)$, we have $\mathcal{H}^1(\partial O \cap (\partial_I^* K \cup \partial_{II}^* K) \setminus \partial U) = \mathcal{H}^1(\partial O \setminus \partial U)$ because $\partial O \setminus \partial U \subset \partial^\circ K$ and $\mathcal{H}^1(\partial^\circ K \setminus (\partial_I^* K \cup \partial_{II}^* K)) = 0$ by Corollary 8.10; therefore the preceding inequality can be rewritten as

$$\sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \leq \mathcal{H}^1(\partial_I^* K \cap U) + 2\mathcal{H}^1(\partial_{II}^* K \cap U)\,.$$

Let $\mathcal{U}$ be a family of pairwise disjoint domains included in A. Summing the preceding inequality over all the domains U in $\mathcal{U}$ we get

$$\sum_{U \in \mathcal{U}} \sum_{O \in \mathcal{C}(K,U)} \mathcal{H}^1(\partial O \setminus \partial U) \leq \mathcal{H}^1(\partial_I^* K \cap A) + 2\mathcal{H}^1(\partial_{II}^* K \cap A)\,.$$

Taking the supremum over all families $\mathcal{U}$, together with the remark stated before the proof, we obtain the claim of the lemma. □

COROLLARY 10.6. *For any x, y in $\mathbb{R}^2$, we have $\mathcal{S}([x, y]) = 2|y - x|_2$.*

PROOF. By Lemma 10.3, we have $\mathcal{S}([x,y]) \geq 2|y-x|_2$. Since $\partial^*_{II}[x,y] =]x,y[$, Lemma 10.5 yields $\mathcal{S}([x,y]) \leq 2\mathcal{H}^1(]x,y[)$. $\square$

LEMMA 10.7. *Let x be a point in $\mathbb{R}^2$ and let θ be an angle. For any positive r, any ε in $]0,1/4[$, any continuum K, we have the implication*

$$\operatorname{diam} K > 2r, \quad D(K(x,r), U_-(x,r,\theta)) \leq r\varepsilon \Longrightarrow \mathcal{S}(K, \overset{\circ}{B}(x,r)) \geq 2r(1-3\varepsilon)\,.$$

PROOF. There exists y in $K(x,r)$ such that $|y-x|_2 \leq r\varepsilon$. Let P be the union of the two segments $P = [y,x] \cup [x, x+rv(\theta)]$. Since $\operatorname{diam} K > 2r$, y is connected by $K \cup [x,y]$ to some point outside $B(x,r)$. Because $\varepsilon < 1/4 < \sin(\pi/8)$, K does not meet $S(x,r,\theta+\pi/8, \pi+\theta-\pi/8)$, so that y is connected by $K(x,r) \cup [x,y]$ to some point of $S(x,r,\pi+\theta-\pi/8, \theta+\pi/8)$. Moreover y is connected in $K(x,r) \cup P$ to $x+rv(\theta)$. Since $D(K(x,r), U_-(x,r,\theta)) \leq r\varepsilon$, then $K(x,r)$ does not intersect the set $\overset{\circ}{B}(x,r) \setminus \mathcal{V}(U_-(x,r,\theta), r\varepsilon)$. This set is disconnected into two components by the segment $[x, x+rv(\theta)]$; let O_1 be the component containing $x+2\varepsilon rv(\theta) - r(1-2\varepsilon)u(\theta)$ and let O_2 be the component containing $x+2\varepsilon rv(\theta)+r(1-2\varepsilon)u(\theta)$. Notice that $K \cap O_1 = K \cap O_2 = \varnothing$. Let O'_1 (respectively O'_2) be the residual domain of K in $\overset{\circ}{B}(x,r) \setminus P$ containing O_1 (respectively O_2). Suppose that $O'_1 = O'_2$. Then there would exist an arc $\gamma : [0,1] \mapsto B(x,r)$ such that: $\gamma(0) = x + ru(\theta+3\pi/4)$, $\gamma(1) = x+ru(\theta+\pi/4)$ and $\gamma(]0,1[) \subset \overset{\circ}{B}(x,r) \setminus K \setminus P$. This arc γ is a cross cut of the sphere $\partial B(x,r)$ disconnecting $x+rv(\theta)$ from $S(x,r,\pi+\theta-\pi/8, \theta+\pi/8)$, which is absurd, since $K \cup P$ realizes this connection. Hence O'_1 and O'_2 are distinct. Let $z = x+2\varepsilon rv(\theta)$. The segment $]z - r\varepsilon u(\theta), z - r(1-2\varepsilon)u(\theta)[$ (respectively $]z + r\varepsilon u(\theta), z + r(1-2\varepsilon)u(\theta)[$) is included in O'_1 (respectively O'_2). Therefore each arc $S(z,s,\pi,0)$, $r\varepsilon < s < r(1-2\varepsilon)$, intersects both $\partial O'_1 \setminus P$ and $\partial O'_2 \setminus P$. It follows that

