

## Line-Energy Ginzburg-Landau Models: Zero-Energy States

PIERRE-EMMANUEL JABIN – FELIX OTTO –  
BENOÎT PERTHAME

**Abstract.** We consider a class of two-dimensional Ginzburg-Landau problems which are characterized by energy density concentrations on a one-dimensional set. In this paper, we investigate the states of vanishing energy. We classify these zero-energy states in the whole space: They are either constant or a vortex. A bounded domain can sustain a zero-energy state only if the domain is a disk and the state a vortex. Our proof is based on specific entropies which lead to a kinetic formulation, and on a careful analysis of the corresponding weak solutions by the method of characteristics.

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### 1. – Introduction

Line-energy Ginzburg-Landau models arise in many physical situations like smectic liquid crystals, soft ferromagnetic films, in blister formation or —more abstractly— in the gradient theory of phase transition (see [6] and the references therein). Roughly speaking, these models come through dimensional reduction of a three dimensional Ginzburg-Landau-type model in a thin film and singularly depend on a small parameter  $\varepsilon$  proportional to the film thickness. These variational problems have in common that in the limit  $\varepsilon \downarrow 0$ , the minimizers converge to a two-dimensional vector field of unit length which is divergence-free. Vector fields of this class generically have line singularities, which typically are imposed by the boundary conditions. This is reflected in the phenomenon that in the limit  $\varepsilon \downarrow 0$ , the energy density of the minimizers concentrates on a one-dimensional set. Point singularities carry only a vanishing fraction of the energy —as opposed to the classical Ginzburg-Landau problem (see F. Béthuel, H. Brézis and F. Hélein [3]).

### 1.1. – The models

Two examples of line-energy Ginzburg-Landau models have been recently considered.

MODEL 1. (Jin and Kohn [10], Ambrosio, De Lellis and Mantegazza [1], DeSimone, Kohn, Müller and Otto [5]). The admissible two-dimensional vector fields are given by

$$(1.1) \quad \operatorname{div} m = 0 \text{ in } \Omega, \quad m \cdot n = 0 \text{ on } \partial\Omega,$$

where  $n$  denotes the outer unit normal to the boundary  $\partial\Omega$  of  $\Omega \subset \mathbb{R}^2$ , and the energy is

$$(1.2) \quad E_\varepsilon^1(m) = \varepsilon \int_\Omega |\nabla m|^2 + \frac{1}{\varepsilon} \int_\Omega (1 - |m|^2)^2.$$

MODEL 2 (Rivière and Serfaty [11]). The constraint is given by

$$(1.3) \quad |m| = 1 \text{ in } \Omega,$$

whereas the functional is

$$(1.4) \quad E_\varepsilon^2(m) = \varepsilon \int_\Omega |\nabla m|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla^{-1} \operatorname{div} m|^2,$$

where for the last term,  $m$  has been trivially (that is, by zero) extended on all  $\mathbb{R}^2$ .

We are interested in the zero energy states  $m$ , that is, all possible limits of sequences  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  with energy vanishing in the limit  $\varepsilon \downarrow 0$ . From this point of view, it is natural to consider the “minimum” of Model 1 and Model 2.

MODEL 0. No constraints and energy functional given by ( $m$  being still extended by 0)

$$(1.5) \quad E_\varepsilon(m) = \varepsilon \int_\Omega |\nabla m|^2 + \frac{1}{\varepsilon} \int_\Omega (1 - |m|^2)^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla^{-1} \operatorname{div} m|^2.$$

### 1.2. – Finite-energy states

It has been proved that a sequence  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  of bounded energy is compact in  $(L^2(\Omega))^2$  ([1] and [5] for Model 1, [11] for Model 2, at least the proof in [5] easily extends to the case of Model 0, see below). Furthermore, any limit  $m$  (a “finite-energy state”) belongs to Sobolev spaces  $W^{s,q}$  for all  $0 \leq s < \frac{1}{5}$ ,  $q < \frac{5}{3}$  (see P.-E. Jabin and B. Perthame [9] for Model 1, the proof adapts without difficulty to Model 2 and Model 0). In fact this last statement is a consequence of a kinetic formulation for  $m$  based on a family of entropies introduced in [5].

In particular, any finite-energy state  $m$  satisfies both constraints, (1.1) in an appropriate weak sense *and* (1.3). Hence, in simply connected domains, we formally have that  $m = \nabla^\perp \phi$  ( $\perp$  denotes the counter clockwise rotation by  $\frac{\pi}{2}$ ), where  $\phi$  satisfies the boundary value problem for the eikonal equation

$$(1.6) \quad \begin{cases} |\nabla \phi| = 1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

The eikonal equation clarifies how the boundary condition enforces singularities and why these are generically line-singularities. (1.6) also highlights the difficulty in understanding the limit  $\varepsilon \downarrow 0$ : To which of the many weak solutions of (1.6) does a sequence of minimizers  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  converge? The notion of *viscosity solution* ensures uniqueness for (1.6) and would be a very convenient candidate.

Unfortunately, for Model 1, it is expected that the limit is not always given by the viscosity solution: Jin and Kohn show that for a specific  $\Omega$ , the viscosity solution is not the only minimizer of the natural candidate for the  $\Gamma$ -limit of Model 1 (see [10]). Aviles and Giga have given an appropriate weak formulation of this candidate of a  $\Gamma$ -limit and proved that it is lower-semicontinuous and that it is at least a lower bound for the true  $\Gamma$ -limit (see [2]).

