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# Continuability in Time of Smooth Solutions of Strong-Nonlinear Nondiagonal Parabolic Systems

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**Abstract.** A class of quasilinear parabolic systems with quadratic nonlinearities in the gradient is considered. It is assumed that the elliptic operator of a system has variational structure. In the multidimensional case, the behavior of solutions of the Cauchy-Dirichlet problem smooth on a time interval $[0, T)$ is studied. Smooth extendibility of the solution up to $t = T$ is proved, provided that "normilized local energies" of the solution are uniformly bounded on $[0, T)$. For the case where $[0, T)$ determines the maximal interval of existence of a smooth solution,the Hausdorff measure of a singular set at the moment $t = T$ is estimated.

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## 1. – Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\partial\Omega$; $(x, t) \in \Omega \times (0, T) = Q$, where $T > 0$ is an arbitrarily fixed number.

Consider $u: Q \to \mathbb{R}^N$, $N > 1$, that is a solution of the Cauchy-Dirichlet problem

$$\begin{aligned} u_t + Lu = 0, &\quad (x, t) \in Q\,, \\ u\big|_{\Gamma} = 0, &\quad \Gamma = \partial\Omega \times (0, T)\,, \\ u\big|_{t=0} = \varphi\,. & \end{aligned} \tag{1}$$

In (1), $L$ is a quasilinear elliptic operator, $\varphi$ is a given smooth function.

To describe $L$, we introduce a scalar function

$$f(x, u, p) = \frac{1}{2}\langle A(x, u)p, p\rangle = \frac{1}{2} \sum_{\substack{\alpha,\beta \leq n \\ k,l \leq N}} A_{kl}^{\alpha\beta}(x, u) p_\beta^l p_\alpha^k \tag{2}$$

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on the set $\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ and assume that the following conditions hold on the set $\mathcal{M} = \overline{\Omega} \times \mathbb{R}^N$:

$\mathbb{A}_1$. $A_{kl}^{\alpha\beta} = A_{lk}^{\beta\alpha}, \quad \alpha, \beta \leq n, \quad k, l \leq N.$

$\mathbb{A}_2$.

$$\begin{aligned} &\langle A(x,u)\xi, \xi\rangle \;\geq\; \nu|\xi|^2 \quad \text{for any} \quad \xi \;\in\; \mathbb{R}^{Nn}, \\ &\sup_{\mathcal{M}} \|A(\cdot,\cdot)\| \leq \mu, \quad \nu, \mu = \text{const} > 0. \end{aligned} \tag{3}$$

$\mathbb{A}_3$. The functions $A_{kl}^{\alpha\beta}$ are twice differentiable with respect to $x$ and $u$ on $\mathcal{M}$ and

$$\begin{aligned} &l_0 = \sup_{\mathcal{M}} \|A'_x\| < +\infty, \quad l_1 = \sup_{\mathcal{M}} \|A'_u\| < +\infty\,, \\ &l_2 = \sup_{\mathcal{M}} \|A''_{uu}\| < +\infty\,. \end{aligned} \tag{4}$$

For $f$ defined in (2) we put

$$E[u] = \int_\Omega f(x, u, u_x)\, dx, \quad u_x = (\nabla u^1, \ldots, \nabla u^N) \in \mathbb{R}^{nN}\,, \tag{5}$$

and denote by $L = \{L^{(k)}\}^{k \leq N}$,

$$L^{(k)} u = -\frac{d}{dx_\alpha} f_{p_\alpha^k}(x, u, u_x) + f_{u^k}(x, u, u_x)\,, \tag{6}$$

the Euler operator of $E[u]$.

Then (1) is the quasilinear parabolic system

$$u_t^k - \left(A_{kl}^{\alpha\beta}(x,u) u_{x_\beta}^l\right)_{x_\alpha} + b^k(x, u, u_x) = 0, \quad k \leq N\,, \tag{7}$$

where

$$\begin{aligned} &b^k(x, u, p) = \frac{1}{2}\left(A_{ml}^{\alpha\beta}(x,u)\right)'_{u^k} p_\beta^l p_\alpha^m\,, \\ &|b(x, u, p)| \leq \frac{l_1}{2}\,|p|^2\,. \end{aligned} \tag{8}$$

In what follows, we *do not impose a smallness condition* on $l_1$. System (7) is an example of a quasilinear nondiagonal parabolic system with a quadratic nonlinearity in the gradient.

The classical local in time solvability of (1), (7), (8) follows from the results of [1] and [8]. Weak global solvability of initial boundary value problems for nondiagonal parabolic systems with quadratic nonlinearities has not yet been proved.

In the case of two spatial variables, the author constructed a weak *global in time* solution of problem (1) with an elliptic operator $L$ of variational structure (6) [2], [3]. Weak solvability for the same class of systems under a Neumann-type boundary condition was proved in [5], [6]. We also mention that in [3]-[5], a more general class of $f$ ($f(x, u, p) \sim |p|^2$ as $|p| \to +\infty$) in comparison with (2) was studied.

In the present paper we study the Cauchy-Dirichlet problem (1) for system of variational structure (7), (8) in the multidimensional case $n = \dim \Omega > 2$. We prove that if the "normalized local energies" of a solution of the system are uniformly sufficiently small on $[0, T)$, then a solution smooth on a time interval $[0, T)$ can be continued up to $t = T$ as a smooth function (Theorem 1).

As a consequence of Theorem 1, we have a description of the singular set of the solution at the moment $T$, provided that $T$ determines the maximal interval $[0, T)$ of existence of a smooth solution (Theorem 2).

It is worth noting that in the papers [2]-[5] the continuability theorem was proved by a different and more cumbersome method. For that method, it is crucial that the dimension of $\Omega$ is equal to two. In contrast, the proposed method is valid for any dimension $n \geq 2$ and is much simpler.

Also, we note that in view of Remark 5 of the present paper it becomes evident that the statements of Theorem 1 (for $n = 2$) and Theorem 0.2 [2] are, in fact, equivalent.

