

## Remarks on Gårding Inequalities for Differential Operators

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**Abstract.** Classical Gårding inequalities such as those of Hörmander, Hörmander-Melin or Fefferman-Phong are proved by pseudo-differential methods which do not allow to keep a good control on the supports of the functions under study nor on the smoothness of the coefficients of the operator. In this paper, we show by very simple calculations that in certain special situations, the results that can be obtained directly are much better than those expected thanks to the general theory.

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One of the main features of the pseudodifferential calculus is to provide links between properties of operators and properties of their symbols. Among these, one of the most important problems is to connect inequalities on the symbols and inequalities on the operators, and this is the purpose of Gårding inequalities. Among these inequalities, let us quote that of Hörmander [5, Th. 18.1.14] also called *the sharp Gårding inequality*, that of Hörmander-Melin [5, Th. 22.3.2] and that of Fefferman-Phong [5, Th. 18.6.8].

These inequalities were extended in several different ways. Here, we want to consider two kinds of such extensions that were discussed in recent papers.

The first one deals with the smoothness assumptions that have to be put on the symbols of the operators. Indeed, the use of Bony's paradifferential calculus [1] and of Hörmander's inequality [6] showed that one can get a Gårding inequality with a gain of  $\delta$  derivative,  $\delta \leq 1$ , for some classes of pseudodifferential operators with  $C^{2\delta}$  symbols. More recently, Bony [2] and Hérau [3] also considered Gårding inequalities for paradifferential or nonsmooth pseudodifferential operators.

The second extension to be discussed here deals with the localness of these inequalities. More precisely, the problem is to prove a Gårding inequality for a function  $u$  assuming that the symbol of the operator is nonnegative *only on*

$\text{supp } u \times \mathbb{R}^n$ . Such an inequality was proved in Lerner-Saint Raymond [7] for functions supported in a half-space and pseudodifferential operators of positive odd integer order, then improved by Hérau [3].

To study this last property of localness, it is natural to consider *differential* operators, since they have the local property of not increasing the supports of functions. For differential operators, it is also easier to control the smoothness assumptions on the symbols since they depend separately on the  $x$  and on the  $\xi$  variables.

This is why this paper deals with such differential operators. But here, we merely want to show, thanks to very elementary computations, that the results quoted above can be much improved in some special situations. In some sense, this completes the result of [7] since our operators here have positive even integer order.

**1. – General notation**

In the whole paper, we denote by  $H^m$  the Sobolev space of order  $m$  on  $\mathbb{R}^n$ , that is the space of all distributions  $u \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\widehat{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$  and that the norm

$$\|u\|_m^2 = \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^m d\xi$$

is finite. We also denote by  $(u, v) = \int u(x)\overline{v(x)}dx$  the  $L^2$  scalar product of the two square integrable functions  $u$  and  $v$ , as well as its various extensions, mainly for  $u \in H^{-m}$  and  $v \in H^m$ , or for  $u \in \mathcal{S}'(\mathbb{R}^n)$  and  $v \in \mathcal{S}(\mathbb{R}^n)$ .

Now, imitating the notation of Section 8.2 of Hörmander [4], if  $m$  and  $M$  are two positive integers with  $m \leq M \leq 2m$ , we consider functions of the form

$$a(x, \zeta, \bar{\zeta}) = \sum_{|\alpha| \leq m, |\beta| \leq m, |\alpha + \beta| \leq M} a_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta$$

where the coefficients  $a_{\alpha\beta}$  are complex valued, bounded functions on  $\mathbb{R}^n$ ,  $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$  and  $\zeta^\alpha = \zeta_1^{\alpha_1} \dots \zeta_n^{\alpha_n}$ . With these functions, that we call *symbols of order  $(M, m)$* , we associate the differential operators

$$a(x, D, \bar{D})u = \sum_{|\alpha| \leq m, |\beta| \leq m, |\alpha + \beta| \leq M} D^\beta (a_{\alpha\beta} D^\alpha u)$$

where  $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$  with  $D_j = -i\partial/\partial x_j$ . Since the operator  $A = a(x, D, \bar{D})$  is continuous from  $H^m$  into  $H^{-m}$ , we have an estimate  $|(Au, u)| \leq C\|u\|_m^2$  whence we get

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \text{Re}(Au, u) \geq -C\|u\|_m^2.$$

When  $\delta > 0$ , the following better estimate

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \operatorname{Re}(Au, u) \geq -C\|u\|_{m-(\delta/2)}^2$$

will be called a *Gårding inequality with a gain of  $\delta$  derivatives*. The goal of this paper is to prove such estimates under assumptions on the symbol of  $A$ .

In the next two sections, we present a first method that consists in writing  $\operatorname{Re}(Au, u)$  as an integral of a nonnegative function, modulo some error terms that can be suitably estimated. To control these error terms, we will assume that some coefficients are Lipschitz continuous, or that they belong to some Hölder classes  $C^r = C^r(\mathbb{R}^n)$ . We will give a precise statement for useful estimates in this context in an appendix at the end of the paper. Using this method, we can prove two Gårding inequalities with a gain of one or two derivatives for second order differential operators (see Section 2), and a Gårding inequality with a gain of one derivative for differential operators in two independent variables (see Section 3).

In the last section, we show that one can use paradifferential methods to get a Gårding inequality with a gain of half a derivative for general differential operators with Lipschitz continuous coefficients, again assuming that the symbol is nonnegative only on  $\operatorname{supp} u \times \mathbb{R}^n$ .

In the statements given in the next sections, we have put the smoothness assumptions that are required to get the best possible gain when using the method under study. However, it is easy to see that these smoothness assumptions can be weakened at the cost of getting Gårding inequalities with a weaker gain. The proof of such variants are left to the reader.

## 2. – Operators of the second order

In this section, we consider second order differential operators  $a(x, D, \bar{D})$ , that is the case  $m = 1$ . In the symbol  $a$ , we put together the terms of same order by setting

$$a_k(x, \zeta, \bar{\zeta}) = \sum_{|\alpha+\beta|=k} a_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta,$$

then, writing  $\zeta_j = \xi_j + i\eta_j$  with  $\xi_j$  and  $\eta_j \in \mathbb{R}$ , we define the operator  $\partial_{(x,\eta)}^2 = \sum_{j=1}^n \partial_{x_j} \partial_{\eta_j}$  and the following polynomials with distribution coefficients

$$p(x, \xi) = a_2(x, \xi, \xi), \quad s(x, \xi) = a_1(x, \xi, \xi) + \frac{1}{2} \partial_{(x,\eta)}^2 a_2(x, \xi + i\eta, \xi - i\eta)|_{\eta=0}$$

and

$$t(x) = a_0(x) + \frac{1}{2} \partial_{(x,\eta)}^2 a_1(x, \xi + i\eta, \xi - i\eta)|_{\eta=0}$$

that are defined for  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ , and will be called the  $\xi$ -symbols of the operator  $a(x, D, \overline{D})$ . If  $s(x, \xi) = \sum_{j=1}^n s_j(x)\xi_j$ , we will also write  $\operatorname{div} s(x) = \sum_{j=1}^n \partial_{x_j} s_j(x)$ .

In the following statements, we give Gårding inequalities under an assumption  $\operatorname{Re} p \geq 0$  or  $\operatorname{Re}(p + s) \geq -C_0$ , and assuming also that certain combinations of the coefficients of the operator have some smoothness that is measured in terms of Hölder classes  $C^r = C^r(\mathbb{R}^n)$ .