$$\mathcal{S}(K, \overset{\circ}{B}(x,r) \setminus P) \geq \mathcal{H}^1(\partial O'_1 \setminus \partial B(x,r) \setminus P) + \mathcal{H}^1(\partial O'_2 \setminus \partial B(x,r) \setminus P) \geq 2r(1-3\varepsilon)\,.$$

Applying the remark after Definition 10.2, we conclude that $\mathcal{S}(K, \overset{\circ}{B}(x,r)) \geq 2r(1-3\varepsilon)$. $\square$

LEMMA 10.8. *Let x be a point in $\mathbb{R}^2$ and let θ be an angle. For any positive r and ε in $]0,1/4[$, any continuum K, we have the implication*

$$\operatorname{diam} K > 2r, \quad D(K(x,r), L(x,r,\theta)) \leq r\varepsilon \Longrightarrow \mathcal{S}(K, \overset{\circ}{B}(x,r)) \geq 4r(1-4\varepsilon)\,.$$

PROOF. We have $K \cap \partial B(x,r) \subset S(x,r,\pi+\theta-2\varepsilon, \pi+\theta+2\varepsilon) \cup S(x,r,\theta-2\varepsilon, \theta+2\varepsilon)$ and $K \cap \partial B(x,r) \neq \varnothing$. Therefore the set $K(x,r) \cup S(x,r,\pi+\theta-2\varepsilon, \pi+\theta+2\varepsilon) \cup S(x,r,\theta-2\varepsilon,\theta+2\varepsilon)$ has either one or two components. Suppose it has two components, and let K_1 (respectively K_2) be the one containing $S(x,r,\pi+\theta-2\varepsilon, \pi+\theta+2\varepsilon)$ (respectively $S(x,r,\theta-2\varepsilon,\theta+2\varepsilon)$). These components are closed sets. Let (y_1,y_2) in $K_1 \times K_2$ be such that $d(K_1,K_2) = |y_1-y_2|_2$. Let $y = (y_1+y_2)/2$ be the middle of y_1 and y_2. Since the set $\mathcal{V}(L(x,r,\theta), r\varepsilon) \cap B(x,r)$ is convex, then y is still in this set, so that the ball

$B(y, r\varepsilon)$ intersects $L(x, r, \theta)$; thus the ball $B(y, 2r\varepsilon)$ intersects $K(x, r)$ and meets either K_1 or K_2. Therefore either $d(y, K_1) \leq 2r\varepsilon$ or $d(y, K_2) \leq 2r\varepsilon$. By the very construction of y, we have $d(K_1, K_2) = 2d(y, K_1) = 2d(y, K_2)$, so that $d(K_1, K_2) \leq 4r\varepsilon$. In case the initial set is connected, we choose $y_1 = y_2$ to be any point of $K(x, r)$ and the end of the argument is the same. The component of $K(x, r) \cup [y_1, y_2]$ containing $[y_1, y_2]$ meets both $S(x, r, \pi + \theta - 2\varepsilon, \pi + \theta + 2\varepsilon)$ and $S(x, r, \theta - 2\varepsilon, \theta + 2\varepsilon)$. Moreover $K(x, r) \cup [y_1, y_2]$ is included in $\mathcal{V}(L(x, r, \theta), r\varepsilon) \cap B(x, r)$. Let $z_1 = x + 2r\varepsilon v(\theta)$ and $z_2 = x - 2r\varepsilon v(\theta)$. Let O_1 (respectively O_2) be the residual domain of $\mathcal{V}(L(x, r, \theta), r\varepsilon)$ inside $\mathring{B}(x, r)$ containing z_1 (respectively z_2). Clearly $K(x, r) \cup [y_1, y_2]$ disconnects O_1 and O_2 inside $B(x, r)$. Let O_1' (respectively O_2') be the residual domain of $K(x, r)$ inside $\mathring{B}(x, r) \setminus [y_1, y_2]$ containing O_1 (respectively O_2). Necessarily, O_1' and O_2' are distinct. The segment $[z_1 - r(1-2\varepsilon)u(\theta), z_1 + r(1-2\varepsilon)u(\theta)]$ (respectively $[z_2 - r(1-2\varepsilon)u(\theta), z_2 + r(1-2\varepsilon)u(\theta)]$) is included in O_1' (respectively O_2'). Therefore each segment $[z_1 + su(\theta), z_2 + su(\theta)]$, $|s| \leq r(1-2\varepsilon)$, meets both $\partial O_1'$ and $\partial O_2'$. It follows that