Model 2 has an implicit topological (thus global) constraint on smooth two-dimensional vector fields  $m$  through  $|m| = 1$ . Rivière and Serfaty (see [11] and [12]) account for this topological constraint by assuming that (locally)  $m = e^{i\varphi}$  with  $\varphi \in H^1$  —this rules out vortices on the  $\varepsilon$ -level, but not in the limit. They prove that indeed the viscosity solution gives the right value of the limiting minimal energy when plugged into the natural candidate for the  $\Gamma$ -limit of (1.4) (which measures jumps in  $\varphi$ )! Our work is oblivious to this topological constraint. Anyhow, this topological constraint seems less appropriate for models of ferromagnetic films, which allow for vortices on the  $\varepsilon$ -level. It also leads to a kinetic formulation, closer to the usual one for conservation laws, see [12].

### 1.3. – Zero-energy states

In this paper we study the possible limits  $m$  of sequences  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  such that  $\lim_{\varepsilon \downarrow 0} E(m_\varepsilon) = 0$  within Model 0 (“zero-energy states”). Our main tool is a kinetic equation for zero-energy states. For this purpose, we introduce

$$(1.7) \quad \chi(x, \xi) = \chi(m(x), \xi) := \begin{cases} 1 & \text{for } m(x) \cdot \xi > 0 \\ 0 & \text{for } m(x) \cdot \xi \leq 0 \end{cases}.$$

The next proposition contains all the information of zero-energy states we will use.

PROPOSITION 1.1. *Consider any zero-energy state  $m$ , that is, a limit of a sequence  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  with  $\lim_{\varepsilon \downarrow 0} E(m_\varepsilon) = 0$ . Then  $m$  satisfies*

$$(1.8) \quad |m(x)| = 1 \quad \text{for a.e. } x \in \Omega,$$

$$(1.9) \quad \operatorname{div} m = 0 \quad \text{distributionally in } \mathbb{R}^2 \text{ if trivially extended,}$$

$$(1.10) \quad \xi \cdot \nabla \chi(\cdot, \xi) = 0 \quad \text{distributionally in } \Omega \text{ for all } \xi \in S^1.$$

REMARKS. 1. Notice that (1.9) is a weak formulation of (1.1).

2. Notice that (1.10) contains part of (1.9): If one integrates (1.10) over  $\xi \in S^1$ , one obtains  $\operatorname{div} m = 0$  distributionally in  $\Omega$ . Hence (1.9) contains the additional information  $m \cdot n = 0$  on  $\partial\Omega$ .

3. Any *smooth*  $m$  which satisfies (1.8) and (1.9) will also satisfy the kinetic equation (1.10). In fact, (1.10) is a weak formulation of the principle of characteristics for  $\operatorname{div} m = 0$  and  $|m| = 1$ . We will use this intuition later on.

Proposition 1.1 is a direct consequence of the results in [9]. For the convenience of the reader, we indicate in Section 2 below an independent proof of Proposition 1.1; (1.8) and (1.9) are true for finite-energy states, whereas the zero right-hand side of the kinetic equation (1.10) is only true for zero-energy states. We now state the main results of this paper. We first classify the zero-energy states in the whole space.

THEOREM 1.1. *Consider any measurable function  $m$ , satisfying (1.8) and (1.10) with  $\Omega = \mathbb{R}^2$ . Then either  $m$  is constant, that is, there exists an  $m_0 \in S^1$  such that*

$$m(x) = m_0 \quad \text{for a.e. } x \in \mathbb{R}^2,$$

*or  $m$  is a vortex, that is, there exists a point  $\mathcal{O} \in \mathbb{R}^2$  and a sign  $\alpha \in \{-1, 1\}$  such that*

$$m(x) = \alpha \frac{(x - \mathcal{O})^\perp}{|x - \mathcal{O}|} \quad \text{for a.e. } x \in \mathbb{R}^2,$$

*where  $\perp$  denotes the counter clockwise rotation by  $\frac{\pi}{2}$ .*

We then identify the domains  $\Omega$  which allow for zero-energy states. We look for them in a specific class, namely

$$(1.11) \quad \Omega \neq \mathbb{R}^2 \text{ is connected, } C^2, \text{ and either } \Omega \text{ is a strip or } \exists \tilde{y}, \tilde{z} \in \partial\Omega \text{ such that the inner normal lines issued from } \tilde{y}, \tilde{z} \text{ are different and intersect in } \Omega \text{ before crossing } \partial\Omega.$$

We point out that any regular, simply connected domain satisfies Property (1.11) (see [8] where it is explained that in any simply connected domain there is always a ball of radius at least the minimal radius of curvature of the boundary) but also for instance the domain included between two balls which do not have the same center.

THEOREM 1.2. *Let  $\Omega$  satisfy Property (1.11). Assume that the measurable function  $m$  satisfies (1.8), (1.9) and (1.10). Then  $\Omega$  is either a disk and  $m$  is a vortex, or  $\Omega$  is a strip and  $m$  is a constant.*

The last result investigates the local behavior of finite-energy states, that is, we disregard the boundary conditions contained in (1.9). The class of solutions of (1.3) and (1.10) is fairly large: It includes all smooth solutions of  $\operatorname{div} m = 0$  and  $|m| = 1$ , hence in particular the rotated gradient of the distance function  $\operatorname{dist}(\cdot, K)$  to any closed set  $K$ , provided it has no singularity within  $\Omega$ . We prove that solutions of (1.3) and (1.10) have certain regularity properties.