**THEOREM 2.1.** *Let  $K$  be a compact subset of  $\mathbb{R}^n$  and  $\varepsilon > 0$ . Assume that:*

- (i) *The coefficients of  $\operatorname{Re} s(x, \xi)$  are in  $C^\varepsilon$ ,  $\operatorname{div}(\operatorname{Re} s) \in C^{-(1/2)+\varepsilon}$  and  $\operatorname{Re} t \in C^{-(1/2)+\varepsilon}$ ;*
- (ii)  *$\operatorname{Re} p(x, \xi) \geq 0$  for almost all  $x \in K$  and all  $\xi \in \mathbb{R}^n$ .*

*Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(K)$ ,  $\operatorname{Re}(a(x, D, \overline{D})u, u) \geq -C\|u\|_{1/2}^2$ .*

**THEOREM 2.2.** *Let  $K$  be a compact subset of  $\mathbb{R}^n$  and  $C_0 \in \mathbb{R}$ . Assume that:*

- (i) *The coefficients of  $\operatorname{Re} s(x, \xi)$  and  $\operatorname{Re} t(x)$  are bounded functions of  $x$ ;*
- (ii)  *$\operatorname{Re}(p(x, \xi) + s(x, \xi)) \geq -C_0$  for almost all  $x \in K$  and all  $\xi \in \mathbb{R}^n$ .*

*Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(K)$ ,  $\operatorname{Re}(a(x, D, \overline{D})u, u) \geq -C\|u\|_0^2$ .*

**COMMENTS.** The important fact here is that the inequality on the symbol is assumed only on  $K \times \mathbb{R}^n$ . As for the smoothness of the coefficients, we could give many variants of these statements, in particular by allowing the coefficients  $a_{\alpha\beta}$  to be distributions instead of bounded functions. Among all these variants, let us quote the following:

- Theorem 2.1 remains true when its assumption (i) is replaced with: *the coefficients  $a_{\alpha\beta}$  are Lipschitz continuous for  $|\alpha + \beta| = 2$ , and belong to  $C^{(1/2)+\varepsilon}$  for  $|\alpha + \beta| = 1$ . Similarly, the assumption (i) in Theorem 2.2 is fulfilled as soon as the coefficients  $a_{\alpha\beta}$  are Lipschitz continuous for  $|\alpha + \beta| \geq 1$ .*
- These results must be compared with what can be proved thanks to Bony-Hörmander’s paradifferential Gårding inequalities [6] and [2] that require that  $a_{\alpha\beta} \in C^{2\delta}$  for a gain of  $\delta$  derivatives: here we get a gain of one or even two derivatives when the coefficients are merely Lipschitz continuous.
- We can even state the following corollary of Theorem 2.2: *if the coefficients in the principal part are Lipschitz continuous and if  $\operatorname{Re}(p+s) \geq -C_0$  on  $K \times \mathbb{R}^n$ , then there exists a distribution  $c$  such that  $\operatorname{Re}((a(x, D, \overline{D}) + c)u, u) \geq 0$  for all  $u \in C_0^\infty(K)$ .*

**PROOF OF THEOREMS 2.1 AND 2.2.** In the proof, we will use *symmetric* (resp. *skew-symmetric*) symbols that are defined as symbols  $b(x, \zeta, \bar{\zeta})$  satisfying  $b_{\alpha\beta} = b_{\beta\alpha}$  (resp.  $b_{\alpha\beta} = -b_{\beta\alpha}$ ) for all multiindices  $\alpha$  and  $\beta$ . The proof of our two theorems then comes from the following three lemmas.

LEMMA 2.3. For any symbol  $a(x, \zeta, \bar{\zeta})$  of order  $(2, 1)$ , there exists a symmetric symbol  $b(x, \zeta, \bar{\zeta})$  with real valued, distribution coefficients such that for all  $u \in \mathcal{S}(\mathbb{R}^n)$ ,  $\mathcal{R}e(a(x, D, \bar{D})u, u) = (b(x, D, \bar{D})u, u)$ . Moreover, if  $p, s$  and  $t$  are the  $\xi$ -symbols of the operator  $a(x, D, \bar{D})$ , then  $b(x, \xi, \bar{\xi}) = \mathcal{R}e(p(x, \xi) + s(x, \xi) + t(x))$  for all  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ .

PROOF. If the symbol of the operator  $A = a(x, D, \bar{D})$  is written  $\sum_{\alpha, \beta} a_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta$ , then the operator  $\frac{1}{2}(A + A^*)$  has a symbol  $\tilde{a}$  equal to

$$\tilde{a}(x, \zeta, \bar{\zeta}) = \sum_{\alpha, \beta} \frac{1}{2} (a_{\alpha\beta}(x) + \overline{a_{\beta\alpha}(x)}) \zeta^\alpha \bar{\zeta}^\beta = \sum_{\alpha, \beta} b_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta + \sum_{\alpha, \beta} i c_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta$$

with  $b_{\alpha\beta} = \frac{1}{2} \mathcal{R}e(a_{\alpha\beta} + \overline{a_{\beta\alpha}}) = b_{\beta\alpha}$  and  $c_{\alpha\beta} = \frac{1}{2} \mathcal{I}m(a_{\alpha\beta} + \overline{a_{\beta\alpha}}) = -c_{\beta\alpha}$ . Then the symbol

$$b(x, \zeta, \bar{\zeta}) = \sum_{\alpha, \beta} b_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta + \sum_{\alpha} \left( \sum_{|\beta|=1} \frac{1}{4} \partial_x^\beta (c_{\alpha\beta} - c_{\beta\alpha})(x) \right) (\zeta^\alpha + \bar{\zeta}^\alpha)$$

is a symmetric symbol with real valued coefficients satisfying

$$\begin{aligned} b(x, \xi, \bar{\xi}) &= \mathcal{R}e \left( a(x, \xi, \bar{\xi}) + \frac{1}{2} \partial_{(x, \eta)}^2 a(x, \xi + i\eta, \bar{\xi} - i\eta) \Big|_{\eta=0} \right) \\ &= \mathcal{R}e(p(x, \xi) + s(x, \xi) + t(x)). \end{aligned}$$

On the other hand, to treat the  $c_{\alpha\beta}$  terms in  $(\tilde{a}(x, D, \bar{D})u, u)$ , we write

$$\begin{aligned} (i c_{\alpha\beta} D^\alpha u, D^\beta u) &= \frac{1}{2} (i c_{\alpha\beta} D^\alpha u, D^\beta u) + \frac{1}{2} (i c_{\alpha\beta} D^\alpha u, D^\beta u) \\ &= \frac{1}{2} ([D^\beta, i c_{\alpha\beta}] D^\alpha u, u) + \frac{1}{2} ([i c_{\alpha\beta}, D^\alpha] u, D^\beta u) \\ &\quad + \frac{1}{2} (i c_{\alpha\beta} D^{\alpha+\beta} u, u) + \frac{1}{2} (i c_{\alpha\beta} u, D^{\alpha+\beta} u). \end{aligned}$$

Since the  $c_{\alpha\beta}$  are the coefficients of a skew-symmetric symbol, the sum of the terms containing  $D^{\alpha+\beta} u$  gives 0, and in view of the identity  $[D^\alpha, ic] = |\alpha| \partial^\alpha c$  for  $|\alpha| \leq 1$ , we thus get

$$\sum_{\alpha, \beta} (i c_{\alpha\beta} D^\alpha u, D^\beta u) = \frac{1}{4} \sum_{\alpha} \sum_{|\beta|=1} \left( (\partial^\beta (c_{\alpha\beta} - c_{\beta\alpha}) D^\alpha u, u) + (\partial^\beta (c_{\alpha\beta} - c_{\beta\alpha}) u, D^\alpha u) \right)$$

that proves that  $\mathcal{R}e(a(x, D, \bar{D})u, u) = (b(x, D, \bar{D})u, u)$  for the symmetric symbol  $b$  already introduced, as required.  $\square$

LEMMA 2.4. *Let  $K$  be a compact subset of  $\mathbb{R}^n$ , and  $b(x, \zeta, \bar{\zeta})$  be a symmetric symbol of order  $(2, 1)$  with bounded, real valued coefficients satisfying  $b(x, \xi, \xi) \geq 0$  for almost all  $x \in K$  and all  $\xi \in \mathbb{R}^n$ . Then we have:  $\forall u \in C_0^\infty(K)$ ,  $(b(x, D, \bar{D})u, u) \geq 0$ .*

PROOF. Writing  $b(x, \xi, \xi) = \sum_{j,k=1}^n b_{jk} \xi_j \xi_k + \sum_{j=1}^n (b_{j0} + b_{0j}) \xi_j + b_{00}$ , we have for (almost) all  $(x, \xi_0, \xi) \in K \times \mathbb{R}^* \times \mathbb{R}^n$ ,

$$\sum_{j,k=0}^n b_{jk}(x) \xi_j \xi_k = \xi_0^2 b(x, (\xi/\xi_0), (\xi/\xi_0)) \geq 0,$$

and this is still true for  $\xi_0 = 0$  by continuity. Therefore, writing  $u(x) = \xi_0(x) + i\eta_0(x)$  and  $D_j u(x) = \xi_j(x) + i\eta_j(x)$ , the symmetry of the symbol  $b$  shows that

$$(b(x, D, \bar{D})u, u) = \int \sum_{j,k=0}^n b_{jk}(x) (\xi_j(x) \xi_k(x) + \eta_j(x) \eta_k(x)) dx,$$

and this is nonnegative provided that  $u \in \mathcal{S}(\mathbb{R}^n)$  is supported in  $K$ . □

Theorem 2.2 simply comes from Lemmas 2.3 and 2.4 since if, on  $K \times \mathbb{R}^n$ , we have  $\operatorname{Re}(p(x, \xi) + s(x, \xi)) \geq -C_0$ , then we also have  $\operatorname{Re}(p(x, \xi) + s(x, \xi) + t(x) + C) \geq 0$  on  $K \times \mathbb{R}^n$  with  $C = C_0 + \sup_K |\operatorname{Re} t|$ .

