

Strong Boundary Values: Independence of the Defining Function and Spaces of Test Functions

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Abstract. The notion of “strong boundary values” was introduced by the authors in the local theory of hyperfunction boundary values (boundary values of functions with unrestricted growth, not necessarily solutions of a PDE). In this paper two points are clarified, at least in the global setting (compact boundaries): independence with respect to the defining function that defines the boundary, and the spaces of test functions to be used. The proofs rely crucially on simple results in spectral asymptotics.

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1. – Introduction

Consider a compact real-analytic manifold \mathcal{M} of dimension d endowed with an analytic Riemannian metric g . We shall denote by \mathbf{dx} the element of volume on \mathcal{M} induced from the metric g . In addition to \mathcal{M} , introduce the product manifold $\mathcal{M}_x = \mathbb{R} \times \mathcal{M}$. It will be convenient to denote by \mathcal{M}_x^+ the half space $(0, \infty) \times \mathcal{M}$ and by \mathcal{M}_x^- the half space $(-\infty, 0) \times \mathcal{M}$.

We denote by $\mathcal{A}(\mathcal{M})$ the space of real-analytic real-valued functions on \mathcal{M} .

The problem we are interested in is the following. Fix a function $U \in \mathcal{C}(\mathcal{M}_x^-)$.

DEFINITION 1. The function U has *strong boundary values* on \mathcal{M} if for each function V defined and real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$ in \mathcal{M}_x , the function ψ_V defined for small $t < 0$ by

$$(1) \quad \psi_V(t) = \int_{\mathcal{M}} U(t, x) V(t, x) \mathbf{dx}$$

extends to be real-analytic in an interval about $0 \in \mathbb{R}$.

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This notion was introduced and studied in some detail in the Memoir [6]⁽¹⁾. It turned out to be very useful in the local theory of hyperfunction boundary values, but this we do not discuss here.

The condition that ψ_V extend to be real-analytic in an interval about $0 \in \mathbb{R}$ is equivalent to the condition that it extend to be holomorphic in a neighborhood of $0 \in \mathbb{C}$.

When the strong boundary value of U exists, the functional $V \mapsto \lim_{t \rightarrow 0^-} \psi_V(t)$ defines an analytic functional, which is an element of the dual space $\mathcal{A}'(\mathcal{M})$.

In the above definition \mathcal{M} , identified with $\{0\} \times \mathcal{M} \subset \mathcal{M}_\times$, is in a suitable sense approximated by $\{t\} \times \mathcal{M}$ ($t < 0$), i.e., by the level sets of the defining function t , that defines \mathcal{M} in \mathcal{M}_\times by the equation $t = 0$. In [6] it was not clear that the notion of strong boundary value is independent of the approximating hypersurfaces, or, equivalently, of the defining function for the boundary. This is of course a very natural question of general type, which arises in various settings, e.g., in the theory of Hardy spaces. More precisely, the following issue was not dealt with. Let R be a function defined, real-valued, and real-analytic on a neighborhood of $\mathcal{M} = \{0\} \times \mathcal{M}$ in \mathcal{M}_\times . Assume that $R(0, x) = 0$ and that $\frac{\partial R(0, x)}{\partial t} > 0$ so that $R(t, x) < 0$ when $t < 0$ and $R(t, x) > 0$ when $t > 0$, provided in both cases that t is sufficiently small. Does the existence of the strong boundary value of U imply that the function ψ_V^R defined by

$$(2) \quad \psi_V^R(t) = \int_{\mathcal{M}} U(R(t, x), x) V(R(t, x), x) \mathbf{d}\mathbf{x},$$

which is defined in an interval $(t_0, 0)$ for sufficiently small, negative t_0 , extends to be analytic in an interval about the point 0?

It is shown below that this implication is correct. Moreover, the value of the extended function at the origin is independent of the choice of R . This is the main result of the paper.

In an attempt to understand what might be the notion of boundary values in the smooth, as opposed to real-analytic, setting we consider in Section 4 of the paper an example that shows there to be no naive analogue of our main result in the smooth setting.