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## 2. – Notation and main results

We use the following notation:

$$u\colon \overline{Q} \to \mathbb{R}^N,\ u = (u^1, \dots, u^N),\ x \in \overline{\Omega},\ x = (x_1, \dots, x_n),\ n \geq 2,\ z = (x, t) \in \overline{Q},$$

$$u_x = \{u^k_{x_\alpha}\}^{k \leq N}_{\alpha \leq n},\ |u_x|^2 = \sum_{\substack{k \leq N \\ \alpha \leq n}} (u^k_{x_\alpha})^2,\ u_{xt} = \{u^k_{x_\alpha t}\}^{k \leq N}_{\alpha \leq n},$$

$$|u_{xt}|^2 = \sum_{\substack{k \leq N \\ \alpha \leq n}} (u^k_{x_\alpha t})^2,\ u_{xx} = \{u^k_{x_\alpha x_\beta}\}^{k \leq N}_{\alpha, \beta \leq n},\ |u_{xx}|^2 = \sum_{\substack{k \leq N \\ \alpha, \beta \leq n}} (u^k_{x_\alpha x_\beta})^2.$$

$$B_R(x^0)=\{x\in\mathbb{R}^n:\ |x-x^0|<R\},\ \ S_R(x^0)=\{x\in\mathbb{R}^n:\ |x-x^0|=R\}\,,$$

$$B_R^+(x^0)=B_R(x^0)\cap\{x_n>x_n^0\},\ \Omega_R(x^0)=B_R(x^0)\cap\Omega,\ Q_R(z^0)=\Omega_R(x^0)\times\Lambda_R(t^0)\,,$$

$$\Lambda_R(t^0)=(t^0-R^2,t^0),\ \partial' Q_R(z^0)=(\partial\Omega_R(x^0)\times\Lambda_R(t^0))\cup(\Omega_R(x^0)\times\{t^0-R^2\})\,,$$

$$\Omega^t=\Omega\times\{t\},\quad |D|=\mathrm{meas}_{n+1}D\,,$$

$$v_R^0=\fint_{Q_R(z^0)}v\,dz=\frac{1}{|Q_R|}\int_{Q_R(z^0)}v\,dz,\ \fint_{\Omega_R(x^0)}|v|^2dx=\frac{1}{R^{n-2}}\int_{\Omega_R(x^0)}|v|^2dx\,.$$

For brevity, we write $B_R, S_R,\dots$ in place of $B_R(0), S_R(0),\dots$ and $u\in\mathcal{B}(Q)$ in place of $u\in\mathcal{B}(Q;\mathbb{R}^N)$.

The definition of the spaces $W_p^k(\Omega)$, $\mathbb{C}^{k+\alpha}(\overline{\Omega})$, $W_p^{l,k}(Q)$, $\mathbb{C}(\overline{Q})$, and $\mathbb{C}^{\alpha,\beta}(\overline{Q})$ can be found in [9]. We denote by $L^{p,\alpha}(Q;\delta)$ and $\mathcal{L}^{p,\alpha}(Q;\delta)$ the Morrey and Campanato spaces in the parabolic metric

$$\delta(z^1,z^2)=\max\left\{|x^1-x^2|,|t^1-t^2|^{1/2}\right\},\quad z^i=(x^i,t^i),\quad i=1,2,$$

(see [6]).

In addition, $\|u\|_{m,D}$ is the norm in the space $L^m(D)$ of $m$-integrable functions, $H_k(\sigma)$ is the $k$-dimensional Hausdorff measure of a set $\sigma$.

To describe a class of smooth solutions, for $\alpha\in(0,1)$ we introduce the space $\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$ of functions $v$ such that $v, v_x, v_t$ and $v_{xx}$ are continuous functions in $\overline{Q}$ and have the following finite norm (see [9]):

$$\begin{aligned}\|v\|_{\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})}=\|v\|_{\mathbb{C}(\overline{Q})}+\|v_x\|_{\mathbb{C}(\overline{Q})}+\|v_t\|_{\mathbb{C}^{\alpha,\alpha/2}(\overline{Q})}\\ +\|v_{xx}\|_{\mathbb{C}^{\alpha,\alpha/2}(\overline{Q})}+\langle v_x\rangle_{t,Q}^{(1+\alpha)/2}\,,\end{aligned}\tag{9}$$

where

$$\langle w\rangle_{t,Q}^{(\beta)}=\sup_{\substack{(x,t'),(x,t'')\in\overline{Q}\\ t'\neq t''}}\frac{\left|w(x,t')-w(x,t'')\right|}{\left|t'-t''\right|^{\beta}}\,.$$

For a fixed number $\alpha\in(0,1)$ we define a class of smooth solutions:

$$\mathcal{K}_\alpha\left\{[t_1,t_2]\right\}=\left\{v:\ \overline{Q}'\to\mathbb{R}^N|\ v\in\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q}'),\, v_{xt}\in L^{2,n+2\alpha}(Q';\delta)\right\},\tag{10}$$

where $Q'=\Omega\times(t_1,t_2)$, $t_1,t_2\in[0,T]$.

We write $v\in\mathcal{K}_\alpha\{[t_1,t_2)\}$, if $v\in\mathcal{K}_\alpha\{[t_1,\tau]\}$ for any $\tau<t_2$.

THEOREM 1. *Let conditions* $\mathbb{A}_1-\mathbb{A}_3$ *hold,* $\partial\Omega\in\mathbb{C}^{2+\alpha}$, $\varphi\in\mathbb{C}^{2+\alpha}(\overline{\Omega})$ *for a fixed* $\alpha\in(0,1)$. *Let* $u\in\mathcal{K}_\alpha\{[0,T)\}$ *be a solution of* (1), (7), (8) *for a fixed* $T>0$. *There exists a number* $\varepsilon_0>0$ *such that if for some* $R_0=R_0(\varepsilon_0)>0$

$$\sup_{\substack{t^0\in[T/2,T)\\ x^0\in\overline{\Omega}}}\ \sup_{\rho\le R_0}\frac{1}{\rho^n}\int_{Q_\rho(z^0)}|u_x(x,t)|^2dz<\varepsilon_0\,,\tag{11}$$

*then* $u\in\mathcal{K}_\alpha\{[0,T]\}$. *The number* $\varepsilon_0$ *depends only on the parameters* $\nu$, $\mu$, $l_0$, $l_1$ *and* $l_2$ *and* $\mathbb{C}^{1+1}$*-characteristics of* $\partial\Omega$.