In the case of Theorem 2.1, Lemma 2.3 gives  $\operatorname{Re}(a(x, D, \bar{D})u, u) = (b(x, D, \bar{D})u, u)$ , and writing  $b = b_2 + b_1 + b_0$  according to the orders of the different terms in the symbol  $b$ , we have  $b_2(x, \xi, \xi) = \operatorname{Re} p(x, \xi)$ ,  $b_1(x, \zeta, \bar{\zeta}) = \operatorname{Re} s(x, \frac{1}{2}(\zeta + \bar{\zeta}))$  and  $b_0(x) = \operatorname{Re} t(x)$ . Therefore assumption (ii) of Theorem 2.1 and Lemma 2.4 show that  $(b_2(x, D, \bar{D})u, u) \geq 0$ , whence

$$\operatorname{Re}(a(x, D, \bar{D})u, u) \geq \left( \operatorname{Re} s \left( x, \frac{1}{2}(D + \bar{D}) \right) u, u \right) + (\operatorname{Re} t(x)u, u).$$

In view of assumption (i) of Theorem 2.1, the result finally comes from the following lemma.

LEMMA 2.5. *Let  $\varepsilon$  be a positive real number,  $a(x, \xi) = \sum_{j=1}^n a_j(x) \xi_j$  be a polynomial of degree 1 with real valued coefficients that belong to  $C^\varepsilon$ , and  $b$  be a real valued distribution. Assume that  $\operatorname{div} a$  and  $b$  belong to  $C^{-(1/2)+\varepsilon}$ . Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(\mathbb{R}^n)$ ,  $|(a(x, D + \bar{D})u, u) + (b(x)u, u)| \leq C \|u\|_{1/2}^2$ .*

PROOF. There exists a function  $A \in C^{(3/2)+\varepsilon}$  satisfying  $\sum_{j=1}^n \partial_j^2 A = \operatorname{div} a$  in a neighborhood of  $K$ , and therefore the functions  $d_j = a_j - \partial_j A \in C^\varepsilon$  and  $d(x, \xi) = \sum_{j=1}^n d_j(x) \xi_j$  satisfy

$$\operatorname{div} d = \sum_{j=1}^n \partial_j (a_j - \partial_j A) = \operatorname{div} a - \sum_{j=1}^n \partial_j^2 A = 0.$$

Then we can choose functions  $B_j \in C^{2+\varepsilon}$  satisfying  $\sum_{k=1}^n \partial_k^2 B_j = d_j$  and  $\sum_{j=1}^n \partial_j B_j = 0$  in a neighborhood of  $K$ , and set  $c_{jk} = 2(\partial_k B_j - \partial_j B_k) \in C^{1+\varepsilon}$ . Since the matrix  $(c_{jk})$  is skew-symmetric, we can repeat the proof of Lemma 2.3 to get the identity

$$\sum_{j,k=1}^n (i c_{jk}(x) D_j u, D_k u) = (d(x, D + \bar{D})u, u).$$

On the other hand, if  $B \in C^{(3/2)+\varepsilon}$  satisfies  $\sum_{j=1}^n \partial_j^2 B = b$  in a neighborhood of  $K$ , we have  $(b(x)u, u) = i \sum_{j=1}^n ((\partial_j B(x)u, D_j u) - (\partial_j B(x) D_j u, u))$ , and therefore we can write

$$\begin{aligned} (a(x, D + \bar{D})u, u) + (b(x)u, u) &= (d(x, D + \bar{D})u, u) \\ &\quad + \sum_{j=1}^n \left( (\partial_j A(x) - i \partial_j B(x)) D_j u, u \right) \\ &\quad + \sum_{j=1}^n \left( (\partial_j A(x) + i \partial_j B(x)) u, D_j u \right) \\ &= (c(x, D, \bar{D})u, u) \end{aligned}$$

if we set  $c(x, \zeta, \bar{\zeta}) = \sum_{j,k=0}^n i c_{jk}(x) \zeta_j \bar{\zeta}_k$  with  $\zeta_0 = \bar{\zeta}_0 = 1$ ,  $i c_{j0} = \partial_j A - i \partial_j B = \overline{i c_{0j}}$  for  $j > 0$ , and  $c_{00} = 0$ .

Since the principal part of the symbol  $c$  is skew-symmetric and has Lipschitz continuous coefficients, it follows from Lemma A.2 (a) in the appendix that the corresponding terms can be estimated by  $C\|u\|_{1/2}^2$ . Since the other terms can be estimated similarly thanks to Lemma A.2 (b), the proof is complete.  $\square$

### 3. – Operators in two independent variables

Throughout this section, we assume that  $n = 2$ , or in other words, we consider operators in two independent variables.

Given a symbol  $a(x, \zeta, \bar{\zeta}) = \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha\beta}(x) \zeta^\alpha \bar{\zeta}^\beta$  of order  $(2m, m)$ , we define the coefficients of the *principal part* as the  $a_{\alpha\beta}$ 's for  $|\alpha| + |\beta| = 2m$ , and the coefficients of the *semi-principal part* as the  $a_{\alpha\beta}$ 's for  $\max\{|\alpha|, |\beta|\} = m$  and  $|\alpha| + |\beta| < 2m$ . As in the previous section, we consider the *principal  $\xi$ -symbol* of the operator  $a(x, D, \bar{D})$  defined as  $p(x, \xi) = \sum_{|\alpha|+|\beta|=2m} a_{\alpha\beta}(x) \xi^{\alpha+\beta}$ . The main assumption we will need is the following.

**DEFINITION 3.1.** Let  $x_0$  be a point in  $\mathbb{R}^2$ , and  $p(x, \xi)$  be a homogeneous polynomial of degree  $2m$  in  $\xi = (\xi_1, \xi_2)$  with Lipschitz continuous, real valued coefficients. Then we say that *its real characteristics can be well factorized at*

$x_0$  if for all  $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$  satisfying  $p(x_0, \xi_0) = 0$ , there exist a neighborhood  $V$  of  $x_0$ , Lipschitz continuous functions  $\zeta^k : V \rightarrow \mathbb{C}^2$  with  $\zeta^k(x_0) = \xi_0$  for  $k \leq L$ , and a polynomial  $q(x, \xi)$  with Lipschitz continuous coefficients and not vanishing at  $(x_0, \xi_0)$ , such that  $p(x, \xi) = q(x, \xi) \prod_{k=1}^L (\zeta_1^k(x)\xi_2 - \zeta_2^k(x)\xi_1)$  in  $V \times \mathbb{R}^2$ .

EXAMPLES. It is easy to show that this assumption is fulfilled in the following two situations:

- When  $p$  is elliptic at  $x_0$ , since this means that there is no real characteristic at  $x_0$ .
- When  $p$  is a nonnegative polynomial with  $C^2$  coefficients, and the real characteristics at  $x_0$  are at most double, since the square root of a nonnegative  $C^2$  function is a Lipschitz continuous function.

Under this assumption, we get the following Gårding inequality.