$$\begin{aligned}
\mathcal{S}(K, \mathring{B}(x, r) \setminus [y_1, y_2]) &\geq \mathcal{H}^1(\partial O_1' \setminus \partial B(x, r) \setminus [y_1, y_2]) \\
&\quad + \mathcal{H}^1(\partial O_2' \setminus \partial B(x, r) \setminus [y_1, y_2]) \\
&\geq \mathcal{H}^1(\partial O_1' \setminus \partial B(x, r)) \\
&\quad + \mathcal{H}^1(\partial O_2' \setminus \partial B(x, r)) - 2\mathcal{H}^1([y_1, y_2]) \\
&\geq 4r(1 - 4\varepsilon)\,.
\end{aligned}$$

Applying the remark after Definition 10.2, we conclude that $\mathcal{S}(K, \mathring{B}(x, r)) \geq 4r(1 - 4\varepsilon)$. □

Proposition 10.9. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. For any domain A we have*

$$\mathcal{S}(K, A) = \mathcal{H}^1(\partial_I K \cap A) + 2\mathcal{H}^1(\partial_{II} K \cap A)\,.$$

In particular, $\mathcal{S}(K) = \mathcal{H}^1(\partial_I K) + 2\mathcal{H}^1(\partial_{II} K)$.

Proof. By Lemma 10.5, we already have $\mathcal{S}(K, A) \leq \mathcal{H}^1(\partial_I K \cap A) + 2\mathcal{H}^1(\partial_{II} K \cap A)$. We now prove the converse inequality. Let ε be positive. By Proposition 7.9, to each point x of $\partial_I^* K \cap A$ we can associate $r_1(x, \varepsilon)$ such that

$$\begin{gathered}
\forall x \in \partial_I^* K \cap A \quad \operatorname{diam} K \cap A > 2r_1(x, \varepsilon), \quad B(x, r_1(x, \varepsilon)) \subset A\,, \\
\forall r < r_1(x, \varepsilon) \quad D(K(x, r), U_-(x, r, \theta)) \leq r\varepsilon\,.
\end{gathered}$$

Similarly, to each point x of $\partial_{II}^* K \cap A$ we can associate $r_2(x, \varepsilon)$ such that

$$\begin{gathered}
\forall x \in \partial_{II}^* K \cap A \quad \operatorname{diam} K \cap A > 2r_2(x, \varepsilon), \quad B(x, r_2(x, \varepsilon)) \subset A\,, \\
\forall r < r_2(x, \varepsilon) \quad D(K(x, r), L(x, r, \theta)) \leq r\varepsilon\,.
\end{gathered}$$

We apply the covering Lemma 10.1 with these functions $r_1(x,\varepsilon)$ and $r_2(x,\varepsilon)$: there exists a finite family of disjoint balls $B(x_i,r_i)$, $i \in I_1 \cup I_2$ such that: for i in I_1, x_i belongs to $\partial_I^* K \cap A$ and $0 < r_i < r_1(x_i,\varepsilon)$, for i in I_2, x_i belongs to $\partial_{II}^* K \cap A$ and $0 < r_i < r_2(x_i,\varepsilon)$, and

$$\mathcal{H}^1(\partial_I^* K \cap A) + 2\mathcal{H}^1(\partial_{II}^* K \cap A) \le (1+2\varepsilon)\left(2\sum_{i\in I_1} r_i + 4\sum_{i\in I_2} r_i\right).$$