**THEOREM 1.3.** *Let  $\Omega$  be any open set of  $\mathbb{R}^2$ . Consider a measurable function  $m$  which satisfies (1.8) and (1.10). Then  $m$  is locally Lipschitz continuous inside  $\Omega$  except at a locally finite number of points.*

There is another possible application. It was shown in [9] that any finite-energy state  $m$  (that is, a limit of a sequence of bounded energy) satisfies a kinetic equation of the form

$$(1.12) \quad \xi \cdot \nabla_x \chi = \operatorname{div}_\xi d_1 + d_2,$$

where  $(d_1, d_2)$  are two unknown “kinetic defect measures”. The only information is that they have bounded mass in  $\Omega \times S^1$ . For any given point  $x_0 \in \Omega$ , we now consider the blow-up  $\tilde{m}$  of  $m$  defined by

$$(1.13) \quad \tilde{m}(x) = \lim_{\delta \rightarrow 0} m(\delta(x - x_0)),$$

(according to the above cited compactness results, we have almost-everywhere convergence for a subsequence.) Applying a classical result in measure theory (see Federer [7], Proposition 2.10.19, page 181, for instance), there exists a set  $E$ , which is one-dimensional in the sense that its Hausdorff dimension is one, with the following property: For any  $x_0 \in \Omega \setminus E$ , the blow-up’s  $\tilde{d}_i$  of the measures  $d_1$  and  $d_2$ , defined as

$$\tilde{d}_i(A) = \lim_{\delta \rightarrow 0} \delta d_i(\delta(A - x_0)), \quad A \subset \mathbb{R}^2,$$

vanish. Consequently all possible blow-up’s  $\tilde{m}$  at any  $x_0 \in \Omega \setminus E$  satisfy (1.8) and (1.10) in  $\Omega = \mathbb{R}^2$ . Theorem 1.1 now implies that  $\tilde{m}$  is either a constant or a vortex. This is a very weak way of saying that the singular set of a finite-energy state  $m$  is one-dimensional.

Eventually, we mention that the blow-up techniques, used by A. Vasseur in [14] to obtain strong traces for scalar conservation laws, may also be applied here for the limit of a sequence with bounded energy. Consequently, any such limit  $m$  has a strong trace on the boundary  $\partial\Omega$  and in particular  $m|_{\partial\Omega}$  is of norm 1 almost everywhere on  $\partial\Omega$ .

The outline of this paper is as follows. In the second section, we recall the derivation of the kinetic equation (1.10) based on the entropies introduced in [5]. In the third section, we prove Theorem 1.1 (the case of the whole space), in the fourth section we establish Theorem 1.2 (the case with boundary) and in the fifth we prove Theorem 1.3 (the local case).

## 2. – Proof of Proposition 1.1

We use the notion of entropy introduced in [5, Definition 2.1]. A function  $\Phi \in C_0^\infty(\mathbb{R}^2)$  is called an entropy if

$$(2.1) \quad z \cdot D\Phi(z) z^\perp = 0 \quad \text{for all } z \in \mathbb{R}^2, \quad \text{and} \quad \Phi(0) = 0, \quad \nabla\Phi(0) = 0.$$

This definition is equivalent to the fact there exists a vector field  $\Psi \in C_0^\infty(\mathbb{R}^2)$  and a function  $\alpha \in C_0^\infty(\mathbb{R}^2)$  such that

$$(2.2) \quad D\Phi(z) = -\Psi(z) \times z + \alpha(z) \text{id} \quad \text{for all } z \in \mathbb{R}^2$$

see [5, Lemma 2.2]. The following formula is a slight generalization of the one given in [5, Lemma 2.3]: Let  $\Phi$ ,  $\Psi$  and  $\alpha$  be related as in (2.2). Then we have for any  $m \in H^1(\Omega)$

$$(2.3) \quad \text{div}(\Phi(m)) = \Psi(m) \cdot \nabla(1 - |m|^2) + \alpha(m) \text{div } m \quad \text{a.e. in } \Omega.$$

Consider now a zero-energy state  $m$ . By definition, there exists a sequence  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$  with  $E_\varepsilon(m_\varepsilon)$  converging to zero. From the representation (2.3) we gather, using the arguments from [5] (and in particular applying Murat’s lemma as in [5, Lemma 3.1]), that

$$(2.4) \quad \{\text{div}(\Phi(m_\varepsilon))\}_{\varepsilon \downarrow 0} \text{ converges to zero strongly in } H^{-1}(\Omega')$$

for all  $\Omega' \subset\subset \Omega$  and any entropy  $\Phi$ . Compactness in  $H^{-1}$  in (2.4) is sufficient to apply a compensated compactness argument inspired by Tartar to conclude that

$$(2.5) \quad \{m_\varepsilon\}_{\varepsilon \downarrow 0} \text{ is compact in } L^2(\Omega),$$

see [5] for the details. For this argument, it is important that the special entropies  $\chi(\cdot, \xi)$  (they play the role of Kruzkov’s entropies) defined in (1.7) can be approximated by smooth entropies: As proved in [5, Lemma 5], for any fixed  $\xi \in S^1$ , the function

$$(2.6) \quad \mathbb{R}^2 \ni z \mapsto \chi(z, \xi) \xi$$

is the pointwise limit of a sequence  $\{\Phi_n\}_{n \uparrow \infty}$  of entropies in the sense of (2.1). We now observe that the weakly vanishing divergence (2.4), the strong compactness (2.5) and the approximation (2.6) yield the desired kinetic equation (1.10) for our zero-energy state  $m$  in the limit  $\varepsilon \downarrow 0$ .