THEOREM 3.2. *Let  $K$  be a compact subset of  $\mathbb{R}^2$ ,  $\varepsilon$  be a positive real number, and  $a(x, \zeta, \bar{\zeta})$  be a symbol of order  $(2m, m)$  in two independent variables with Lipschitz continuous coefficients in the principal part and  $C^{(1/2)+\varepsilon}$  coefficients in the semi-principal part. Assume that the real part of its principal  $\xi$ -symbol satisfies  $\operatorname{Re} p(x, \xi) \geq 0$  on  $K \times \mathbb{R}^2$ , and that the real characteristics of  $\operatorname{Re} p$  can be well factorized at every point  $x \in K$ . Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(K)$ ,  $\operatorname{Re}(a(x, D, \bar{D})u, u) \geq -C\|u\|_{m-(1/2)}^2$ .*

COMMENTS. We could give here several corollaries of this theorem, in particular thanks to the examples following Definition 3.1. Indeed, Theorem 3.2 shows that a Gårding inequality with a gain of one derivative is true for an elliptic operator with Lipschitz continuous coefficients, but this result can also be proved more simply by “freezing” the coefficients at different points and estimating the difference in sufficiently small neighborhoods. Another consequence is that this inequality holds for operators with  $C^2$  coefficients when the real characteristics are at most double, but here again such a result can be proved more simply by using Bony-Hörmander’s paradifferential Gårding inequality [6]. However, let us point out that here our symbols are assumed to be nonnegative only on  $K \times \mathbb{R}^2$ .

Writing  $A = a(x, D, \bar{D})$ , the proof of Theorem 3.2 essentially consists in a factorization  $\frac{1}{2}(A + A^*) = B^*B + R$  where the error terms  $R$  can be well estimated. This is why we first establish several factorization lemmas.

LEMMA 3.3. *For  $a \in \mathbb{C}^m$ ,  $b \in \mathbb{C}^n$  and  $c \in \mathbb{C}^{m+n}$ , let’s set  $p(a, \tau) = \tau^m + a_1\tau^{m-1} + \dots + a_m$ ,  $q(b, \tau) = \tau^n + b_1\tau^{n-1} + \dots + b_n$  and  $r(c, \tau) = \tau^{m+n} + c_1\tau^{m+n-1} + \dots + c_{m+n}$ . Assume that the variables  $a, b$  and  $c$  are linked by the relation  $r(c, \tau) = p(a, \tau)q(b, \tau)$  and that for some  $a^0 \in \mathbb{C}^m$  and  $b^0 \in \mathbb{C}^n$ , the polynomials  $p(a^0, \tau)$  and  $q(b^0, \tau)$  have no common root  $\tau \in \mathbb{C}$ . Then the coefficients  $a$  and  $b$  of the polynomials  $p$  and  $q$  can be written near  $a^0$  and  $b^0$  as holomorphic functions of the coefficients  $c$ .*



PROOF. The relation  $r(c, \tau) = p(a, \tau)q(b, \tau)$  shows that the coefficients  $c$  can be written as polynomials, and therefore holomorphic functions of the coefficients  $a$  and  $b$ . Moreover, the jacobian determinant  $\det(\partial c/\partial(a, b))$  is nonzero at  $(a^0, b^0)$  since it is equal to the resultant of  $p$  and  $q$ , and that  $p(a^0, \tau)$  and  $q(b^0, \tau)$  have no common root  $\tau \in \mathbb{C}$ . Therefore, the lemma follows from the holomorphic inverse mapping theorem.  $\square$

LEMMA 3.4. *Let  $p(x, \tau) = \tau^{2m} + a_1(x)\tau^{2m-1} + \dots + a_{2m}(x)$  be a polynomial with Lipschitz continuous, real valued coefficients, and  $x_0 \in \mathbb{R}^2$ . Assume that  $p(x_0, \tau) \geq 0$  for all  $\tau \in \mathbb{R}$ , and that for any real root  $\tau_0$  of  $p(x_0, \tau) = 0$ , there exist a neighborhood  $V$  of  $x_0$ , Lipschitz continuous functions  $\theta^k : V \rightarrow \mathbb{C}$  with  $\theta^k(x_0) = \tau_0$  for  $k \leq L$ , and a polynomial  $q(x, \tau)$  with Lipschitz continuous coefficients and not vanishing at  $(x_0, \tau_0)$ , such that  $p(x, \tau) = q(x, \tau) \prod_{k=1}^L (\tau - \theta^k(x))$  for all  $x \in V$ . Then there exists in some neighborhood  $W$  of  $x_0$  a factorization  $p(x, \tau) = p_1(x, \tau)p_2(x, \tau)$  as a product of two polynomials of degree  $m$  with Lipschitz continuous, complex valued coefficients, and such that for all  $x \in W$  satisfying  $p(x, \tau) \geq 0$  on  $\mathbb{R}$ ,  $p_2(x, \tau) = \overline{p_1(x, \tau)}$ .*

PROOF. Since  $p$  has real valued coefficients, its nonreal roots appear as conjugate pairs. Let us denote by  $\theta_0^1, \dots, \theta_0^n$  the roots of  $p(x_0, \tau)$  that have a positive imaginary part, and set  $\varepsilon = \frac{1}{2} \inf_{k \leq n} \Im \theta_0^k > 0$ . Then, there is a neighborhood  $U$  of  $x_0$  such that for all  $x \in U$ , the polynomial  $p(x, \tau)$  has exactly  $n$  roots  $\theta^1(x), \dots, \theta^n(x)$  with an imaginary part greater than  $\varepsilon$ . This remark gives a first factorization  $p(x, \tau) = q_1(x, \tau)q_2(x, \tau)q_3(x, \tau)$  where  $q_1(x, \tau) = \prod_{k=1}^n (\tau - \theta^k(x))$ ,  $q_2(x, \tau) = \overline{q_1(x, \tau)}$  and  $q_3$  has only real roots at  $x_0$ . Since  $p$  has Lipschitz continuous coefficients and  $q_1, q_2$  and  $q_3$  have no common root at  $x_0$ , these polynomials also have Lipschitz continuous coefficients in some neighborhood of  $x_0$  thanks to Lemma 3.3.

It is much more difficult to handle the roots of  $q_3$ , that are the roots that happen to be real at  $x_0$ . By assumption, we already know that  $q_3(x, \tau) = \prod_{k=1}^{2\ell} (\tau - \theta^k(x))$  for some Lipschitz continuous functions  $\theta^k$  in a neighborhood  $V$  of  $x_0$ . For every fixed  $x \in V$ , let us then put the *real* roots  $\theta^k(x)$  in decreasing order, and call  $n(k, x)$  the number of the root  $\theta^k(x)$  in this order (when several  $\theta^k(x)$  are equal real roots at some point  $x \in V$ , the corresponding numbers  $n(k, x)$  are chosen arbitrarily). Finally, let us set

$$I(x) = \left\{ k \leq 2\ell; \Im \theta^k(x) > 0 \text{ or } (\Im \theta^k(x) = 0 \text{ and } n(k, x) \text{ is odd}) \right\},$$

$$r_1(x, \tau) = \prod_{k \in I(x)} (\tau - \theta^k(x)) \text{ and } r_2(x, \tau) = \prod_{k \notin I(x)} (\tau - \theta^k(x)).$$

Since  $q_3$  has real valued coefficients,  $I(x)$  has exactly  $\ell$  elements at each point  $x \in V$ , and we thus get a factorization  $q_3(x, \tau) = r_1(x, \tau)r_2(x, \tau)$  in two polynomials of degree  $\ell$  with the following properties:

- The coefficients of  $r_1$  and  $r_2$  are Lipschitz continuous in each domain  $V_J$  defined as  $V_J = \{x \in V; I(x) = J\}$ , where  $J$  denotes any subset of  $\llbracket 1, 2\ell \rrbracket$  with  $\ell$  elements, because of the smoothness of the  $\theta^k$ 's.