In Section 5 we show that to establish the existence of strong boundary values it suffices to use test functions from certain spaces of real-analytic functions smaller than the space of all functions real-analytic near \mathcal{M} in \mathcal{M}_\times . These results are in the spirit of Proposition 3.8 of [6].

In Section 6 we give some examples of functions with strong boundary values, functions that are shown to satisfy no noncharacteristic partial differential equation.

Finally, we note that the local version of our main result remains open. In [6] a local notion of strong boundary value is introduced. Whether it is independent of the defining function for the domain seems to be a considerably more subtle question than the corresponding question in the global (compact) case, which is settled by our main theorem.

⁽¹⁾In fact, in [6] the compact \mathcal{M} was always the boundary of a bounded domain in \mathbb{R}^N .

2. – Preliminaries

The mechanism for our development is the theory of the Laplace-Beltrami operator on \mathcal{M} . A good general reference is [2]. Instead of the Laplace-Beltrami operator, we could as well use any positive second order elliptic operator with real-analytic coefficients, with no real change in the proof.

Denote by Δ the Laplace-Beltrami operator on \mathcal{M} determined by the metric g . Thus, if in local coordinates the metric g has the expression $\sum_{1 \leq i, j \leq d} g_{i,j} dx_i \otimes dx_j$, then with $(g^{i,j})$ the inverse of the matrix $(g_{i,j})$ and with $\bar{g} = \det(g_{i,j})$, the differential operator Δ is given by $\Delta(u) = -\frac{1}{\sqrt{\bar{g}}} \sum_{1 \leq i, j \leq d} \frac{\partial}{\partial x_i} (g^{i,j} \sqrt{\bar{g}} \frac{\partial u}{\partial x_j})$. It is a second order elliptic operator the leading terms of which are $-\sum_{1 \leq i, j \leq d} g^{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$.

Let the eigenvalues of Δ be $0 = \lambda_1 \leq \lambda_2 \leq \dots$. For each eigenvalue λ_k , let ϕ_k be a corresponding eigenfunction. The ϕ_k 's are real-analytic. They may be chosen so that $\{\phi_k\}_{k=1, \dots}$ is a complete orthonormal basis in the real space $L^2(\mathbf{dx})$. On the real-analytic manifold \mathcal{M}_\times introduce the elliptic operator $\Delta_\times = \Delta - \frac{\partial^2}{\partial t^2}$. This is the Laplace-Beltrami operator for the manifold \mathcal{M}_\times endowed with the metric $g_\times = dt \otimes dt + g$ in which we take t to be the usual parameter on the factor \mathbb{R} of \mathcal{M}_\times . Observe that the functions $\Phi_k^\pm \in \mathcal{A}(\mathcal{M}_\times)$ given by $\Phi_k^\pm(t, x) = e^{\pm \sqrt{\lambda_k} t} \phi_k(x)$ satisfy $\Delta_\times(\Phi_k^\pm) = 0$; they are harmonic on \mathcal{M}_\times .

Let $\mathcal{H} \subset \mathcal{A}(\mathcal{M}_\times)$ be the linear span of the functions Φ_k^+ and Φ_k^- , $k = 1, 2, \dots$.

For $R > 0$ denote by $A(R)$ the annular region $\{(t, x) : -R < t < R\}$ in \mathcal{M}_\times . Denote by $\|\cdot\|_{2,R}$ the L^2 norm on \mathcal{H} given by

$$\|H\|_{2,R} = \left(\int_{A(R)} |H(t, x)|^2 dt \mathbf{dx} \right)^{\frac{1}{2}}.$$

DEFINITION 2. $\mathcal{H}_{2,R}$ is the Hilbert space which is closure of \mathcal{H} with respect to the norm $\|\cdot\|_{2,R}$.

The space $\mathcal{H}_{2,R}$ will provide us with a convenient Hilbert space of functions. The following lemma summarizes what we need from standard elliptic theory.

LEMMA 3. $\mathcal{H}_{2,R}$ is the space of square integrable harmonic functions on $A(R)$. Every function in $\mathcal{H}_{2,R}$ is real-analytic on $A(R)$. If K is a compact subset of $A(R)$, there is a constant c_K such that for each $H \in \mathcal{H}_{2,R}$

$$|H(p)| \leq c_K \|H\|_{2,R}$$

for every $p \in K$.