From the (2.5) we obtain that  $m$  is the pointwise almost-everywhere limit of a subsequence of  $\{m_\varepsilon\}_{\varepsilon \downarrow 0}$ . In particular, the increasing penalization of  $|m_\varepsilon| \neq 1$  through the second term in the functional (1.5) turns into the constraint (1.8). Finally, the increasing penalization of  $\text{div } m_\varepsilon$  (where  $m_\varepsilon$  is already extended trivially on all of  $\mathbb{R}^2$ ) through the third term in the functional (1.5) turns right into (1.9) in the limit  $\varepsilon \downarrow 0$ . This establishes Proposition 1.1.

**3. – Proof of Theorem 1.1**

The proof is divided into four parts. First of all, we express the kinetic equation (1.10) for  $\chi$  in terms of  $m$ . Secondly, we show that  $m$  has a single trace on any line segment. This allows us in a third step to extend the classical notion of characteristics for smooth solutions of  $\operatorname{div} m = 0$  and  $|m| = 1$  to our weak solutions. Once we can use characteristics, the proof follows in a fourth step from elementary geometric arguments.

**3.1. – General properties of  $\chi$**

We recall that  $x$  is called a Lebesgue point of  $m$  if there exists an  $m(x)$  such that

$$(3.1) \quad \lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(x)} |m(y) - m(x)| dy = 0.$$

It is well known that there exists a set  $E \subset \Omega$  of vanishing Lebesgue measure such that  $x$  is a Lebesgue point of  $m$  for all  $x \in \Omega \setminus E$ . In particular we have  $|m(x)| = 1$  for  $x \in \Omega \setminus E$ .

*PROPOSITION 3.1. Let the points  $x_1, x_2 \in \Omega \setminus E$  be such that the connecting line segment is contained in  $\Omega$ . Let  $\xi := x_2 - x_1$  denote the tangent to the line segment. Then the following implication holds*

$$m(x_1) \cdot \xi > 0 \implies m(x_2) \cdot \xi > 0.$$

*PROOF.* Since  $m(x_1) \cdot \xi \neq 0$ ,  $x_1$  is also a Lebesgue point of  $\chi(\cdot, \xi)$  with  $\chi(x_1, \xi) = 1$ . Indeed,  $m(x_1) \cdot \xi > 0$  entails via (3.1) that the set  $\{m \cdot \xi > 0\}$  has density one in  $x_1$ . By the definition (1,7) of  $\chi$ , this means that  $x_1$  is a Lebesgue point of  $\chi(\cdot, \xi)$  with  $\chi(x_1, \xi) = 1$ . On the other hand, it follows from (1.10) that

$$\chi(x + \xi, \xi) = \chi(x, \xi) \quad \text{for a.e. } x$$

in a neighborhood of the line segment. Since  $x_1$  is a Lebesgue point of  $\chi(\cdot, \xi)$ , this implies in particular that also  $x_2 = x_1 + \xi$  is a Lebesgue point and that  $\chi(x_2, \xi) = \chi(x_1, \xi) = 1$ . This means that the set  $\{m \cdot \xi > 0\}$  has density one in  $x_2$ . Since  $x_2$  is Lebesgue point of  $m$ , this implies as desired  $m(x_2) \cdot \xi > 0$ .

**3.2. – Existence of traces for  $m$**

From classical kinetic averaging results one obtains that  $m$  belongs to  $H^{1/2}$ , which is just not enough to define a trace. A general theory of traces which would also work here has been developed by Ukai [13] and Cessenat [4]. We prefer to give a simple independent proof based on (1.10), which shows the existence of traces in the sense of Lebesgue points in  $L^1$ .

LEMMA 3.1. *Let the line segment  $L := \{0\} \times [-1, 1]$  be contained in  $\Omega$ . Then there exists a bounded measurable function  $\tilde{m}$  of  $x_2 \in [-1, 1]$  which is the trace in the sense of*

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |m(x_1, x_2) - \tilde{m}(x_2)| dx_2 dx_1 = 0.$$

*In particular, we have  $|\tilde{m}(x_2)| = 1$  for a. e.  $x_2$ , and for every  $x = (0, x_2) \notin E$ ,  $m(x) = \tilde{m}(x_2)$ .*

PROOF. Let  $\xi \in S^1$  be fixed for the moment. According to (1.10), there exists a bounded and measurable function  $\tilde{\chi}(\cdot, \xi)$  in a single variable such that

$$(3.2) \quad \chi(x, \xi) = \tilde{\chi}(x \cdot \xi^\perp, \xi) \quad \text{for a.e. } x$$

in a neighborhood of the line segment  $L$ . Provided that  $\xi_1 \neq 0$ , this implies that  $x_2 \mapsto \tilde{\chi}(\xi_1 x_2, \xi)$  is the trace of  $\chi(\cdot, \xi)$  on  $L$  in the sense of

$$(3.3) \quad \lim_{r \downarrow 0} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |\chi((x_1, x_2), \xi) - \tilde{\chi}(\xi_1 x_2, \xi)| dx_2 dx_1 = 0.$$

Indeed, this follows from (3.2) and the inequality

$$\begin{aligned} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |\chi((x_1, x_2), \xi) - \tilde{\chi}(\xi_1 x_2, \xi)| dx_2 dx_1 \\ \stackrel{(3.2)}{\leq} \frac{1}{|\xi_1|} \sup_{|y_1| \leq r} \int_{-1}^1 |\tilde{\chi}(y_1 + y_2, \xi) - \tilde{\chi}(y_2, \xi)| dy_2 \end{aligned}$$

and the fact that a bounded measurable function has an  $L^1$ -modulus of continuity.