- The coefficients of  $r_1$  and  $r_2$  are also Lipschitz continuous on the closure of each domain  $V_J$ , since formulas  $r_1(x) = \prod_{k \in J} (\tau - \theta^k(x))$  and  $r_2(x) = \prod_{k \notin J} (\tau - \theta^k(x))$  are valid even at boundary points of  $V_J$ . Indeed, if  $z$  is a boundary point of  $V_J$ , then we have  $\mathcal{J}m \theta^k(z) \geq 0$  for all  $k \in J$  thanks to the continuity of the roots  $\theta^k$ . If  $\mathcal{J}m \theta^k(z) > 0$ , or if  $\mathcal{J}m \theta^k(z) = 0$  and  $n(k, z)$  is odd, then we have  $k \in I(z)$ . The remaining case ( $\mathcal{J}m \theta^k(z) = 0$  and  $n(k, z)$  even) can occur only when  $\theta^k(z)$  is a multiple real root of  $q_3(z, \tau)$ , and then it is easy to check that  $J$  and  $I(z)$  contain the same number of elements  $j$  such that  $\theta^j(z) = \theta^k(z)$ , whence the formulas for  $r_1$  and  $r_2$ .
- The coefficients of  $r_1$  and  $r_2$  are Lipschitz continuous on the whole of  $V$ . Indeed, if  $a$  is one of these coefficients and if  $C$  denotes the largest Lipschitz constant of  $a$  as a function on the closures of the various  $V_J$  's, then  $|a(x) - a(y)| \leq C|x - y|$  for all  $x$  and  $y \in V$ , because otherwise, there would be two points  $x^1$  and  $y^1 \in V$  such that  $|a(x^1) - a(y^1)| > C|x^1 - y^1|$ ; then it would be possible to construct by dichotomy a sequence  $[x^n, y^n]$  of segments converging to some point  $z \in V$  and satisfying  $|a(x^n) - a(y^n)| > C|x^n - y^n|$ ; and finally, after extraction of a subsequence such that all the  $x^{\varphi(n)}$  belong to the same  $V_J$  and all the  $y^{\varphi(n)}$  belong to the same  $V_I$ , we would have  $|a(x^{\varphi(1)}) - a(z)| \leq C|x^{\varphi(1)} - z|$  and  $|a(z) - a(y^{\varphi(1)})| \leq C|z - y^{\varphi(1)}|$  since  $z = \lim x^{\varphi(n)} = \lim y^{\varphi(n)}$  belongs to the closures of  $V_J$  and of  $V_I$ , and this would lead to a contradiction since  $z \in [x^{\varphi(1)}, y^{\varphi(1)}]$  implies that  $|x^{\varphi(1)} - z| + |z - y^{\varphi(1)}| = |x^{\varphi(1)} - y^{\varphi(1)}|$  (here, we implicitly assumed that  $V$  is convex, which we may).
- Finally, if  $x \in V$  is such that  $p(x, \tau) \geq 0$  on  $\mathbb{R}$ , then all its real roots have even multiplicities, and therefore it follows that  $r_2(x, \tau) = r_1(x, \tau)$ .

Now, we just have to set  $p_1(x, \tau) = q_1(x, \tau)r_1(x, \tau)$  and  $p_2(x, \tau) = q_2(x, \tau)r_2(x, \tau)$  to complete the proof of our lemma. □

LEMMA 3.5. *Let  $K$  be a compact subset of  $\mathbb{R}^2$ , and  $p(x, \xi)$  be a homogeneous polynomial of degree  $2m$  in  $\xi \in \mathbb{R}^2$  with Lipschitz continuous, real valued coefficients. Assume that  $p(x, \xi) \geq 0$  on  $K \times \mathbb{R}^2$  and that the real characteristics of  $p$  can be well factorized at every point  $x \in K$ . Then there exist a neighborhood  $V$  of  $K$  and polynomials  $p_1^k(x, \xi)$  and  $p_2^k(x, \xi)$ , for  $k \leq N$ , of degree  $m$  with Lipschitz continuous, complex valued coefficients such that  $p(x, \xi) = \sum_{k=1}^N p_1^k(x, \xi) p_2^k(x, \xi)$  in  $V \times \mathbb{R}^2$ , and that  $p_2^k(x, \xi) = \overline{p_1^k(x, \xi)}$  for all  $x \in K$ .*

PROOF. Let us fix any point  $x_0 \in K$ . Because of the assumption on the real characteristics of  $p$ , the polynomial  $p(x_0, \xi)$  is not identically zero, and therefore, after a possible rotation in  $\xi$ , we may assume that  $p(x, \xi) = \sum_{k=0}^{2m} a_k(x) \xi_1^k \xi_2^{2m-k}$  with  $a_0(x_0) > 0$ . Therefore, there is a neighborhood of  $x_0$  where we can consider the polynomial  $\tilde{p}(x, \tau) = \tau^{2m} + \sum_{k=1}^{2m} (a_k(x)/a_0(x)) \tau^{2m-k}$ . This polynomial  $\tilde{p}$  has Lipschitz continuous, real valued coefficients, and it is nonnegative on  $K \times \mathbb{R}$  (near  $x_0$ ) since  $\tilde{p}(x, \tau) = p(x, \xi_\tau)/a_0(x)$  for  $\xi_\tau = (1, \tau)$ .

Let us now prove that the polynomial  $\tilde{p}$  also satisfies the assumption of Lemma 3.4 on the real roots of  $\tilde{p}(x_0, \tau)$ . If  $\tau_0$  is such a real root, then

$\xi_{\tau_0} = (1, \tau_0)$  is a real characteristic of  $p$ , and therefore there exist by assumption a neighborhood of  $x_0$ , and in this neighborhood a Lipschitz continuous factorization  $p(x, \xi) = q(x, \xi) \prod_{k=1}^L (\zeta_1^k(x)\xi_2 - \zeta_2^k(x)\xi_1)$  with  $q(x_0, \xi_{\tau_0}) \neq 0$  and  $\zeta^k(x_0) = \xi_{\tau_0} = (1, \tau_0)$  for all  $k \leq L$ . If  $q(x, \xi) = \sum_{\ell=0}^{2m-L} b_\ell(x) \xi_1^\ell \xi_2^{2m-L-\ell}$ , we can write  $a_0(x) = b_0(x) \prod_{k=1}^L \zeta_1^k(x)$  whence  $b_0(x_0) \neq 0$ . On the other hand, the functions  $\theta^k(x) = \zeta_2^k(x)/\zeta_1^k(x)$  are Lipschitz continuous in a neighborhood of  $x_0$ , and this is also true for the coefficients of the polynomial  $\tilde{q}(x, \tau) = \tau^{2m-L} + \sum_{\ell=1}^{2m-L} (b_\ell(x)/b_0(x)) \tau^{2m-L-\ell}$ . Since  $\tilde{q}(x, \tau) = q(x, \xi_\tau)/b_0(x)$ , we have  $\tilde{q}(x_0, \tau_0) \neq 0$  and

$$\begin{aligned} \tilde{p}(x, \tau) &= \frac{1}{a_0(x)} p(x, \xi_\tau) = \frac{q(x, \xi_\tau)}{a_0(x)} \prod_{k=1}^L (\zeta_1^k(x)\tau - \zeta_2^k(x)) \\ &= \frac{\prod_{k=1}^L \zeta_1^k(x)}{a_0(x)} q(x, \xi_\tau) \prod_{k=1}^L (\tau - \theta^k(x)) = \tilde{q}(x, \tau) \prod_{k=1}^L (\tau - \theta^k(x)). \end{aligned}$$

This proves that the polynomial  $\tilde{p}$  satisfies at  $x_0$  all the assumptions of Lemma 3.4.

Thanks to Lemma 3.4, we then have a Lipschitz continuous factorization  $\tilde{p}(x, \tau) = \tilde{p}_1(x, \tau)\tilde{p}_2(x, \tau)$  in some neighborhood of  $x_0$  such that  $\tilde{p}_2(x, \tau) = \tilde{p}_1(x, \tau)$  for all  $x$  satisfying  $\tilde{p}(x, \tau) \geq 0$  on  $\mathbb{R}$ . Since  $a_0(x_0) > 0$ , the function  $\sqrt{a_0(x)}$  is Lipschitz continuous in a neighborhood of  $x_0$ , and therefore we get a similar Lipschitz continuous factorization  $p(x, \xi) = q_1(x, \xi)q_2(x, \xi)$  for  $p$  in this neighborhood by setting  $q_1(x, \xi) = \sqrt{a_0(x)}\xi_1^m \tilde{p}_1(x, \xi_2/\xi_1)$  and  $q_2(x, \xi) = \sqrt{a_0(x)}\xi_1^m \tilde{p}_2(x, \xi_2/\xi_1)$ .