By Lemmas 10.4, 10.7, 10.8, we have

$$\mathcal{S}(K,A) \ge \sum_{i\in I_1\cup I_2} \mathcal{S}(K, \mathring{B}(x_i,r_i)) \ge \sum_{i\in I_1} 2r_i(1-3\varepsilon) + \sum_{i\in I_2} 4r_i(1-4\varepsilon).$$

Therefore we have $\mathcal{S}(K,A) \ge (\mathcal{H}^1(\partial_I^* K\cap A) + 2\mathcal{H}^1(\partial_{II}^* K \cap A))(1-4\varepsilon)/(1+2\varepsilon)$ for any positive ε. Letting ε go to zero, we get $\mathcal{S}(K,A) \ge \mathcal{H}^1(\partial_I^* K \cap A) + 2\mathcal{H}^1(\partial_{II}^* K \cap A)$. $\square$

COROLLARY 10.10. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. Then $\mathcal{H}^1(\partial^\circ K) \le \mathcal{S}(K) \le 2\mathcal{H}^1(\partial^\circ K)$.*

PROPOSITION 10.11. *Let K_1, K_2 be any continua. We have $\mathcal{S}(K_1 \cup K_2) \le \mathcal{S}(K_1) + \mathcal{S}(K_2)$. For any domain A, we have also $\mathcal{S}(K_1\cup K_2, A) \le \mathcal{S}(K_1,A) + \mathcal{S}(K_2,A)$.*

PROOF. We do the proof only for the case $A = \mathbb{R}^2$: the general case is similar, just by considering the intersections of the sets with A. We need only to consider the case where $\mathcal{S}(K_1) < \infty$ and $\mathcal{S}(K_2) < \infty$, otherwise there is nothing to prove. By Corollary 10.10, $\mathcal{H}^1(\partial^\circ K_1)$ and $\mathcal{H}^1(\partial^\circ K_2)$ are finite. By Corollary 5.3, $\mathcal{H}^1(\partial^\circ(K_1\cup K_2))$ is also finite. By Lemma 5.2 and Corollary 8.10, we have

$$\begin{aligned}\mathcal{H}^1(\partial_I^*(K_1\cup K_2)) &= \mathcal{H}^1(\partial_I^*(K_1\cup K_2) \cap (\partial^\circ K_1 \cup \partial^\circ K_2))\\ &= \mathcal{H}^1(\partial_I^*(K_1\cup K_2) \cap (\partial_I^* K_1 \cup \partial_I^* K_2))\\ &\quad + \mathcal{H}^1(\partial_I^*(K_1\cup K_2) \cap (\partial_{II}^* K_1 \cup \partial_{II}^* K_2)).\end{aligned}$$

By Lemmas 5.2, 7.14 and Corollary 8.10, we have also

$$\begin{aligned}\mathcal{H}^1(\partial_{II}^*(K_1\cup K_2)) &= \mathcal{H}^1(\partial_{II}^*(K_1\cup K_2) \cap (\partial^\circ K_1 \cup \partial^\circ K_2))\\ &= \mathcal{H}^1(\partial_{II}^*(K_1\cup K_2) \cap (\partial_{II}^* K_1 \cup \partial_{II}^* K_2)).\end{aligned}$$

The two previous equalities yield

$$\begin{aligned}&\mathcal{H}^1(\partial_I^*(K_1\cup K_2)) + 2\mathcal{H}^1(\partial_{II}^*(K_1\cup K_2))\\ &\quad\le \mathcal{H}^1(\partial_I^* K_1 \cup \partial_I^* K_2) + \mathcal{H}^1(\partial_I^*(K_1\cup K_2)\cap(\partial_{II}^* K_1 \cup \partial_{II}^* K_2))\\ &\qquad + 2\mathcal{H}^1(\partial_{II}^*(K_1\cup K_2)\cap(\partial_{II}^* K_1 \cup \partial_{II}^* K_2))\\ &\quad\le \mathcal{H}^1(\partial_I^* K_1 \cup \partial_I^* K_2) + 2\mathcal{H}^1(\partial_{II}^* K_1 \cup \partial_{II}^* K_2)\\ &\quad\le \mathcal{H}^1(\partial_I^* K_1) + 2\mathcal{H}^1(\partial_{II}^* K_1) + \mathcal{H}^1(\partial_I^* K_2) + 2\mathcal{H}^1(\partial_{II}^* K_2)\end{aligned}$$

whence $\mathcal{S}(K_1\cup K_2) \le \mathcal{S}(K_1) + \mathcal{S}(K_2)$. $\square$

REMARK. There is a natural way to extend the surface energy $\mathcal{S}$ to sets which are a countable union of pairwise disjoint continua, by simply summing the surface energy of all the continua. One should then define a suitable metric on these sets in order to ensure the lower semicontinuity of this functional. A possible way would be to build a metric using a technique similar to the one used for Caccioppoli partitions [5], [10].