THEOREM 2. *Let* $u \in \mathcal{K}_\alpha\{[0,T)\}$ *be a solution of the problem* (1), (7), (8), *and let the number* $T$ *determine the maximal interval of existence of the smooth solution* $u$. *Let* $\Sigma(T) = \sigma \times \{T\}$ *be the singular set of* $u$*; then for any* $x^0 \in \sigma \subset \overline{\Omega}$

$$\overline{\lim_{t \nearrow T}} \;\; ⨍_{\Omega_R(x^0)} |u_x(y,t)|^2 dy \geq \varepsilon_0 \quad \textit{for a sequence} \quad R \to 0\,, \tag{12}$$

*and* $H_{n-2}(\sigma) \leq c_0$. *The constant* $c_0$ *depends on the same characteristics as* $\varepsilon_0$ ($\varepsilon_0$ *is defined in* (11)). *The function* $u$ *has a smooth continuation to the set* $(\overline{\Omega} \setminus \sigma) \times \{T\}$.

## 3. – Proof of Theorem 1

We subdivide the proof into several lemmas.

In what follows, we denote by $c$ and $c_i$ different constants depending of the parameters $\nu$, $\mu$, $l_0$, $l_1$, $l_2$ and $n$.

LEMMA 1. *The following estimates hold for a solution* $u \in \mathcal{K}_\alpha\{[0,T)\}$ *of problem* (1):

$$\int_{t_1}^{t_2} \int_\Omega |u_t|^2 dx\, dt + E[u(t_2)] \leq E[u(t_1)],\ t_1 \leq t_2 < T\,, \tag{13}$$

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\Omega_R(x^0)} |u_t|^2 dx\, dt + \frac{\nu}{2} \int_{\Omega_R(x^0)} |u_x(x,t_2)|^2 dx \leq c_1(\mu) \int_{\Omega_{2R}(x^0)} |u_x(x,t_1)|^2 dx \\ &\quad + \frac{c_2(\mu)}{R^2} \int_{t_1}^{t_2} \int_{\Omega_{2R}(x^0)} |u_x(x,\tau)|^2 dx\, d\tau,\ x^0 \in \overline{\Omega},\ t_1 \leq t_2 < T,\ R > 0\,. \end{aligned} \tag{14}$$

PROOF. In order to prove Lemma 1, we exploit the variational structure (6) of the operator $L$ and argue precisely in the same way as in [2]. The function $u$ satisfies the identity

$$\int_{t_1}^{t_2} \int_\Omega [u_t^k h^k + f_{p_\alpha^k}(x,u,u_x) h_{x_\alpha}^k + f_{u^k}(x,u,u_x) h^k]\, dx\, dt = 0,\ 0 \leq t_1 \leq t_2 < T\,, \tag{15}$$

for any smooth function $h$, $h\big|_{\partial\Omega \times (t_1,t_2)} = 0$.

Inequality (13) follows from (15) with $h = u_t$. To derive (14), we put $h = u_t \xi^2$, where $\xi = \xi(x)$ is a cut-off function for $B_{2R}(x^0)$, $\xi = 1$ in $B_R(x^0)$.

REMARK 1. From (13) it follows that

$$\int_Q |u_t|^2 dz \le E[\varphi]\,; \tag{16}$$

$$E[u(T)] \le \liminf_{t\nearrow T} E[u(t)] \le E[\varphi], \qquad \sup_{t\in[0,T]} E[u(t)] \le E[\varphi]\,; \tag{17}$$

$$E[u(t_2)] \le E[u(t_1)], \quad t_1 < t_2 \le T\,; \tag{18}$$

$$u(\cdot,T) \in \overset{\circ}{W}{}^1_2(\Omega), \quad \|u(\cdot,T)\|_{W^1_2(\Omega)} \le c(\nu,\mu)\|\varphi_x\|_{2,\Omega}\,. \tag{19}$$

REMARK 2. Several important relations follow from inequality (14).

Let us fix $t^0 \in (0,T]$ and $R > 0$ such that $t^0 - 4R^2 > 0$. We put $t_1 \in (t^0-4R^2, t^0-2R^2)$, $t_2 \in \Lambda_R(t^0) = (t^0-R^2, t^0)$ in (14) and have

$$\begin{aligned}\int_{t^0-2R^2}^{t_2}\int_{\Omega_R(x^0)} |u_t|^2 dx\,dt + \nu\|u_x(\cdot,t_2)\|^2_{2,\Omega_R(x^0)}&\\ \le c_1\|u_x(\cdot,t_1)\|^2_{2,\Omega_{2R}(x^0)} + \frac{c_2}{R^2}\int_{\Lambda_{2R}(t^0)}\int_{\Omega_{2R}(x^0)} |u_x|^2 dx\,dt\,.&\end{aligned}$$

Now we integrate this inequality over $t_1 \in (t^0-4R^2, t^0-2R^2)$ and divide by $2R^2$:

$$\int_{t^0-2R^2}^{t_2}\int_{\Omega_R(x^0)} |u_t|^2 dx\,dt + \nu\|u_x(\cdot,t_2)\|^2_{2,\Omega_R(x^0)} \le \frac{c_3}{R^2}\int_{Q_{2R}(z^0)} |u_x|^2 dz\,. \tag{20}$$

From (20) it follows that

$$R^2\int_{Q_R(z^0)} |u_t|^2 dz \le c_3\int_{Q_{2R}(z^0)} |u_x|^2 dz\,, \tag{21}$$

$$\sup_{\Lambda_R(t^0)} \|u_x(\cdot,t)\|^2_{2,\Omega_R(x^0)} \le \frac{c_4}{R^2}\int_{Q_{2R}(z^0)} |u_x|^2 dz\,. \tag{22}$$

By the Poincaré inequality and (21), we also obtain

$$\int_{Q_R(z^0)} |u-u^0_R|^2 dz \le c_* R^2\int_{Q_{2R}(z^0)} |u_x|^2 dz\,, \tag{23}$$

where $u^0_R = \fint_{Q_R(z^0)} u\,dz$, $z^0 \in \overline{\Omega}\times(0,T]$, $R \le \frac{\sqrt{t_0}}{2}$.