At this point, we have proved the following result: each point  $x^k \in K$  has an open neighborhood  $V^k$  where  $p$  can be factorized in the form  $p(x, \xi) = q_1^k(x, \xi)q_2^k(x, \xi)$  with Lipschitz continuous factors satisfying  $q_2^k(x, \xi) = q_1^k(x, \xi)$  for all  $x \in V^k \cap K$ . Since these  $V^k$  cover the compact set  $K$ , we can choose a finite number  $V^1, \dots, V^N$  of them that still cover  $K$ , and real valued functions  $\varphi^k \in C_0^\infty(V^k)$  such that  $\sum_{k=1}^N (\varphi^k(x))^2 = 1$  in a neighborhood  $V$  of  $K$ . Therefore, we complete the proof of our lemma by setting  $p_1^k(x, \xi) = \varphi^k(x)q_1^k(x, \xi)$  and  $p_2^k(x, \xi) = \varphi^k(x)q_2^k(x, \xi)$ .  $\square$

PROOF OF THEOREM 3.2. Because of our assumption on the real characteristics of  $\operatorname{Re} p$ , Lemma 3.5 shows that we can write  $\operatorname{Re} p(x, \xi) = \sum_{k=1}^N p_1^k(x, \xi) p_2^k(x, \xi)$  in a neighborhood  $V$  of  $K$  for polynomials  $p_1^k$  and  $p_2^k$  with Lipschitz continuous, complex valued coefficients such that  $p_2^k(x, \xi) = \overline{p_1^k(x, \xi)}$  on  $K \times \mathbb{R}^2$ . Then the symbol  $b(x, \zeta, \bar{\zeta}) = \sum_{k=1}^N p_1^k(x, \zeta) p_2^k(x, \bar{\zeta})$  is a symbol of order  $(2m, m)$  with Lipschitz continuous coefficients satisfying

$$\begin{aligned} \forall u \in C_0^\infty(K), \quad (b(x, D, \bar{D})u, u) &= \int_K \sum_{k=1}^N p_1^k(x, D)u(x) p_2^k(x, -D)\bar{u}(x) dx \\ &= \sum_{k=1}^N \int_K |p_1^k(x, D)u(x)|^2 dx \geq 0. \end{aligned}$$

On the other hand, we can write  $\operatorname{Re}(a(x, D, \overline{D})u, u) = (\tilde{a}(x, D, \overline{D})u, u)$  for a symbol  $\tilde{a}$  that can be written  $\tilde{a}(x, \zeta, \bar{\zeta}) = \tilde{b}(x, \zeta, \bar{\zeta}) + \tilde{c}(x, \zeta, \bar{\zeta}) + \tilde{d}(x, \zeta, \bar{\zeta})$  where  $\tilde{b}$  is a homogeneous symbol of order  $(2m, m)$  with Lipschitz continuous coefficients,  $\tilde{c}$  is a symbol of order  $(2m - 1, m)$  with  $C^{(1/2)+\varepsilon}$  coefficients, and  $\tilde{d}$  is a symbol of order  $(2m - 2, m - 1)$ . Since we can write  $\tilde{b}(x, \xi, \bar{\xi}) = \operatorname{Re} p(x, \xi) = b(x, \xi, \bar{\xi})$ , we have

$$\begin{aligned} (\tilde{b}(x, D, \overline{D})u, u) &= (b(x, D, \overline{D})u, u) + ((\tilde{b} - b)(x, D, \overline{D})u, u) \\ &\geq ((\tilde{b} - b)(x, D, \overline{D})u, u), \end{aligned}$$

and this last expression can be estimated thanks to Lemma A.2 (a) in the appendix since the symbol  $\tilde{b} - b$  has Lipschitz continuous coefficients and satisfies  $(\tilde{b} - b)(x, \xi, \bar{\xi}) \equiv 0$  for all  $(x, \xi) \in V \times \mathbb{R}^n$ . The terms coming from the symbol  $\tilde{c}$  can be estimated by  $C\|u\|_{m-(1/2)}^2$  thanks to Lemma A.2 (b), and the terms coming from the symbol  $\tilde{d}$  can be obviously estimated by  $C\|u\|_{m-1}^2$ . This completes the proof.  $\square$

#### 4. – A general result

In this section, we return to general symbols  $a(x, \zeta, \bar{\zeta})$  of order  $(2m, m)$  in  $n$  variables. In this situation, we define the coefficients of the principal part, the coefficients of the semi-principal part, and the principal  $\xi$ -symbol exactly as in the previous section.

**THEOREM 4.1.** *Let  $K$  be a compact subset of  $\mathbb{R}^n$ ,  $\varepsilon$  be a positive real number, and  $a(x, \zeta, \bar{\zeta})$  be a symbol of order  $(2m, m)$  with Lipschitz continuous coefficients in the principal part and  $C^{(1/4)+\varepsilon}$  coefficients in the semi-principal part. Assume that the real part of its principal symbol satisfies  $\operatorname{Re} p(x, \xi) \geq 0$  on  $K \times \mathbb{R}^n$ . Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(K)$ ,  $\operatorname{Re}(a(x, D, \overline{D})u, u) \geq -C\|u\|_{m-(1/4)}^2$ .*

To make use of the paradifferential Gårding inequality, we need an everywhere nonnegative symbol. This comes from the following lemma, where it can be seen that it is useless to assume more smoothness on the coefficients of the principal part, and therefore that the gain of half a derivative cannot be improved.

**LEMMA 4.2.** *Under the assumptions of Theorem 4.1, there exist a  $C^\infty$  function  $\chi$  and a Lipschitz continuous function  $c$ , both real valued and compactly supported in  $\mathbb{R}^n$ , satisfying  $1 - \chi \equiv c \equiv 0$  on  $K$ , and  $\operatorname{Re}(\chi(x)p(x, \xi)) + c(x)|\xi|^{2m} \geq 0$  for all  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ .*

**PROOF.** Choose a real valued function  $\chi \in C_0^\infty(\mathbb{R}^n)$  satisfying  $\chi \equiv 1$  in a neighborhood of  $K$ , and set  $c(x) = C\chi(x)\operatorname{dist}(x, K)$ . These functions have the required smoothness since the distance to a compact set is a Lipschitz

continuous function, and the condition  $\operatorname{Re}(\chi p) + c|\xi|^{2m} \geq 0$  is fulfilled as soon as  $C$  is chosen sufficiently large, since the coefficients of  $p$  are Lipschitz continuous and  $\operatorname{Re} p \geq 0$  on  $K \times \mathbb{R}^n$ .  $\square$

PROOF OF THEOREM 4.1 Let  $\chi$  and  $c$  be as in Lemma 4.2. Then, for  $|\alpha| = |\beta| = m$  let us set  $\tilde{b}_{\alpha\beta}(x) = \frac{1}{2}(a_{\alpha\beta}(x) + \overline{a_{\beta\alpha}(x)})$ ,  $\delta_{\alpha\beta} = 0$  (resp.  $\delta_{\alpha\beta} = 1$ ) if  $\alpha \neq \beta$  (resp. if  $\alpha = \beta$ ),  $b_{\alpha\beta}(x) = \chi(x)\tilde{b}_{\alpha\beta}(x) + \delta_{\alpha\beta}\frac{m!}{\alpha!}c(x)$ , and

$$q(x, \xi) = \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}(x)\xi^{\alpha+\beta} = \operatorname{Re}(\chi(x)p(x, \xi)) + c(x)|\xi|^{2m}.$$

Since  $q$  is (everywhere) nonnegative and has Lipschitz continuous coefficients, the paradifferential operator  $T_q$  with symbol  $q$  in the sense of Bony [1] satisfies Hörmander's Gårding inequality [6] that can be written

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \operatorname{Re}(T_q u, u) \geq -C\|u\|_{m-(1/4)}^2.$$

On the other hand, and again since the coefficients  $b_{\alpha\beta}$  are Lipschitz continuous, it follows from Bony's paradifferential symbolic calculus [1] that, up to error terms that can be suitably estimated,  $T_q$  is equal to the paradifferential operator  $B$  defined as  $Bu = \sum_{|\alpha|=|\beta|=m} D^\beta(T_{b_{\alpha\beta}} D^\alpha u)$ , and also equal to the differential operator  $b(x, D, \overline{D})$  with symbol  $b(x, \zeta, \bar{\zeta}) = \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}(x)\zeta^\alpha \bar{\zeta}^\beta$  since  $\|(b - T_b)v\|_{1/4} \leq C\|v\|_{-1/4}$  whenever the coefficient  $b$  belongs to  $C^{1/2}$ , thanks to Lemma A.1 in the appendix.