Thanks to the identity

$$(3.4) \quad m(x) = \int_{S^1} \xi \chi(x, \xi) d\xi,$$

which follows immediately from the definition (1.7) of  $\chi$ , a trace for  $\chi$  yields a trace for  $m$ . More precisely, we claim that

$$(3.5) \quad \tilde{m}(x_2) = \int_{S^1} \xi \tilde{\chi}(\xi_1 x_2, \xi) d\xi$$

is the trace of  $m$  on  $L$  in the sense of the statement of the lemma. Indeed, this follows from (3.4), (3.5) via the inequality

$$\begin{aligned} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |m(x_1, x_2) - \tilde{m}(x_2)| dx_2 dx_1 \\ \stackrel{(3.4), (3.5)}{\leq} \int_{S^1} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |\chi((x_1, x_2), \xi) - \tilde{\chi}(\xi_1 x_2, \xi)| dx_2 dx_1 d\xi \end{aligned}$$

from (3.3) with help of the principle of dominated convergence.



**3.3. – Lines orthogonal to  $m$**

The purpose of this section is to clarify in which sense the classical characteristics of  $\operatorname{div} m = 0$  and  $|m| = 1$  are characteristics for our weak solutions. We show that if  $L$  is the line orthogonal to  $m(x_0)$ , where  $x_0$  is a Lebesgue point of  $m$ , then the trace  $\tilde{m}$  of  $m$  on  $L$  is almost everywhere orthogonal to  $L$ , see figure 1.

PROPOSITION 3.2. *Suppose that the line segment  $L = \{0\} \times [0, 1]$  lies in  $\Omega$  and that  $\mathcal{O} := (0, 0) \notin E$  with  $m(\mathcal{O}) = (1, 0)$ . Then the trace  $\tilde{m}$  defined in Lemma 3.1 satisfies*

$$\tilde{m}_2(x_2) = 0 \quad \text{for a.e. } x_2 \in [0, 1].$$

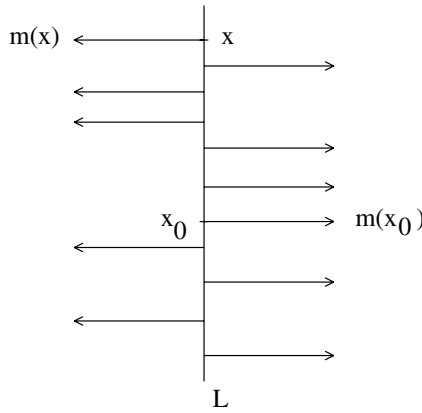


Fig. 1. Orthogonality of  $m$  to the line  $L$  orthogonal to  $m(x_0)$ .

PROOF. The proposition is a consequence of the last subsection and of LEMMA 3.2 (see figure 2). Let  $\varepsilon > 0$ , then

$$x \in \{0 < x_1 < \varepsilon|x|, 0 \leq x_2 \leq 1\} \setminus E \implies m_2(x) \geq -\varepsilon.$$

According to Lemma 3.1, Lemma 3.2 implies that  $\tilde{m}_2 \geq 0$  a.e. on  $L$ . Considering instead the sets  $\{-\varepsilon|x| < x_1 < 0, 0 \leq x_2 \leq 1\}$  we obtain  $\tilde{m}_2 \leq 0$  a.e. on  $L$  in an analogous manner. Hence Proposition (3.2) follows.

PROOF OF LEMMA 3.2. We observe that for all  $\xi \in S^1$  with  $\xi_1 > 0$

$$\xi_1 = m(\mathcal{O}) \cdot \xi > 0.$$

Hence we have by Proposition 3.1

$$m(x) \cdot \frac{x}{|x|} > 0 \quad \text{for all } x \notin E \text{ with } x_1 > 0,$$

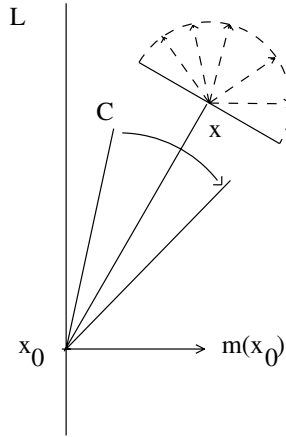


Fig. 2. For any  $x$  in the cone  $C$ ,  $m(x)$  is necessarily in the half-sphere.

which implies

$$m_2(x) \geq -\varepsilon \quad \text{for all } x \notin E \text{ with } 0 < \frac{x_1}{|x|} < \varepsilon, x_2 > 0.$$

**3.4. – Conclusion of the proof of Theorem 1.1**

We assume that  $m$  is not constant on  $\mathbb{R}^2 \setminus E$  and we will prove that it is a vortex.