Our next goal is to prove that the surface energy $\mathcal{S}$ is lower semicontinuous.

THEOREM 10.12. *The map $K \in \mathcal{K}_c \mapsto \mathcal{S}(K)$ is lower semicontinuous with respect to the Hausdorff metric i.e. for any sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}_c$ such that $D(K_n, K)$ converges to 0 as n goes to ∞, we have $\liminf_{n\to\infty} \mathcal{S}(K_n) \geq \mathcal{S}(K)$.*

PROOF. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of continua converging for the Hausdorff distance to a compact connected set K. We may suppose that $\liminf_{n\to\infty} \mathcal{S}(K_n)$ is finite and that $\operatorname{diam} K > 0$ (otherwise there is nothing to prove). We have by Proposition 5.5 and Corollary 10.10

$$\frac{1}{2}\mathcal{S}(K) \leq \mathcal{H}^1(\partial^\circ K) \leq \liminf_{n\to\infty} \mathcal{H}^1(\partial^\circ K_n) \leq \liminf_{n\to\infty} \mathcal{S}(K_n)$$

so that $\mathcal{H}^1(\partial^\circ K)$ is finite, as well as $\mathcal{S}(K)$. Let ε be positive smaller than $1/16$. To each point x of $\partial^*_I K$ (respectively $\partial^*_{II} K$) we associate $r_1(x,\varepsilon)$ (respectively $r_2(x,\varepsilon)$) as in Lemma 7.12 (respectively Lemma 7.13). We impose the additional conditions:

$$\forall x \in \partial^*_I K \quad r_1(x,\varepsilon) < \operatorname{diam} K/4, \qquad \forall x \in \partial^*_{II} K \quad r_2(x,\varepsilon) < \operatorname{diam} K/4\,.$$

We apply the covering Lemma 10.1 with these functions $r_1(x,\varepsilon)$ and $r_2(x,\varepsilon)$: there exists a finite family of disjoint balls $B(x_i, r_i)$, $i \in I_1 \cup I_2$, such that: for i in I_1, x_i belongs to $\partial^*_I K$ and $0 < r_i < r_1(x_i,\varepsilon)$, for i in I_2, x_i belongs to $\partial^*_{II} K$ and $0 < r_i < r_2(x_i,\varepsilon)$, and

$$\mathcal{S}(K) \;=\; \mathcal{H}^1(\partial^*_I K) + 2\mathcal{H}^1(\partial^*_{II} K) \;\leq\; (1+2\varepsilon)\left(2\sum_{i\in I_1} r_i + 4\sum_{i\in I_2} r_i\right).$$

Let $\eta = \varepsilon \min\{r_i : i \in I_1 \cup I_2\}$. Let n_0 be such that $\operatorname{diam} K_n > \operatorname{diam} K/2$ and $D(K_n, K) < \eta$ for n larger than n_0. Fix an integer n larger than n_0. Let i belong to I_1. By construction, we have

$$\operatorname{diam} K_n > 2r_i\,, \qquad D(K_n(x_i,r_i), U_-(x_i,r_i,\theta(x_i))) \leq 4r_i\varepsilon\,.$$

Lemma 10.7 implies that $\mathcal{S}(K_n, \mathring{B}(x_i,r_i)) \geq 2r_i(1-4\varepsilon)$.