In conclusion, we have proved that there is a constant  $C$  such that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (b(x, D, \overline{D})u, u) \geq -C\|u\|_{m-(1/4)}^2.$$

Now, as in the proof of Theorem 3.2, we can write  $\operatorname{Re}(a(x, D, \overline{D})u, u) = (\tilde{a}(x, D, \overline{D})u, u)$  for a symbol  $\tilde{a}$  that can be written  $\tilde{a}(x, \zeta, \bar{\zeta}) = \tilde{b}(x, \zeta, \bar{\zeta}) + \tilde{c}(x, \zeta, \bar{\zeta}) + \tilde{d}(x, \zeta, \bar{\zeta})$  where  $\tilde{b}(x, \zeta, \bar{\zeta}) = \sum_{|\alpha|=|\beta|=m} \tilde{b}_{\alpha\beta}(x)\zeta^\alpha \bar{\zeta}^\beta$ ,  $\tilde{c}$  is a symbol of order  $(2m-1, m)$  with  $C^{(1/4)+\varepsilon}$  coefficients, and  $\tilde{d}$  is a symbol of order  $(2m-2, m-1)$ . Since  $1-\chi \equiv c \equiv 0$  on  $K$ , we have  $\tilde{b}(x, D, \overline{D})u = b(x, D, \overline{D})u$  for all  $u \in C_0^\infty(K)$ , and since the terms coming from the symbols  $\tilde{c}$  and  $\tilde{d}$  can be estimated in the same way as in the proof of Theorem 3.2, the proof is complete.  $\square$

FINAL REMARK. The method described in this section can be extended to some pseudodifferential operators with nonsmooth symbols, but still assuming that these symbols are nonnegative only on  $K \times \mathbb{R}^n$ .

More precisely, if for  $m \in \mathbb{R}$  and  $r \in (0, 1]$  we consider the space  $\Sigma_r^m$  of functions  $a(x, \xi)$  defined on  $\mathbb{R}^n \times \mathbb{R}^n$  and satisfying estimates

$$|\partial_\xi^\alpha a(x, \xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$$

and

$$|\partial_\xi^\alpha a(x, \xi) - \partial_\xi^\alpha a(y, \xi)| \leq C_\alpha |x - y|^r (1 + |\xi|)^{m-|\alpha|}$$

uniformly for  $x, y$  and  $\xi \in \mathbb{R}^n$ , and for all multiindices  $\alpha$ , and the corresponding pseudodifferential operators  $a(x, D)$  defined on  $\mathcal{S}(\mathbb{R}^n)$  by

$$a(x, D)u(x) = (2\pi)^{-n} \int e^{i(x,\xi)} a(x, \xi) \widehat{u}(\xi) d\xi,$$

then we can state the following result, the proof of which is left to the reader.

**THEOREM 4.3.** *Let  $m$  be a positive real number and  $\{m\} = \min_{k \in \mathbb{Z}} |m - k|$ ,  $\delta \in (\frac{2}{3}\{m\}, \frac{1}{2}]$ , and  $K$  be a compact subset of  $\mathbb{R}^n$ . Let  $a^k$  and  $b^k$  be functions from the space  $\Sigma_{2\delta}^m$  satisfying  $\Re(\sum_{k=1}^N a^k(x, \xi) \overline{b^k(x, \xi)}) \geq 0$  on  $K \times \mathbb{R}^n$ . Then there exists a constant  $C$  such that:  $\forall u \in C_0^\infty(K)$ ,  $\Re \sum_{k=1}^N (a^k(x, D)u, b^k(x, D)u) \geq -C \|u\|_{m-(\delta/2)}^2$ .*

### Appendix: Bony’s paraproduct and related estimates

To prove estimates we used in the previous sections, we need to give in this appendix a short reminder of Bony’s construction of the paraproduct operator [1].

Let us choose a function  $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$  such that  $\widehat{\varphi_0}(\xi) = 0$  for  $|\xi| \geq \frac{4}{3}$  and  $\widehat{\varphi_0}(\xi) = 1$  for  $|\xi| \leq 1$ . Then let us set  $\varphi_p(x) = 2^{pn} \varphi_0(2^p x)$ ,  $\psi_0(x) = \varphi_0(x)$  and for all integers  $p \geq 1$ ,  $\psi_p(x) = \varphi_p(x) - \varphi_{p-1}(x)$ . Since the sequence  $(\varphi_p)$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to the Dirac distribution, it follows that for any distribution  $u \in \mathcal{S}'(\mathbb{R}^n)$ , the convolution products  $u * \psi_p$  are  $C^\infty$  functions satisfying  $u = \sum_{p \geq 0} (u * \psi_p)$ , where the series converges in  $\mathcal{S}'(\mathbb{R}^n)$ .

This decomposition gives information on the smoothness of the distribution  $u$ . Indeed, if  $u$  belongs to the Sobolev space  $H^s$  for some  $s \in \mathbb{R}$ , then  $u * \psi_p \in L^2(\mathbb{R}^n)$  for all integers  $p \geq 0$ , and we have  $\sum_{p \geq 0} 2^{2ps} \|u * \psi_p\|_0^2 \leq C \|u\|_s^2 < \infty$ . Similarly, if  $a$  belongs to the Hölder space  $C^r$  for some  $r \in \mathbb{R}$ , then  $a * \psi_p \in L^\infty(\mathbb{R}^n)$  for all integers  $p \geq 0$ , and writing  $|a|_0 = \sup_{\mathbb{R}^n} |a|$ , we have  $\sup_{p \geq 0} 2^{pr} \|a * \psi_p\|_0 < \infty$ . To state a converse, let  $B$  be a compact subset of  $\mathbb{R}^n$ , and assume that  $0 \notin B$  or  $s > 0$ : if  $(u_p)$  is a sequence of smooth square integrable functions such that  $\text{supp } \widehat{u_p} \subset 2^p B$  and  $\sum_{p \geq 0} 2^{2ps} \|u_p\|_0^2 < \infty$ , then the series  $\sum_{p \geq 0} u_p$  is convergent in  $H^s$  (for all this, see [1] where a similar result is also proved for  $C^r$  distributions).

Since  $a = \sum_{p \geq 0} (a * \psi_p)$  and  $u = \sum_{q \geq 0} (u * \psi_q)$ , we are led to define the product  $au$  formally by setting

$$au = \sum_{p \geq 0} \sum_{q \geq 0} (a * \psi_p)(u * \psi_q).$$

Bony’s paraproduct operator can be defined by choosing half of these terms, as follows

$$T_a u = \sum_{q \geq 2} \left( \sum_{p < q-1} (a * \psi_p) \right) (u * \psi_q),$$

where the series is convergent in  $\mathcal{S}'(\mathbb{R}^n)$  whenever  $a \in C^r$  and  $u \in H^s$  for some real numbers  $r$  and  $s$ . Therefore, we will say that the product  $au$  is well defined if the complementary series

$$au - T_a u = \sum_{p \geq 0} \left( \sum_{q \leq p+1} (u * \psi_q) \right) (a * \psi_p)$$

is also convergent in  $\mathcal{S}'(\mathbb{R}^n)$ .

It is worth noting that this product coincides with the usual product when it is classically defined, namely when  $a \in L^\infty$  and  $u \in L^2$ , or when  $a \in \cap_{r \in \mathbb{R}} C^r$ , or when  $u \in \cap_{s \in \mathbb{R}} H^s$ . Moreover, when the products  $(D_j a)u$  and  $a(D_j u)$  are well defined, this is also true for the product  $au$ , and we then have  $D_j(au) = (D_j a)u + a(D_j u)$ .

In the present paper, we use the following estimates.