We first point out that if  $m$  is not constant, it cannot be constant up to a sign. Indeed, assume we had  $m(x) = (\pm 1, 0)$  for almost every  $x \notin E$ . Then for every couple  $x, y \notin E$  such that  $x_1 \neq y_1$ , we would have  $m(x) = m(y)$  according to Proposition 3.1 —hence  $m$  would be constant on  $\mathbb{R}^2 \setminus E$ .

Hence there exist two points  $x_0$  and  $y_0$  in  $\mathbb{R}^2 \setminus E$  with

$$(3.6) \quad m(x_0) \neq m(y_0) \text{ and } m(x_0) \neq -m(y_0).$$

We denote  $L_1$  and  $L_2$  the lines orthogonal to  $m(x_0)$  resp.  $m(y_0)$  and passing through  $x_0$  resp.  $y_0$ . According to (3.6), these lines intersect in a point  $\mathcal{O}$ . In addition, we may assume

$$(3.7) \quad y_0 \neq \mathcal{O}.$$

Indeed, if this should not be the case, we fix a point  $\tilde{y}_0 \in L_2 \setminus \mathcal{O}$ . According to Proposition 3.2, the trace of  $m$  on  $L_2$  agrees with  $\pm m(y_0)$  almost everywhere. Hence there exist points  $\hat{y}_0 \notin E$  arbitrarily close to  $\tilde{y}_0$  with  $m(\hat{y}_0)$  arbitrarily close to  $\pm m(y_0)$ . Hence the corresponding  $\hat{L}_2$  and  $\hat{\mathcal{O}}$  differ as little from  $L_2$  resp.  $\mathcal{O}$  as we wish; in particular we can arrange for  $\hat{y}_0 \neq \hat{\mathcal{O}}$ .

In the following lemma, we show that the trace  $\tilde{m}$  of  $m$  on  $L_1$  is uniquely determined in such a constellation. We will then argue that this forces  $m$  to

be a vortex with center  $\mathcal{O}$ . Thanks to a translation, we may assume that  $\mathcal{O}$  is the origin. Thanks to a rotation, we may assume  $L_1 = \{0\} \times \mathbb{R}$  so that  $m(x_0) = \pm(1, 0)$ . Thanks to a reflection at the  $x_1, m_1$ -axis resp. the  $x_2, m_2$ -axis, we may assume that (3.6) and (3.7) can be specified to

$$(3.8) \quad m_2(y_0) > 0 \text{ and } (y_0)_1 < 0,$$

so that Figure 3 applies.

LEMMA 3.3. *The trace  $\tilde{m}$  of  $m$  on  $L_1$  defined in Lemma 3.1 satisfies*

$$\tilde{m}(x_2) = \left\{ \begin{array}{ll} (1, 0) & \text{for a.e. } x_2 > 0 \\ (-1, 0) & \text{for a.e. } x_2 < 0 \end{array} \right\}.$$

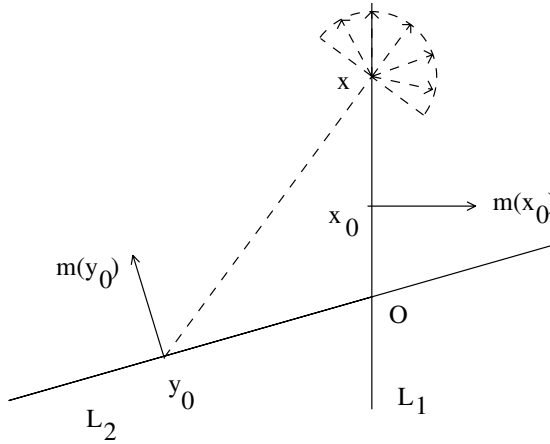


Fig. 3. Values of  $m$  on the line  $L_1$  compatible with  $m(y_0)$ . Following the characteristic from  $y_0$  to any  $x$ , we indeed deduce that  $m(x)$  belongs to the drawn half-sphere.

PROOF. Consider the half space above the line  $L_2$ . According to (3.8) we have

$$m(y_0) \cdot (x - y_0) > 0 \text{ for all } x \text{ above } L_2.$$

Together with Proposition 3.1, this implies

$$m(x) \cdot (x - y_0) > 0 \text{ for all } x \notin E \text{ above } L_2.$$

According to Lemma 3.1, this yields for the trace

$$(3.9) \quad \tilde{m}(x_2) \cdot ((0, x_2) - y_0) \geq 0 \text{ for a.e. } x_2 > 0.$$

On the other hand, we have in view of Proposition 3.2

$$(3.10) \quad \tilde{m}_2(x_2) = 0 \text{ for a.e. } x_2,$$

so that (3.9) turns into

$$-\tilde{m}_1(x_2)(y_0)_1 \geq 0 \quad \text{for a.e. } x_2 > 0.$$

Because of (3.8), this means

$$(3.11) \quad \tilde{m}_1(x_2) \geq 0 \quad \text{for a.e. } x_2 > 0.$$

Since Lemma 3.1 implies in particular that  $|\tilde{m}| = 1$  a.e., we gather from (3.10) and (3.11) that

$$\tilde{m}_1(x_2) = (1, 0) \quad \text{for a.e. } x_2 > 0;$$

the second half of the statement of the lemma is established analogously.