Let i belong to I_2. By construction, we have

$$\operatorname{diam} K_n > 2r_i, \qquad D(K_n(x_i,r_i), L(x_i,r_i,\theta(x_i))) \leq 4r_i\varepsilon\,.$$

Lemma 10.8 implies that $\mathcal{S}(K_n, \mathring{B}(x_i,r_i)) \geq 4r_i(1-16\varepsilon)$. Therefore for any n larger than n_0, by Lemma 10.4,

$$\mathcal{S}(K_n) \geq \sum_{i\in I_1\cup I_2} \mathcal{S}(K_n, \mathring{B}(x_i,r_i)) \geq (1-16\varepsilon)\left(\sum_{i\in I_1} 2r_i + \sum_{i\in I_2} 4r_i\right) \geq \frac{1-16\varepsilon}{1+2\varepsilon}\mathcal{S}(K)\,.$$

The result follows by letting n go to ∞ and then ε go to 0. □

We finally prove an important approximation result, namely, a continuum can be approximated simultaneously in the sense of the Hausdorff metric and in the sense of surface energy by a set belonging to a simple class, for instance a polygon.

PROPOSITION 10.13. *Let K be a continuum such that $\mathcal{H}^1(\partial^\circ K) < \infty$. For any positive ε, there exists a continuum F such that ∂F is a finite union of segments and circular arcs, every point of ∂F apart the vertices is of type* I*, and*

$$K \subset F \subset \mathcal{V}(K, \varepsilon), \quad |\mathcal{S}(K) - \mathcal{S}(F)| < \varepsilon .$$

PROOF. Since $\mathcal{H}^1(\partial^\circ K) < \infty$, for any $\delta > 0$, there exists at most a finite number of residual domains $O_1, \dots, O_n$ of K having diameter larger than δ. Let O_∞ be the unbounded residual domain of K and let $K'(\delta) = \mathbb{R}^2 \setminus (O_\infty \cup O_1 \cup \cdots \cup O_n)$. Clearly, we have $K \subset K'(\delta)$ and

$$\lim_{\delta \to 0} D(K, K'(\delta)) = 0 , \quad \lim_{\delta \to 0} \mathcal{S}(K'(\delta)) = \mathcal{S}(K) .$$

Therefore we need only to consider the case where K itself has a finite number of residual components. We shall next approximate conveniently each residual domain of K from inside by a suitable domain. Let $O_1, \dots, O_n$ be the residual domains of K. Let ε be positive smaller than $1/16$. By Proposition 7.9, to each point x of $\partial_I^* K$ we can associate $r_1(x, \varepsilon)$ such that

$$\forall r < r_1(x, \varepsilon) \qquad D(K(x, r), U_-(x, r, \theta(x))) < r\varepsilon .$$

Similarly, to each point x of $\partial_{II}^* K$ we can associate $r_2(x, \varepsilon)$ such that

$$\forall r < r_2(x, \varepsilon) \qquad D(K(x, r), L(x, r, \theta(x))) < r\varepsilon .$$

Let α be the angle in $]0, \pi/2[$ such that $\sin\alpha = \varepsilon$. By Definition 4.4, to each point x of $\partial_I^* K \cup \partial_{II}^* K$ we can associate $r(x, \varepsilon) > 0$ such that for any $r < r(x, \varepsilon)$

$$\mathcal{H}^1(\partial^\circ K \cap U(x, r, \theta(x), \alpha)) \; < \; r\varepsilon/8 .$$

We impose in addition that

$$\forall x \in \partial_I^* K \cup \partial_{II}^* K \quad r(x, \varepsilon) < \frac{1}{3} \min\{\operatorname{diam} O_1, \dots, \operatorname{diam} O_n, 1\} .$$

We apply the covering Lemma 10.1 with the functions $(1+\varepsilon)^{-1} \min\{ r_1(x, \varepsilon), r(x, \varepsilon) \}$ and $(1+\varepsilon)^{-1} \min\{ r_2(x, \varepsilon), r(x, \varepsilon) \}$: there exists a finite family of disjoint balls $B(x_i, r_i)$, $i \in I_1 \cup I_2$, such that: for i in I_1, x_i belongs to $\partial_I^* K$ and $0 < r_i < r_1(x_i, \varepsilon)/(1+\varepsilon)$, for i in I_2, x_i belongs to $\partial_{II}^* K$ and $0 < r_i < r_2(x_i, \varepsilon)/(1+\varepsilon)$, and

$$\mathcal{H}^1 \left(\partial^\circ K \setminus \bigcup_{i \in I_1 \cup I_2} B(x_i, r_i) \right) \; \le \; 2\varepsilon \sum_{i \in I_1 \cup I_2} r_i .$$