PROOF. If $\{f_j\}$ is a sequence of harmonic functions on $A(R)$, which is the case if $f_j \in \mathcal{H}_{2,R}$, then L^2 convergence on $A(R)$ implies convergence in any Sobolev norm (hence C^∞ convergence) on compact sets. (See e.g., [3] Chapter 6 Section C.) So if this sequence has a limit, its limit is a smooth function that is harmonic, with a bound on K , as desired.

By the theory of elliptic operators with real-analytic coefficients –see [4] 7.5, every harmonic function on $A(R)$ is real-analytic. Although it is an easy Hilbert space exercise, we omit the proof that $\mathcal{H}_{2,R}$ is the space of all square integrable harmonic function on $A(R)$. We will not need it.

Notice that given a function $h \in \mathcal{A}(\mathcal{M})$, there is an $H \in \mathcal{H}_{2,R}$ for some $R > 0$ with $H|_{\mathcal{M}} = h$, as follows from the Cauchy-Kovalevsky theorem.

Given V real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$ in \mathcal{M}_\times , there is an expansion

$$(3) \quad V(t, x) = \sum_{k=1, \dots} v_k(t) \phi_k(x)$$

in which the coefficients are the Fourier coefficients

$$(4) \quad v_k(t) = \int_{\mathcal{M}} V(t, x) \phi_k(x) \mathbf{d}\mathbf{x}.$$

Similarly, given $U \in \mathcal{C}(\mathcal{M}_\times^-)$ there is an expansion

$$(5) \quad U(t, x) = \sum_{k=1, \dots} u_k(t) \phi_k(x).$$

Again, for fixed $t < 0$, the coefficients $u_k(t)$ are the Fourier coefficients. The expansions (3) and (5) converge at least in the sense of the $L^2(\mathbf{d}\mathbf{x})$ norm for a fixed value of t .

Considerably more is true of the series (3). First of all, the coefficients all extend, with a uniform bound, as holomorphic functions into a fixed neighborhood $\mathbb{U}(\eta)$ of the origin in \mathbb{C} . This is so, for as the function V is real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$, it extends to be holomorphic in a neighborhood of $\{0\} \times \mathcal{M}$ in the complexification of $\mathbb{R} \times \mathcal{M}$. These extensions are given by the integral formula (4), which show the uniform boundedness of the v_k in some disc centered at the origin.

These holomorphic functions admit exponential bounds. For suitable τ_0 , $\eta_0 > 0$, there is the bound that, uniformly in k

$$(6) \quad |v_k(\zeta)| \leq \text{const } e^{-\sqrt{\lambda_k} \tau_0} \quad \text{for all } \zeta \in \mathbb{U}(\eta_1).$$

To see this, take $\tau_0, \eta_0 > 0$ small enough that for every $s \in (-2\eta_0, 2\eta_0)$, there is a function \tilde{V}_s defined on the annular domain $\{(t, x) \in \mathcal{M}_\times : |t - s| < 2\tau_0\}$ that satisfies there the equation $\Delta_\times \tilde{V}_s = 0$ and also the conditions that

$\tilde{V}_s(s, x) = V(s, x)$ and that $\frac{\partial \tilde{V}}{\partial t}(s, x) = 0$. For a fixed constant L , $|\tilde{V}_s(t, x)| < L$ for all choices of $s \in (-\eta_0, \eta_0)$ and for all (t, x) with $|t - s| \leq \tau_0$.

The function \tilde{V}_s admits an expansion

$$\tilde{V}_s(t, x) = \sum_{k=1, \dots} \tilde{v}_{s,k}(t) \phi_k(x)$$

in which the coefficients $\tilde{v}_{s,k}$ satisfy the equation

$$\tilde{v}_{s,k}''(t) = \lambda_k \tilde{v}_{s,k}(t)$$

subject to the initial conditions that $\tilde{v}_{s,k}'(s) = 0$ and $\tilde{v}_{s,k}(s) = v_k(s)$. The solution of this problem is

$$\tilde{v}_{s,k}(t) = v_k(s) \cosh \sqrt{\lambda_k}(t - s)$$

The coefficients $\tilde{v}_{s,k}(t)$ are given by integrals

$$\tilde{v}_{s,k}(t) = \int_{\mathcal{M}} V_s(t, x) \phi_k(x) \mathbf{d}\mathbf{x}$$

and so are uniformly bounded. Consequently, there is an estimate that

$$|v_k(s)| \leq \text{const} [\cosh \sqrt{\lambda_k}(t - s)]^{-1} \quad \text{for } t \text{ with } |t - s| \leq \tau_0.$$