Hence, the trace of  $m$  is completely determined on the line  $L_1$ . In the same way, it is completely determined on  $L_2$ . Consider now another point  $z_0 \notin E$ . Then, we have either  $m(z_0) \neq \pm m(x_0)$  or  $m(z_0) \neq \pm m(y_0)$ . Let  $L_3$  be the line through  $z_0$  normal to  $m(z_0)$ . Then earlier arguments apply; And they imply that  $L_3$  cannot intersect  $L_1$  and  $L_2$  in distinct points. So either it intersects at  $\mathcal{O}$ , or else it is parallel say to  $L_1$ . If the latter possibility holds, then for any other line  $L_4$  passing through point  $w_0$  and normal to  $m(w_0)$ ,  $L_4$  cannot intersect any two  $L_1, L_2, L_3$  in two distinct points. This implies that  $L_4$  must also be parallel to  $L_1$ . So in fact *a.e* line is parallel to  $L_1$ , contradicting the assumption that  $m$  is not constant. Eventually  $L_3, L_2$  and  $L_1$  intersect at  $\mathcal{O}$  and the explicit form of  $m(z_0)$  follows,

$$m(y_0) = -\frac{y_0^\perp}{|y_0|}.$$

This completes the proof of Theorem 1.1.

#### 4. – Proof of Theorem 1.2

A simple geometric remark enables us to use the proof of Theorem 1.1. In all this section, we will use the representation and properties of  $m$  and  $\chi$  detailed in Subsection 3.1.

We first notice that

LEMMA 4.1. *The function  $m$  has a trace  $m|_\Gamma \in L^\infty(\partial\Omega)$ , parallel to the tangent, in the sense of Lemma 3.1, but locally defined with curves parallel to  $\partial\Omega$ .*

PROOF. The proof is a technical variant of that in Lemma 3.1 and we do not repeat it here.

If  $\Omega$  is a not a strip then denote by  $\tilde{y}, \tilde{z}$  two points of  $\partial\Omega$  whose normal lines intersect before crossing  $\partial\Omega$  the first time.

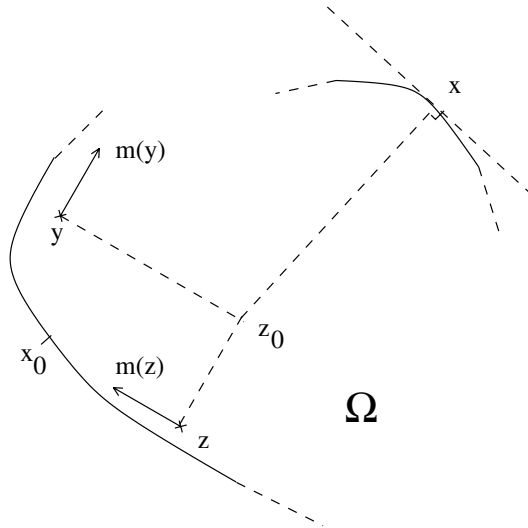


Fig. 4. Vortex structure implied at  $z_0$  by the boundary condition. Consequently, at any point like  $x$ , the tangent must be orthogonal to  $x - z_0$ .

Because of the existence of a tangential trace, there are points  $y, z \in \Omega \setminus E$ , arbitrarily close to  $\tilde{y}$ , resp.  $\tilde{z}$ , such that  $m(y), m(z)$  are arbitrarily close to  $\pm$  the tangent in  $\tilde{y}$ , resp.  $\tilde{z}$ . Therefore there are points  $y, z \in \Omega \setminus E$  such that the lines normal to  $m(y)$ , resp.  $m(z)$  intersect before leaving  $\Omega$  the first time (Figure 4).

The proof of Theorem 1.1 can be reproduced here and it shows that for some  $\alpha = \pm 1$ , for any point  $x \in \Omega \setminus E$  such that the segment  $[x, z_0]$  is included in  $\Omega$ , we have

$$m(x) = \alpha \frac{(x - z_0)^\perp}{|x - z_0|}.$$

But of course, because of the condition at the boundary, the tangent to the boundary, at the first point of intersection with the boundary of any line passing through  $z_0$ , is orthogonal to this line. Together with the regularity of  $\Omega$ , this proves that  $\Omega$  is a disk centered at  $z_0$ .

It only remains to explain why  $m$  is a constant if  $\Omega$  is a strip. Indeed at any point  $x \in \Omega \setminus E$ ,  $m(x)$  is parallel to the boundary. If this was not the case, the line orthogonal to  $m(x)$  would have one single intersection point with a line orthogonal to the boundary, i.e. a line orthogonal to some  $m(y)$  with  $y \in \partial\Omega$ . The proof of Theorem 1.1 would imply a vortex structure for  $m$ , which is incompatible with the form of  $\Omega$  and the boundary condition.

Now if the direction of  $m$  is constant then we may conclude that  $m$  is a constant just as we did in the beginning of Subsection 3.4.

### 5. – Proof of Theorem 1.3

Consider any open convex subset  $\omega$  of  $\Omega$  with  $d = d(\omega, \partial\Omega) > 0$ . It is enough to prove Theorem 1.3 in any such  $\omega$ .

We again use here the definition and properties given in Subsection 3.1. The proof of Theorem 1.1 implies the following lemma which shows that the direction of  $m$  is Lipschitz in  $\omega$ .

LEMMA 5.1. *Either  $m$  is a vortex in  $\omega$  or for any couple  $(x_0, y_0)$  in  $\omega \setminus E$ , for some  $\alpha = \pm 1$*

$$|m(x_0) - \alpha m(y_0)| \leq \frac{1}{d} |x_0 - y_0|.$$

PROOF. If  $m(x_0)$  and  $m(y_0)$  are parallel then we are done. Otherwise let us consider  $z_0$  the point of intersection of the two lines respectively orthogonal to  $m(x_0)$  and  $m(y_0)$  and passing through these two points.