REMARK 3. The variational structure is essentially used only in proving Lemma 1 and relations (16)-(23). Stronger norms of $u$ will be estimated in the vicinity of $t = T$, in a local coordinate system. For a fixed point $x^0 \in \partial\Omega$, we consider a neibourhood $V(x^0)$ and a $\mathbb{C}^{2+\alpha}$-diffeomorphism $y = y(x)$ such that $y(V \cap \Omega) = B_2^+(0)$, $y(V \cap \partial\Omega) = \gamma_2 = B_2 \cap \{y_n = 0\}$. (For more detailed information on the local setting, see Remarks 2.1 and 2.2 in [2].)

In the local setting, the function $v(y,t) = u(x(y),t)$ is a solution of the problem

$$v_t^k - (a_{kl}^{\alpha\beta}(y,v)v_{y_\beta}^l)_{y_\alpha} + \mathbb{B}^k(y,v,v_y) = 0, \quad (y,t) \in Q_2^+ = B_2^+ \times (0,T), \quad k \leq N, \tag{24}$$
$$v\big|_{\gamma_2 \times (0,T)} = 0, \quad v\big|_{\substack{t=0 \\ y \in B_2^+}} = \varphi(x(y)).$$

Here, the functions $a_{kl}^{\alpha\beta}$ and $\mathbb{B}^k$ satisfy the conditions

$$a_{kl}^{\alpha\beta}(y,v)\eta_\alpha^k \eta_\beta^l \geq \nu_*|\eta|^2, \quad \forall\, \eta \in \mathbb{R}^{Nn}, \qquad \sup_{B_2^+ \times \mathbb{R}^N} \|a(\cdot,\cdot)\| \leq \mu_*, \tag{25}$$
$$|\mathbb{B}(y,v,q)| \leq l_*(1 + |q|^2), \quad q \in \mathbb{R}^{Nn},$$

where the constants $\nu_*$, $\mu_*$, $l_*$ depend on $\nu$, $\mu$, $l_1$, and $\mathbb{C}^{1+1}$-characteristics of $\partial\Omega$.

Let $\{V^j,\ y^j(x)\}_{j=0}^M$ be a finite atlas of $\overline{\Omega}$, $\bigcup_j V^j \supset \overline{\Omega}$; $y^j(V^j \cap \Omega) = B_2^+$, $y^j(V^j \cap \partial\Omega) = \gamma_2$, $j = 1, \dots, M$; $V^0 \subset \Omega$, $y^0(x) \equiv x$; $y^j \in \mathbb{C}^{2+\alpha}(V^j)$.

In what follows, we put

$$\lambda = \sup_{j \leq M} \left\{ \sup_{V^j} \left\| \frac{\partial y^j(x)}{\partial x} \right\|, \ \sup_{B_2^+} \left\| \frac{\partial x^j(y)}{\partial y} \right\| \right\}, \tag{26}$$
$$\lambda_1 = \sup_{j \leq M} \left\{ \|y^j\|_{\mathbb{C}^{1+1}(V^j)}, \ \|x^j\|_{\mathbb{C}^{1+1}(B_2^+)} \right\},$$

where $x^j = x^j(y)$ is the inverse transformation to $y^j$.

We put

$$\omega_R(z^0) = \operatorname*{osc}_{Q_R(z^0)} u, \tag{27}$$

and

$$\psi(\rho, z^0) = \frac{1}{\rho^{n+2\beta}} \int_{Q_\rho(z^0)} |u_x|^2 dz \tag{28}$$

for a fixed $\beta \in (0,1)$, u is the solution of (1),(7),(8) under consideration.

LEMMA 2. *There exist positive numbers $\omega_1$ and $R_1$ such that if for some $R \leq R_1$ in the cylinder $Q_R(z^0)$, $z^0 \in \overline{\Omega} \times [\frac{T}{2}, T)$, the inequality*

$$\omega_R(z^0) \leq \omega_1 \tag{29}$$

*holds, then*

$$\sup_{\rho \leq R} \psi(\rho, z^0) \leq \mathbb{K}_1 \left\{ \psi(R, z^0) + R^{2(2-\beta)} \right\}. \tag{30}$$

*The constants $\omega_1$, $R_1$ and $\mathbb{K}_1$ depend only on $\nu$, $\mu$, $l_0$, $l_1$ and $\lambda_1$.*

PROOF OF LEMMA 2. First, we shall derive a local version of (30). Let $v(y,t)$ be a solution of (24). We fix $t^0 \in [\frac{T}{2}, T)$, $y^0 \in \overline{B}^+_{3/2}$ and $R < \frac{1}{s}\min\{\frac{1}{2}, \sqrt{\frac{T}{2}}\}$, where the number $s = s(\lambda) > 1$ will be chosen later. (The restriction $sR < \sqrt{\frac{T}{2}}$ is imposed only to avoid the situation $Q_{sR}(z^0) \cap \{t = 0\} \neq \emptyset$.)