Apply this inequality with $t = s + \tau_0$ to get, as desired,

$$(7) \quad |v_k(s)| \leq \text{const } e^{-\sqrt{\lambda_k} \tau_0}$$

when $|s| \leq \eta_0$. So far s is real. Our situation is the following: The coefficients $v_k(t)$ are holomorphic and uniformly bounded in a disc $\mathbb{U}(\eta)$, and on the interval $-\eta_0 < t < \eta_0$ they have the bounds (7). It follows that on a smaller disc $U(\eta_1)$ they admit the desired bounds

$$(8) \quad |v_k(\zeta)| < \text{const } e^{-\sqrt{\lambda_k} \tau_1}$$

for a suitable $\tau_1 \in (0, \tau_0)$.

In terms of the expansions (3) and (5), the integral that defines $\psi_V(t)$, for $t < 0$, has the expression

$$(9) \quad \psi_V(t) = \sum_{k=1, \dots} u_k(t) v_k(t),$$

which, for fixed t , converges.

Before we can deal with the main question mentioned above, some preliminaries are in order.

PROPOSITION 4. *If $U \in \mathcal{C}(\mathcal{M}_x^-)$, then the limit $\lim_{t \rightarrow 0^-} \psi_V(t)$ exists for all $V \in \bigcup \{\mathcal{H}_{2,R} : R > 0\}$ if and only if*

- a) *for each $k = 1, \dots$ the limit $\lim_{t \rightarrow 0^-} u_k(t)$ exists, and*
- b) *for every $r > 0$, there is a constant C_r large enough that for all t in an interval $(t_0, 0)$ to the left of 0, and for all $k = 1, \dots$*

$$(10) \quad |u_k(t)| \leq C_r e^{\sqrt{\lambda_k} r}.$$

PROOF. If $\lim_{t \rightarrow 0^-} \psi_V(t)$ exists for all $V \in \mathcal{H}_{2,R}$, then it exists in particular when $V = \Phi_k^+$. With this choice of V , the existence of the limit $\lim_{t \rightarrow 0^-} u_k(t)$ follows.

For b) a little more is required. For each $t \in (-R, 0)$, the functional

$$H \mapsto \int_{\mathcal{M}} U(t, x) H(t, x) \mathbf{d}\mathbf{x}$$

is a bounded linear functional on $\mathcal{H}_{2,R}$, as follows from Lemma 3. As the limit

$$\lim_{t \rightarrow 0^-} \int_{\mathcal{M}} U(t, x) H(t, x) \mathbf{d}\mathbf{x}$$

exists for all h , the Banach-Steinhaus theorem implies that this limit defines a continuous linear functional on $\mathcal{H}_{2,R}$ and that the functionals $v \mapsto \psi_V(t)$ for $-\frac{R}{2} \leq t < 0$ are equicontinuous: There is a constant C_R such that for all $H \in \mathcal{H}_{2,R}$ and for all $t \in [-\frac{R}{2}, 0)$

$$\left| \int_{\mathcal{M}} U(t, x) H(t, x) \mathbf{d}\mathbf{x} \right| \leq C_R \|H\|_{2,R}.$$

If we apply this with $H = \Phi_k^+$, we reach the inequality

$$|e^{\sqrt{\lambda_k} t} u_k(t)| \leq C_R \left\{ \int_{A(R)} |e^{\sqrt{\lambda_k} t} \phi_k(x)|^2 dt \mathbf{d}\mathbf{x} \right\}^{\frac{1}{2}}.$$

The right side is bounded by $C_R e^{\sqrt{\lambda_k} R}$. Take $R = \frac{r}{2}$ and $t_0 = -\frac{r}{4}$.