Applying Lemmas 10.4, 10.7, 10.8, we get

$$\mathcal{S}(K) \geq \sum_{i \in I_1 \cup I_2} \mathcal{S}(K, \mathring{B}(x_i, r_i)) \geq (1-4\varepsilon) \left(2 \sum_{i \in I_1} r_i + 4 \sum_{i \in I_2} r_i \right) \geq \frac{3}{2} \sum_{i \in I_1 \cup I_2} r_i \,.$$

Let

$$A = \bigcup_{1 \leq k \leq n} \partial O_k \setminus \bigcup_{i \in I_1 \cup I_2} \mathring{B}(x_i, r_i)\,.$$

The set A is closed and $\mathcal{H}^1(A) \leq 2\varepsilon \mathcal{S}(K)$. Let $\delta = (\varepsilon/2) \min\{r_i : i \in I_1 \cup I_2\}$. If $A_1, \dots, A_m$ are connected components of A, we have $\mathcal{H}^1(A) \geq \mathcal{H}^1(A_1) + \cdots + \mathcal{H}^1(A_m) \geq \operatorname{diam} A_1 + \cdots + \operatorname{diam} A_m$. Therefore there is at most a finite number of connected components of A of diameter larger than δ. Since the sets $\partial O_1, \dots, \partial O_n$ are connected, then each connected component of A intersects the set $\bigcup_{i \in I_1 \cup I_2} \partial B(x_i, r_i)$. It follows that there is at most a finite number of components of A, say $A_1, \dots, A_m$, which are not included in $\bigcup_{i \in I_1 \cup I_2} \mathring{B}(x_i, r_i + \delta)$. For i in I_1, we set

$$\begin{aligned} \gamma_i = &[x_i + (r_i+\delta)u(\pi + \theta_i - \alpha), x_i + (r_i + \delta)u(\theta_i + \alpha)] \\ &\cup S(x_i, r_i + \delta, \pi + \theta_i - \alpha, \theta_i + \alpha)\,. \end{aligned}$$

For i in I_2, we set

$$\begin{aligned} \gamma_i = &S(x_i, r_i + \delta, \pi + \theta_i - \alpha, \pi + \theta_i + \alpha) \\ &\cup [x_i + (r_i+\delta)u(\pi + \theta_i - \alpha), x_i + (r_i + \delta)u(\theta_i + \alpha)] \\ &\cup S(x_i, r_i + \delta, \theta_i - \alpha, \theta_i + \alpha) \\ &\cup [x_i + (r_i+\delta)u(\pi + \theta_i + \alpha), x_i + (r_i + \delta)u(\theta_i - \alpha)]\,. \end{aligned}$$

The sets γ_i, $i \in I_1 \cup I_2$, are Jordan curves. We denote by $\operatorname{int} \gamma_i$ the bounded component of $\mathbb{R}^2 \setminus \gamma_i$ for $i \in I_1 \cup I_2$. For l in $\{1 \cdots m\}$, we choose a point a_l in A_l. We define finally

$$F = K \cup \bigcup_{i \in I_1 \cup I_2} \overline{\operatorname{int} \gamma_i} \cup \bigcup_{1 \leq l \leq m} B(a_l, 2 \operatorname{diam} A_l)\,.$$

By construction, for i in I_1, we have $e(U_-(x_i, r_i + \delta, \theta(x_i)), \gamma_i) \leq (r_i + \delta)\varepsilon$ and also $r_i + \delta \leq r_i(1 + \varepsilon/2) < r_1(x_i, \varepsilon)$ whence $D(K(x_i, r_i + \delta), U_-(x_i, r_i + \delta, \theta(x_i))) < (r_i + \delta)\varepsilon < \varepsilon$. Similarly, for i in I_2, we have $e(L(x_i, r_i + \delta, \theta(x_i)), \gamma_i) \leq (r_i + \delta)\varepsilon$ and also $r_i + \delta \leq r_i(1 + \varepsilon/2) < r_2(x_i, \varepsilon)$ whence $D(K(x_i, r_i + \delta), L(x_i, r_i + \delta, \theta(x_i))) < (r_i + \delta)\varepsilon < \varepsilon$. For l in $\{1 \cdots m\}$, we have also $e(K, B(a_l, 2 \operatorname{diam} A_l)) \leq 4 \operatorname{diam} A_l \leq 4\delta < 2\varepsilon$. Therefore $e(K, F) < 2\varepsilon$ (notice here that it was necessary to perform the covering with the functions $r_1(x, \varepsilon)/(1+\varepsilon)$, $r_2(x, \varepsilon)/(1+\varepsilon)$ in order to get this inequality). The previous considerations show also that for any i in $I_1 \cup I_2$, we have $K \cap \mathring{B}(x_i, r_i + \delta) \subset \operatorname{int} \gamma_i$, therefore