Now we put $\widetilde{\Omega}_R(y^0) = B_2^+ \cap B_R(y^0)$ and $\hat{Q}_R(\xi^0) = \widetilde{\Omega}_R(y^0) \times \Lambda_R(t^0)$, $\xi^0 = (y^0, t^0)$, and consider the following model problem:

$$
\begin{aligned}
&\theta^k_t - a^{\alpha\beta}_{kl}(y^0, v^0)\theta^l_{y_\beta y_\alpha} = 0 \quad \text{in} \quad \hat{Q}_R(\xi^0)\,, \\
&\theta\big|_{\partial' \hat{Q}_R(\xi^0)} = v\,,
\end{aligned} \tag{31}
$$

$v^0 = \fint_{\hat{Q}_R(\xi^0)} v\, d\xi$. Note that $\theta\big|_{\gamma_R(y^0)\times\Lambda_R(t^0)} = 0$, $\gamma_R(y^0) = B_R(y^0) \cap \{y_n = 0\}$. For a solution $\theta$ of (31), the following integral estimate is known (see [6]):

$$
\int_{\hat{Q}_\rho(\xi^0)} |\theta_y|^2 d\xi \le c\left(\frac{\rho}{R}\right)^{n+2} \int_{\hat{Q}_R(\xi^0)} |\theta_y|^2 d\xi, \quad \rho \le R\,, \tag{32}
$$

$c = c(\nu_*, \mu_*)$.

The function $w = v - \theta$, $w\big|_{\partial'\hat{Q}_R} = 0$, satisfies the identity

$$
\int_{\hat{Q}_R(\xi^0)} [w^k_t h^k + a^{\alpha\beta}_{kl}(y^0, v^0) w^l_{y_\beta} h^k_{y_\alpha} + \Delta a^{\alpha\beta}_{kl} v^l_{y_\beta} h^k_{y_\alpha} + \mathbb{B}^k(y, v, v_y)h^k] d\xi = 0 \tag{33}
$$

for any smooth function $h$ with $h\big|_{\partial\hat{\Omega}_R\times\Lambda_R} = 0$.

From (33) with $h = w$, we deduce the inequality

$$
\int_{\hat{Q}_R(\xi^0)} |w_y|^2 d\xi \le c(\nu_*, \mu_*) \int_{\hat{Q}_R(\xi^0)} [|\Delta a|^2 |v_y|^2 + (1 + |v_y|^2)|w|] d\xi\,,
$$

where $|\Delta a| = |a(y, v) - a(y^0, v^0)| \le c(|y - y| + |v - v^0|)$, $c = c(l_0, l_1, \lambda)$.

It yields the relation

$$
\begin{aligned}
\int_{\hat{Q}_R(\xi^0)} |w_y|^2 d\xi &\le c_1(R^2 + \hat{\omega}^2_R(\xi^0)) \int_{\hat{Q}_R(\xi^0)} |v_y|^2 d\xi + c_2 R^{n+4} \\
&\quad + c_3 \int_{\hat{Q}_R(\xi^0)} |v_y|^2 |w|\, d\xi, \quad \hat{\omega}_R(\xi_0) = \operatorname*{osc}_{\hat{Q}_R(\xi^0)} v\,.
\end{aligned} \tag{34}
$$

To estimate the integral $J_R(\xi^0) = \int_{\hat{Q}_R(\xi^0)} |v_y|^2 |w|\, d\xi$, we apply the integral identity for the solution $v$ of (24) with the test function $\eta = (v - v^0)|w|$.

As a result, we obtain the inequality

$$
\begin{aligned}
\nu_* J_R(\xi^0) \le \hat{\omega}_R(\xi^0) &\int_{\hat{Q}_R(\xi^0)} |v_t||w|\, ds + \mu_* \hat{\omega}_R(\xi^0) \int_{\hat{Q}_R(\xi^0)} |v_y||w_y|\, d\xi \\
&+ l_* \hat{\omega}_R(\xi^0) J_R(\xi^0) + l_* \hat{\omega}_R(\xi^0) \int_{\hat{Q}_R(\xi^0)} |w|\, ds\,.
\end{aligned}
$$

Now assume that

$$\hat{\omega}_R(\xi^0) \le \frac{\nu_*}{2l_*}, \tag{35}$$

and, using the Cauchy inequality with a small parameter, we derive

$$\begin{aligned} J_R(\xi^0) &\le \frac{1}{2c_3} \int_{Q_R(\xi^0)} |w_y|^2 d\xi + c_4 R^{n+4} \\ &\quad + c_5 \hat{\omega}_R^2(\xi^0) \left( P_R(\xi^0) + \int_{\hat{Q}_R(\xi^0)} |v_y|^2 d\xi \right), \\ P_R(\xi^0) &= R^2 \int_{\hat{Q}_R(\xi^0)} |v_t|^2 d\xi . \end{aligned} \tag{36}$$

From (34) and (36) it follows that

$$\begin{aligned} \int_{\hat{Q}_R(\xi^0)} |w_y|^2 d\xi &\le c_6 (R^2 + \hat{\omega}_R^2(\xi^0)) \int_{\hat{Q}_R(\xi^0)} |v_y|^2 d\xi + c_7 R^{n+4} \\ &\quad + c_8 \hat{\omega}_R^2(\xi^0) P_R(\xi^0) . \end{aligned} \tag{37}$$

To estimate $P_R(\xi^0)$ we make use of inequality (21). More precisely, we change the coordinates "$y$" by "$x$" in the expression for $P_R(\xi^0)$, apply (21), and then make the inverse transformation to the coordinates "$y$". As a result, we obtain the inequality

$$P_R(\xi^0) \le c(\lambda, \mu) \int_{\hat{Q}_{sR}(\xi^0)} |v_y|^2 d\xi \tag{38}$$

with some number $s = s(\lambda) > 1$.