If the distance between  $z_0$  and  $\omega$  is less than  $d$ , the arguments in the proof of Theorem 1.1 show that  $m$  is a vortex in  $\omega$ , since the set of points at a distance less than  $d$  from  $\omega$  is also convex.

If the distance between  $z_0$  and  $\omega$  is larger than  $d$ , the idea is that, for  $x_0$  and  $y_0$  close, the direction of  $m(x_0)$  and  $m(y_0)$  must be close too. A rigorous argument is as follows. For simplicity we assume that  $z_0 = 0$ . Both  $|x_0|$  and  $|y_0|$  are larger than  $d$ , thus we have

$$\begin{aligned} \left| \frac{x_0}{|x_0|} - \frac{y_0}{|y_0|} \right|^2 &= 2 - 2 \frac{x_0}{|x_0|} \cdot \frac{y_0}{|y_0|} \leq 2 + \frac{|x_0 - y_0|^2}{|x_0| |y_0|} - \frac{|x_0|}{|y_0|} - \frac{|y_0|}{|x_0|} \\ &\leq \frac{1}{d^2} |x_0 - y_0|^2 + 2 - \frac{|x_0|}{|y_0|} - \frac{|y_0|}{|x_0|} \\ &\leq \frac{1}{d^2} |x_0 - y_0|^2. \end{aligned}$$

Together with the fact that  $m(x_0)$  and  $m(y_0)$  are of norm 1 and orthogonal to  $x_0$  and  $y_0$ , this proves the lemma.

Now, we can conclude the proof of Theorem 1.3. Of course if  $m$  is a vortex in  $\omega$ , it is Lipschitz continuous at every points except the center of the vortex (up to a redefinition on a negligible set).

Otherwise, it is easy to check that for two points  $x_0$  and  $y_0$  close enough,  $m(x_0)$  and  $m(y_0)$  must have the same orientation (see Figure 5).

Indeed, consider any  $x_0 \in \omega \setminus E$ . For any  $y_0 \in \omega \setminus E$  with  $|x_0 - y_0| < d\sqrt{2 - \sqrt{3}}$ , the lemma implies that  $m(y_0)$  makes an angle strictly less than  $\pi/3$  with either  $m(x_0)$  or  $-m(x_0)$ .

It is now always possible to find a Lebesgue point  $z$  of  $m$  in one of the two equilateral triangles of base  $(x_0, y - 0)$  such that both  $z - x_0$  and  $z - y_0$  make with either  $m(x_0)$  or  $-m(x_0)$  an angle less than  $\pi/3$ , since the limiting case corresponds to the situation where  $m(x_0)$  is orthogonal to  $x_0 - y_0$  and then  $(x_0, y_0, z)$  exactly forms an equilateral triangle. Without any loss of generality,

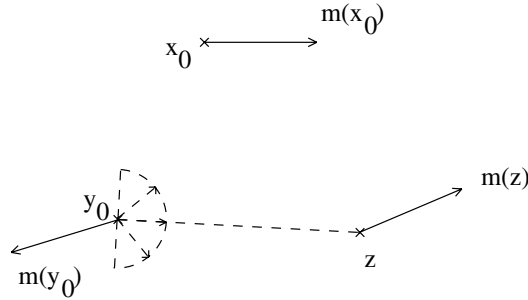


Fig. 5. An impossible configuration. The vector  $m(y_0)$  should be in the drawn half-sphere.

we may assume that it is the angle between  $z - x_0$  and  $m(x_0)$  which is no more than  $\pi/3$ .

Notice first that  $|z - x_0|$  and  $|z - y_0|$  are strictly less than  $d\sqrt{2 - \sqrt{3}}$ , so  $m(z)$  makes with  $\pm m(x_0)$  and  $\pm m(y_0)$  an angle strictly less than  $\pi/3$ .

Applying first Proposition 3.1 to  $x_1 = x_0$  and  $x_2 = z$ , we deduce that  $m(z)$  makes an angle less than  $\pi/2$  with  $z - x_0$ , so less than  $\pi/2 + \pi/3 < \pi$  with  $m(x_0)$ , this last angle being finally at most  $\pi/3$  according to the previous remark.

We again apply Proposition 3.1 but to  $x_1 = y_0$  and  $x_2 = z$ . By the same argument, it has for consequence that the angle between  $m(z)$  and  $m(y_0)$  is also strictly less than  $\pi/3$ . Therefore the angle between  $m(x_0)$  and  $m(y_0)$  is strictly less than  $2\pi/3$  and so than  $\pi/3$ .

This proves that for any couple  $(x_0, y_0)$  in  $\omega \setminus E$ , we have

$$|m(x_0) - m(y_0)| \leq \frac{1}{d}|x_0 - y_0|.$$

And thus, after a redefinition if necessary,  $m$  is Lipschitz continuous everywhere in  $\omega$ .

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Département de Mathématiques et Applications  
École Normale Supérieure  
45 rue d’Ulm  
75230 Paris Cedex 05, France  
jabin@dma.ens.fr  
benoit.perthame@ens.fr

Institut für Angewandte Mathematik  
Universität Bonn  
Wegelerstrasse 10  
53115 Bonn, Germany  
otto@iam.uni-bonn.de