$$\bigcup_{1 \leq k \leq n} \partial O_k \subset \bigcup_{i \in I_1 \cup I_2} \operatorname{int} \gamma_i \cup \bigcup_{1 \leq l \leq m} \mathring{B}(a_l, 2 \operatorname{diam} A_l)$$

whence in particular

$$\partial F \subset \bigcup_{i \in I_1 \cup I_2} \gamma_i \cup \bigcup_{1 \le l \le m} \partial B(a_l, 2\mathrm{diam}\, A_l)\,.$$

The definition of F implies furthermore that $\partial F \cap \partial K = \varnothing$, and since F is built by adding to K a finite number of sets delimited by circular arcs and segments, then ∂F is a finite union of segments and circular arcs, and every point of ∂F apart the vertices is of type I. Let i belong to I_1. We apply Lemma 7.2 with the sets

$$U_-(x_i, r_i(1+\varepsilon), \theta(x_i), \alpha)\,, \qquad V_-(x_i, r_i(1+\varepsilon), \varepsilon/2, \theta(x_i), \alpha)\,.$$

Since

$$\mathcal{H}^1(\partial^\circ K \cap U(x_i, r_i(1+\varepsilon), \theta(x_i), \alpha)) < r_i(1+\varepsilon)\varepsilon/8 < r_i\varepsilon/2\,,$$

and since no residual component of K is contained in $B(x_i, r_i(1+\varepsilon))$, then

$$V_-(x_i, r_i(1+\varepsilon), \varepsilon/2, \theta(x_i), \alpha) \subset \overset{\circ}{K}\,.$$

Thus ∂F does not intersect $S(x_i, r_i+\delta, \pi+\theta_i+3\alpha, \theta_i-3\alpha)$. It follows that

$$\begin{aligned}
\mathcal{S}(F) &\le \sum_{i\in I_1} \mathcal{H}^1(\gamma_i \setminus S(x_i, r_i+\delta, \pi+\theta_i+3\alpha, \theta_i-3\alpha)) \\
&\quad + \sum_{i\in I_2} \mathcal{H}^1(\gamma_i) + \sum_{1\le l\le m} \mathcal{H}^1(\partial B(a_l, 2\mathrm{diam}\, A_l)) \\
&\le \sum_{i\in I_1} 2(r_i+\delta)(1+4\alpha) + \sum_{i\in I_2} 4(r_i+\delta)(1+\alpha) + \sum_{1\le l\le m} 4\pi\,\mathrm{diam}\, A_l \\
&\le (1+\varepsilon)(1+4\alpha)\left(\sum_{i\in I_1} 2r_i + \sum_{i\in I_2} 4r_i\right) + 8\pi\varepsilon\mathcal{S}(K) \\
&\le \mathcal{S}(K)((1+\varepsilon)(1+4\alpha)/(1-4\varepsilon) + 8\pi\varepsilon)\,.
\end{aligned}$$

Recalling that $\sin\alpha = \varepsilon$, we have the desired estimate and the set F answers the problem. □

COROLLARY 10.14. *For any continuum K, the surface energy $\mathcal{S}(K)$ is equal to*

$$\mathcal{S}(K) = \inf\left\{ \liminf_{n\to\infty} \mathcal{H}^1(\partial K_n) : (K_n)_{n\in\mathbb{N}} \in (\mathcal{K}_c^J)^{\mathbb{N}},\ \lim_{n\to\infty} D(K, K_n) = 0 \right\}$$

where $\mathcal{K}_c^J$ is the class of the connected compact sets K such that $\mathbb{R}^2 \setminus K$ has a finite number of bounded components, the boundaries of which are disjoint Jordan curves. The equality is still valid if we require that these Jordan curves are polygonal, i.e., they consist of a finite number of segments.

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