Now from (32), (37), (38), for the function $\Phi(\rho, \xi^0) = \int_{\hat{Q}_\rho(\xi^0)} |v_y|^2 d\xi$ we deduce that

$$\begin{aligned} \Phi(\rho, \xi^0) &\le c_9 \left[ \left( \frac{\rho}{R} \right)^{n+2} + R^2 + \hat{\omega}_R^2(\xi^0) \right] \Phi(R, \xi^0) \\ &\quad + c_{10} \hat{\omega}_R^2(\xi^0) \Phi(sR, \xi^0) + c_{11} R^{n+4}, \quad \rho \le R . \end{aligned} \tag{39}$$

By assumption, $r = sR < \min\{\frac{1}{2}, \sqrt{\frac{T}{2}}\}$ and (39) implies the inequality

$$\Phi(\rho, \xi^0) \le c_{12} \left[ \left( \frac{\rho}{r} \right)^{n+2} + \omega_0^2 \right] \Phi(r, \xi^0) + c_{13} r^{n+4} \tag{40}$$

if

$$\max\left\{ r^2, \hat{\omega}_r^2(\xi^0) \right\} \le \frac{\omega_0^2}{3} . \tag{41}$$

Now we choose $\omega_0$. In accordance with a well-known algebraic lemma (see, for example, [7]), there exists a positive number $\omega_0 = \omega_0(n, c_{12})$ small enough such that if (40) holds with such a $\omega_0$, then the inequality

$$\Phi(\rho,\xi^0) \le c_{14}\left[\left(\frac{\rho}{r}\right)^{n+2\beta}\Phi(r,\xi^0) + \rho^{n+2\beta}r^{2(2-\beta)}\right], \quad \rho \le r \le r_0 \tag{42}$$

is valid.

Taking into account (35), (41), we conclude that (42) holds if

$$\hat{\omega}_r(\xi^0) \le \min\left\{\frac{\nu_*}{2l_*}, \frac{\omega_0}{\sqrt{3}}\right\}, \quad r \le r_0 = \min\left\{\frac{1}{2}, \frac{\omega_0}{\sqrt{3}}, \sqrt{\frac{T}{2}}\right\}. \tag{43}$$

In the coordinates $(x_1, \dots, x_n)$, from (42) for the function $\psi(\rho, z^0)$ (see (28)) we obtain the estimate

$$\psi(\rho, z^0) \le c_{15}\big[\psi(R, z^0) + R^{2(2-\beta)}\big], \quad \rho \le R \le R_1 = R_1(\lambda, r_0),$$

if $\omega_R(z^0) \le \omega_1 = \min\{\frac{\nu_*}{2l_*}, \frac{\omega_0}{\sqrt{3}}\}$. Thus, we have arrived at (30).

The next assertion is a local version in the parabolic metric of a well-known estimate (see [7]).

LEMMA 3. *For a function $u \in \mathcal{L}^{2,n+2+2\beta}(Q;\delta)$ and a cylinder $Q_{2R}(z^0) \subset Q$, $z^0 \in \overline{\Omega} \times (0, T]$, the following inequality holds:*

$$\begin{gathered}|u(z^1) - u(z^2)| \le c(n) \sup_{\substack{\xi \in Q_R(z^0)\\ \rho \le R}} \left(\frac{1}{\rho^{n+2+2\beta}}\int_{Q_\rho(\xi)} |u - u^0_{\xi,\rho}|^2 dz\right)^{1/2} \delta(z^1, z^2)^\beta, \\ \forall\, z^1, z^2 \in \overline{Q_R(z^0)}, \quad u^0_{\xi,\rho} = ⨍_{Q_\rho(\xi)} u\, dz.\end{gathered} \tag{44}$$

REMARK 4. From (23), (44) it follows that for the solution $u \in \mathcal{K}\{[0, T)\}$ under study the estimate

$$\omega_R^2(z^0) \equiv \left(\operatorname*{osc}_{Q_R(z^0)} u\right)^2 \le c_0 \left(\sup_{\substack{\xi \in Q_R(z^0)\\ \rho \le 2R}} \psi(\rho, \xi)\right) R^{2\beta} \tag{45}$$

is valid if $x^0 \in \overline{\Omega}$, $t^0 \in \left[\frac{T}{2}, T\right)$, $2R < \sqrt{\frac{T}{2}}$, $c_0 = c_0(n, \mu)$.

PROOF OF THEOREM 1. Now we put

$$\varepsilon_0 = \frac{\omega_1^2}{8c_0\mathbb{K}_1}, \qquad R_0 = \min\left\{1, R_1, \frac{\omega_1}{\sqrt{\mathbb{K}_1 c_0}}\right\} \tag{46}$$

in assumption (11) of Theorem 1.

In (46) and below, $\omega_1$ and $R_1$ are the constants from Lemma 2, $\mathbb{K}_1$ is the constant in (30), and $c_0$ is given in (45). (We may assume that $c_0$, $\mathbb{K}_1 \geq 1$.)

For a fixed $\tau \in (0, \frac{T}{4})$ we put $Q(\tau) = \Omega \times (\frac{T}{2}, T - \tau), Q_1 = \Omega \times (\frac{T}{2}, \frac{3T}{4})$. Since $u \in \mathbb{C}(\overline{\Omega} \times [0, T - \tau])$, we may fix

$$R_* = \max \left\{ R \leq \frac{\sqrt{T}}{2} \,\middle|\, \sup_{z \in \overline{Q}_1} \omega_R(z) \leq \omega_1 \right\}, \tag{47}$$

$$\hat{R}(\tau) = \max \left\{ R \leq \frac{\sqrt{T}}{2} \,\middle|\, \sup_{z \in \overline{Q(\tau)}} \omega_R(z) \leq \omega_1 \right\}. \tag{48}$$

If $u$ loses smoothness as $t$ tends to $T$, then $\hat{R}(\tau) \underset{\tau \to 0}{\to} 0$. We prove in the sequel that this is impossible provided that condition (11) holds with chosen $\varepsilon_0 > 0$.