LEMMA A.1. *Let  $a \in C^r$  and  $u \in H^s$  for some real numbers  $r$  and  $s$ . Then there exist constants  $C$  independent of  $u$  but depending on  $a$ ,  $r$  and  $s$  such that:*

- (a) *If  $a \in L^\infty(\mathbb{R}^n)$ , then  $\|T_a u\|_s \leq C \|u\|_s$ .*
- (b) *If  $r + s > 0$  and  $r > 0$ , then the product  $au$  is well defined and we have the following estimates:  $\|(a - T_a)u\|_{r+s} \leq C \|u\|_s$  if  $s \in (-r, 0)$ , and  $\|au\|_s \leq C \|u\|_s$  if  $s \in (-r, r)$ .*

PROOF. Estimate (a) was proved in Bony [1].

For the proof of statement (b), let us first assume that  $s \in (-r, 0)$ . We have  $\sum_{q \leq p+1} (u * \psi_q) = u * \varphi_{p+1}$ , and since  $r + s > 0$  and the Fourier transform of  $(u * \varphi_{p+1})(a * \psi_p)$  is supported in  $2^p B$  for a compact subset  $B$  of  $\mathbb{R}^n$ , we just have to estimate  $\sum_{p \geq 0} 2^{2p(r+s)} \|(u * \varphi_{p+1})(a * \psi_p)\|_0^2$ . But because of the localisation of the Fourier transforms of the  $u * \psi_q$ 's, we can write  $\|u * \varphi_{p+1}\|_0^2 \leq C_1 \sum_{q \leq p+1} \|u * \psi_q\|_0^2$ , whence  $\|(u * \varphi_{p+1})(a * \psi_p)\|_0^2 \leq C_1 \sum_{q \leq p+1} \|u * \psi_q\|_0^2 \|a * \psi_p\|_0^2$ . Therefore

$$\begin{aligned} \sum_{p \geq 0} 2^{2p(r+s)} \|(u * \varphi_{p+1})(a * \psi_p)\|_0^2 &\leq C_1 \sum_{p \geq 0} 2^{2p(r+s)} \|a * \psi_p\|_0^2 \left( \sum_{q \leq p+1} \|u * \psi_q\|_0^2 \right) \\ &\leq C_2 \sum_{q \geq 0} \|u * \psi_q\|_0^2 \left( \sum_{p \geq q-1} 2^{2ps} \right) \\ &= C_3 \sum_{q \geq 0} 2^{2qs} \|u * \psi_q\|_0^2 \leq C_4 \|u\|_s^2. \end{aligned}$$

Therefore, in this case, the product  $au$  is well defined and  $\|(a - T_a)u\|_{r+s} \leq C \|u\|_s$ .

Now, if  $s \geq 0$  we have  $u \in H^s$  implies  $u \in H^t$  with  $t = -r/2 < 0$ . But since  $r + t = r/2 > 0$ , it follows from the previous case that the product  $au$  is also well defined in this case.



Finally, if  $s \in (-r, r)$ , let us set  $t = \frac{1}{2}(s - r)$ . Then we have:  $t < s$  whence  $u \in H^s$  implies  $u \in H^t$ ,  $t \in (-r, 0)$  whence  $\|(a - T_a)u\|_{r+t} \leq C\|u\|_t$ , and  $t < s < r + t$  whence

$$\|(a - T_a)u\|_s \leq \|(a - T_a)u\|_{r+t} \leq C\|u\|_t \leq C\|u\|_s .$$

On the other hand, statement (a) shows that  $\|T_a u\|_s \leq C\|u\|_s$ , and therefore, we can conclude that  $\|au\|_s \leq C\|u\|_s$ .  $\square$

Let us point out that, in particular, estimates (b) above are true with  $r = 1$  when the function  $a$  is Lipschitz continuous. We use this fact to prove the following lemma.

LEMMA A.2. *Let  $V$  be an open subset of  $\mathbb{R}^n$ , and  $a(x, \zeta, \bar{\zeta})$  be a symbol of order  $(2m, m)$ .*

- (a) *If the coefficients  $a_{\alpha\beta}$  are Lipschitz continuous in  $V$ , and if  $a(x, \xi, \xi) = 0$  for all  $(x, \xi) \in V \times \mathbb{R}^n$ , then there exists a constant  $C$  such that: for all  $u \in C_0^\infty(V)$ ,  $|a(x, D, \bar{D})u, u| \leq C\|u\|_{m-(1/2)}^2$ .*
- (b) *If  $a$  is a symbol of order  $(2m - 1, m)$  and if the coefficients  $a_{\alpha\beta}$  belong to  $C^{(\delta/2)+\varepsilon}$  for some real numbers  $\delta \leq 1$  and  $\varepsilon > 0$ , then there exists a constant  $C$  such that: for all  $u \in C_0^\infty(V)$ ,  $|a(x, D, \bar{D})u, u| \leq C\|u\|_{m-(\delta/2)}^2$ .*

PROOF. Let us first prove property (a). If  $a$  has Lipschitz continuous coefficients and if  $a(x, \xi, \xi) = 0$  on  $V \times \mathbb{R}^n$ , there exist symbols  $b_{jk}$  and  $c_j$  of order  $(2m - 2, m - 1)$  with Lipschitz continuous coefficients such that

$$a(x, \zeta, \bar{\zeta}) = \sum_{j,k=1}^n (\zeta_j \bar{\zeta}_k - \zeta_k \bar{\zeta}_j) b_{jk}(x, \zeta, \bar{\zeta}) + \sum_{j=1}^n (\zeta_j - \bar{\zeta}_j) c_j(x, \zeta, \bar{\zeta}) .$$

Therefore, we have to estimate terms of two different forms. For the first kind of terms, we use Bony's paraproduct operator  $T_b$  to write, for  $|\alpha| < m$  and  $|\beta| < m$ ,

$$\begin{aligned} & |(bD_j D^\alpha u, D_k D^\beta u) - (bD_k D^\alpha u, D_j D^\beta u)| = \left| ((b - T_b)D_j D^\alpha u, D_k D^\beta u) \right. \\ & \quad + ([T_b, D_j]D^\alpha u, D_k D^\beta u) + ([D_k, T_b]D^\alpha u, D_j D^\beta u) \\ & \quad \left. + ((T_b - b)D_k D^\alpha u, D_j D^\beta u) \right| \leq \|(b - T_b)D_j D^\alpha u\|_{1/2} \|D_k D^\beta u\|_{-1/2} \\ & \quad + \|T_{\partial_j b} D^\alpha u\|_{1/2} \|D_k D^\beta u\|_{-1/2} + \|T_{\partial_k b} D^\alpha u\|_{1/2} \|D_j D^\beta u\|_{-1/2} \\ & \quad + \|(b - T_b)D_k D^\alpha u\|_{1/2} \|D_j D^\beta u\|_{-1/2} \end{aligned}$$

because  $[T_b, D_j] = iT_{\partial_j b}$ , and all these terms can be estimated by a constant times  $\|u\|_{m-(1/2)}^2$  since  $\|(b - T_b)v\|_{1/2} \leq C\|v\|_{-1/2}$  for a Lipschitz continuous coefficient  $b$ , and  $\|T_c v\|_{1/2} \leq C\|v\|_{1/2}$  for a bounded coefficient  $c$  thanks to



Lemma A.1. For the second kind of terms, we simply write, again for  $|\alpha| < m$  and  $|\beta| < m$ ,

$$|(cD_j D^\alpha u, D^\beta u) - (cD^\alpha u, D_j D^\beta u)| = |([c, D_j]D^\alpha u, D^\beta u)| \leq C\|u\|_{m-1}^2$$

since the operator  $[c, D_j]$  is just a multiplication by a bounded function.

Let us now prove property (b). Since a symbol of order  $(2m - 1, m)$  is a linear combination of terms of the form  $a_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta + a_{\beta\alpha}\zeta^\beta\bar{\zeta}^\alpha$  with  $|\alpha| < m$  and  $|\beta| \leq m$ , we just have to write

$$|(aD^\alpha u, D^\beta u)| \leq \|aD^\alpha u\|_{\delta/2}\|D^\beta u\|_{-\delta/2} \leq C\|u\|_{m-(\delta/2)}^2$$

for such  $\alpha$  and  $\beta$ 's, since  $\|av\|_{\delta/2} \leq C\|v\|_{\delta/2}$  thanks to Lemma A.1 when the coefficient  $a$  belongs to  $C^{(\delta/2)+\varepsilon}$ , and  $\|v\|_{\delta/2} \leq \|v\|_{1-(\delta/2)}$  when  $\delta \leq 1$ .  $\square$

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