Let us assume that

$$\hat{R}(\tau) < R_2 = \frac{1}{4} \min\{R_0, R_*\}, \tag{49}$$

and fix $R = 2\hat{R}(\tau)$. By the definition of $\hat{R}(\tau)$, there exists an element $z^* \in \overline{Q(\tau)}$ such that the inequality $\omega_1 < \omega_R(z^*)$ holds and

$$\omega_1^2 < \omega_R^2(z^*) \underset{(45)}{\leq} c_0 \left( \sup_{\substack{\xi \in Q_R(z^*) \\ \rho \leq 2R}} \psi(\rho, \xi) \right) R^{2\beta}.$$

First, we suppose that

$$\sup_{\substack{\xi \in Q_R(z^*) \\ \rho \leq 2R}} \psi(\rho, \xi) = \psi(\hat{r}, \hat{\xi}),$$

where $\hat{r} = (\hat{R}, 2R]$, $\hat{\xi} \in \overline{Q_R(z^*)}$. Then $\frac{1}{\hat{r}} < \frac{2}{R}$ and

$$\omega_1^2 < c_0 \left( \frac{2}{R} \right)^{2\beta} \frac{1}{\hat{r}^n} \int_{Q_{\hat{r}}(\hat{\xi})} |u_x|^2 dz \cdot R^{2\beta} \underset{(11)}{\leq} 4c_0\varepsilon_0 \underset{(46)}{<} \frac{\omega_1^2}{2}.$$

This leads to a contradiction.

It means that

$$\sup_{\substack{\xi \in Q_R(z^*) \\ \rho \leq 2R}} \psi(\rho, \xi) = \sup_{\substack{\xi \in Q_R(z^*) \\ \rho \leq \hat{R}(\tau)}} \psi(\rho, \xi). \tag{50}$$

There are two possibilities: $z^* \in \overline{Q_1}$ and $z^* \in \overline{Q(\tau)} \setminus \overline{Q_1}$.

If $z^* \in \overline{Q_1}$ then $Q_{\hat{R}(\tau)}(\xi) \subset Q_{3\hat{R}(\tau)}(z^*) \overset{(49)}{\subset} Q_{R_*}(z^*)$ for any $\xi \in Q_R(z^*)$, and $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_{R_*}(z^*) \leq \omega_1$.

By Lemma 2,

$$\psi(\rho, \xi) \leq \mathbb{K}_1[\psi(\hat{R}(\tau), \xi) + \hat{R}^{2(2-\beta)}], \ \ \rho \leq \hat{R}(\tau), \ \ \xi \in Q_R(z^*)\,. \tag{51}$$

If $z^* \in \overline{Q(\tau)} \setminus \overline{Q_1}$ then $\xi \in \overline{Q(\tau)}$ for $\xi \in Q_R(z^*)$, and by definition (48), $\omega_{\hat{R}(\tau)}(\xi) \leq \omega_1$. We can apply Lemma 2 to set (51).

In any case, due to (50) and (51), we arrive at the inequalities:

$$\begin{aligned}\omega_1^2 < \omega_R^2(z^*) &\leq c_0\mathbb{K}_1 \sup_{\xi \in Q_R(z^*)} [\psi(\hat{R}(\tau), \xi) + \hat{R}^{2(2-\beta)}](2\hat{R})^{2\beta} \\ &\underset{(11),(46)}{\leq} 4\mathbb{K}_1 c_0 \varepsilon_0 + 4\mathbb{K}_1 c_0 \hat{R}^4 \underset{(46),(49)}{\leq} \frac{\omega_1^2}{2} + \frac{\omega_1^2}{4} < \omega_1^2\,.\end{aligned}$$

As a result, under the assumptions of the theorem with $R_0, \varepsilon_0$ chosen, we have a contradiction to inequality (48) and thus, we claim that

$$\hat{R}(\tau) \geq R_2 = \frac{1}{4}\min\{R_0, R_*\}, \quad \tau \in \left(0, \frac{T}{4}\right)\,.$$

This shows that

$$\omega_R(z^0) \equiv \operatorname*{osc}_{Q_R(z^0)} u \leq \omega_1 \quad \text{for any} \quad R \leq R_2,\ z^0 \in \overline{\Omega} \times \left[\frac{T}{2}, T\right)\,.$$

Now, by Lemma 2, we have the inequality

$$\psi(\rho, z^0) \leq \mathbb{K}_1\big\{\psi(R_2, z^0) + R_2^{2(2-\beta)}\big\} \text{ for any } \quad \rho \leq R_2,\ \ z^0 \in \overline{\Omega} \times \left[\frac{T}{2}, T\right)\,,$$

where

$$\psi(R_2, z^0) \leq \frac{1}{R_2^{n-2+2\beta}} \sup_{[0,T]} \|u_x(t)\|_{2,\Omega}^2 \overset{(17)}{\leq} \frac{c}{R_2^{n-2+2\beta}} \|\varphi_x\|_{2,\Omega}^2\,.$$

Hence

$$\sup_{\substack{\rho \leq R_2 \\ z^0 \in \bar{\Omega}\times[T/2,T)}} \psi(\rho, z^0) \leq \mathbb{K}_2, \tag{52}$$

where $\mathbb{K}_2$ depends on $R_2^{-1}$, $\|\varphi_x\|_{2,\Omega}$ and on the same parameters as $\mathbb{K}_1$ in (30). From (44), (23) and (52) we derive the estimate

$$\sup_{\substack{x,y \in \bar{\Omega} \\ t,\tau \in [T/2,T)}} |u(x,t) - u(y,\tau)| \leq \mathbb{K}_3\big(|x-y|^\beta + |t-\tau|^{\beta/2}\big)\,, \tag{53}$$

whence $u \in \mathbb{C}^\beta(\overline{Q}; \delta)$ and

$$\|u\|_{\mathbb{C}^\beta(\overline{Q};\delta)} \leq \mathbb{K}_4, \quad \beta \in (0,1)\,. \tag{54}$$

Now, analyzing the proof of Lemma 2, with the help of estimates (52), (54) it is not difficult to deduce (in the local setting) the following estimate for a solution $v$ of (24):

$$\sup_{\xi^0 \in B_1^+ \times [T/2,T)} \sup_{\rho \leq R_0} \frac{1}{\rho^{n+2+2\gamma}} \int_{\hat{Q}_\rho(\xi^0)} |v_y - (v_y)_{\rho,\xi^0}|^2 d\xi \leq \mathbb{K}_5 \tag{55}$$

with some $R_0 > 0$ for any $\gamma \in (0,1)$.

Inequality (55) implies the estimate

$$\|u_x\|_{C^\gamma(\overline{Q};\delta)} \leq \mathbb{K}_8 \tag{56}$$

for the solution $u$.

Now by (54) and (56), we may regard our problem as a linear one, and we conclude that $u \in \mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$, $u_{xt} \in \mathcal{L}^{2,n+2\alpha}(Q;\delta)$ (see Lemma 7 in [4]). Thus, $u \in \mathcal{K}_\alpha\{[0,T]\}$ and Theorem 1 is proved.

## 4. – The singular set of u. The proof of Theorem 2

We start with the following remark.

REMARK 5. Let condition (11) hold for $x^0 \in \overline{\Omega}$, $\rho \leq R_0/2$, and $t^0 \in [\frac{T}{2}, T)$. Then

$$\sup_{\Lambda_\rho(t^0)} \fint_{\Omega_\rho(x^0)} |u_x(x,t)|^2 dx \overset{(22)}{\leq} \frac{c}{\rho^n} \int_{Q_{2\rho}(z^0)} |u_x|^2 dz \overset{(11)}{<} c\varepsilon_0 \equiv \varepsilon_1\,, \tag{57}$$

$c = c(\nu, \mu)$.

Obviously, relation $\sup_{\Lambda_\rho(t^0)} \fint_{\Omega_\rho(x^0)} |u_x|^2 dx < \varepsilon_0$, $\rho \leq R_0$, implies the inequality

$$\fint_{Q_\rho(z^0)} |u_x|^2 dz < \varepsilon_0, \quad \rho \leq R_0\,.$$

Consequently, the "smallness" condition (11) is equivalent to the inequality

$$\sup_{\Lambda_\rho(t^0)} \fint_{\Omega_\rho(x^0)} |u_x(x,t)|^2 dx < \varepsilon_1, \quad \rho \leq R_1\,, \tag{58}$$

$x^0 \in \overline{\Omega}$, $t^0 \in [\frac{T}{2}, T)$, for some $\varepsilon_1, R_1 > 0$. Thus, Theorem 1 is valid under condition (58).

Now assume that $T > 0$ determines a maximal interval of existence of a smooth solution $u$ of (1), (7), (8). The existence of such an interval $[0, T)$ follows from the known classical solvability results (see [1] and [8]). Theorem 1 and Remark 5 yield a description of the singular set $\Sigma = \sigma \times \{T\}$ of the solution $u$:

$$\sigma = \left\{ \hat{x} \in \overline{\Omega} : \overline{\lim_{t \nearrow T}} \fint_{\Omega_\rho(\hat{x})} |u_x(x,t)|^2 dx \geq \varepsilon_1, \text{ for a sequence } \rho \to 0 \right\}. \tag{59}$$

Thus, for any $\hat{x} \in \sigma$ and some fixed $\rho > 0$ there exists a sequence of $\{t^k\}$, $t^k \nearrow T$, such that

$$\fint_{\Omega_\rho(\hat{x})} |u_x(x,t^k)|^2 dx \geq \frac{\varepsilon_1}{2} \quad \text{for any} \quad k \geq k_0 , \tag{60}$$

with certain number $k_0 \in \mathbb{N}$.

For a fixed number $\eta > 0$, there exist sequences of $x^j \in \sigma$ and $r_j = r(x^j) < \eta$, (we fix $r_j$ in the way that $r_j/2$ belongs to the sequence of $\{\rho\}$ in (59)), such that

a) $B_{r_j}(x^j) \cap B_{r_i}(x^i) = \emptyset, \quad i \neq j$,

b) $\sigma \subset \bigcup\limits_i B_{3r_i}(x^i)$,

(see, for example, [7], Ch.IV, Lemma 2.1).

Now we fix a number $p \in \mathbb{N}$ and points $x^1, \dots, x^p \in \sigma$. Let $\hat{r}_p = \min_{j \leq p} r_j$, and $\hat{t} = T - \hat{r}_p^2$. Note that $T - \hat{t} \leq r_j^2$ for any $j \leq p$.

From (60) with $\rho = r_j/2$, $t^j(\rho) > \hat{t}$, we have the estimate

$$\fint_{\Omega_{r_j/2}(x^j)} |u_x(x,t^j)|^2 dx \geq \frac{\varepsilon_1}{2} . \tag{61}$$

Local energy estimate (14) with $R = r_j/2$, $t_1 = \hat{t}$, and estimate (61) imply the inequalitites

$$\begin{aligned} \frac{\varepsilon_1}{2} \left( \frac{r_j}{2} \right)^{n-2} &\leq \int_{\Omega_{r_j/2}(x^j)} |u_x(x,t^j)|^2 dx \leq c_1 \int_{\Omega_{r_j}(x^j)} |u_x(x,\hat{t})|^2 dx \\ &+ \frac{c_2}{r_j^2} \int_{\hat{t}}^T \int_{\Omega_{r_j}(x^j)} |u_x(x,\tau)|^2 dx\, d\tau . \end{aligned} \tag{62}$$

From (62) and (17) it follows that

$$\sum_{j=1}^p r_j^{n-2} \leq \frac{c(\nu,\mu)}{\varepsilon_1} \|\varphi_x\|^2_{2,\Omega} \equiv E_1 .$$

Since $E_1$ does not depend on $p$, we obtain the estimate

(63) $$\sum_{j=1}^{\infty} r_j^{n-2} \leq E_1 \,.$$

By the definition of the Hausdorff measure and property b) of the sequences $x^j$, $r_j$, from (63) we conclude that

(64) $$H_{n-2}(\sigma) \leq c(n) E_1 \,.$$

Since $\sigma$ is closed and all considerations in the proof of Theorem 1 are of local nature, one may state that $u$ is a smooth function up to the set $(\overline{\Omega} \setminus \sigma) \times \{T\}$. Theorem 2 is proved.

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