For the converse, suppose the coefficients $u_k(t)$ to satisfy the conditions a) and b) of the lemma. For $V \in \mathcal{H}_{2,R}$ there is an expansion

$$(11) \quad \begin{aligned} V(t, x) &= \sum_{k=1, \dots} \{ \alpha_k^+ \Phi_k^+(t, x) + \alpha_k^- \Phi_k^-(t, x) \} \\ &= \sum_{k=1, \dots} \{ \alpha_k^+ e^{\sqrt{\lambda_k} t} + \alpha_k^- e^{-\sqrt{\lambda_k} t} \} \phi_k(x). \end{aligned}$$

The α 's are computable in the following way: For $t = 0$ the expansion (11) is the expansion of the function $V(0, \cdot) \in \mathcal{A}(\mathcal{M})$ in terms of the eigenfunctions ϕ_k . Consequently, $|\alpha_k^+ + \alpha_k^-| \leq \text{const} e^{-\sqrt{\lambda_k} r'}$ for some $r' > 0$. Similarly, if we differentiate the series with respect to t and then set $t = 0$, we find that $|\sqrt{\lambda_k}(\alpha_k^+ - \alpha_k^-)| \leq \text{const} e^{-\sqrt{\lambda_k} r''}$ for some $r'' > 0$. These estimates yield that

$$|\alpha_k^+|, |\alpha_k^-| \leq \text{const} e^{-\sqrt{\lambda_k} r_1}$$

for some $r_1 > 0$.

With these estimates in mind, choose an $r_0 \in (0, r_1)$. Condition b) provides a constant C_{r_0} such that all $t \in (t_0, 0)$,

$$|u_k(t)| \leq C_{r_0} e^{\sqrt{\lambda_k} r_0}, \quad k = 1, \dots$$

Then for $t < 0$,

$$(12) \quad \int_{\mathcal{M}} U(t, x) V(t, x) \mathbf{d}\mathbf{x} = \sum_{k=1, \dots} u_k(t) (\alpha_k^+ e^{\sqrt{\lambda_k} t} + \alpha_k^- e^{-\sqrt{\lambda_k} t}).$$

Thus, for $t < 0$,

$$|u_k(t) (\alpha_k^+ e^{\sqrt{\lambda_k} t} + \alpha_k^- e^{-\sqrt{\lambda_k} t})| \leq \text{const} e^{(r_0 - r_1 - t)\sqrt{\lambda_k}}.$$

so that, when $t < 0$ satisfies $r_0 - r_1 - t < 0$, the series on the right of (12) is dominated term-by-term by the series $\sum_{k=1, \dots} e^{(r_0 - r_1 - t)\sqrt{\lambda_k}}$. It is known [2] that $\lambda_k \sim \text{const} k^{\frac{2}{d}}$. (This is Weyl's asymptotic formula; see the appendix. The constant has geometric significance.) It follows that the series in (12) converges uniformly on a sufficiently short interval abutting 0 from the left. As we also have the hypothesis a), it follows that the limit $\lim_{t \rightarrow 0^-} \psi_V(t)$ exists, as we wished to show.

Now we consider the case that U has strong boundary values. The result is this.

PROPOSITION 5. *The conditions a)-e) below are equivalent.*

- a) *For each $V \in \bigcup\{\mathcal{H}_{2,R} : R > 0\}$ the function ψ_V extends to be real-analytic on a neighborhood of $0 \in \mathbb{R}$*
- b) *For each $r > 0$, there is a constant $\varrho(r) > 0$ so small that each of the functions u_k extends to be holomorphic in the disc $\mathbb{U}(\varrho(r)) = \{\zeta \in \mathbb{C} : |\zeta| < \varrho(r)\}$ and satisfies the inequality $|u_k(\zeta)| < C_r e^{\sqrt{\lambda_k} r}$ when $\zeta \in \mathbb{U}(\varrho(r))$.*
- c) *There exist $R_0 > 0$ and $\varrho_0 > 0$ such that each of the functions u_k extends holomorphically into the disc $\mathbb{U}(\varrho_0)$ and satisfies the inequality $|u_k(\zeta)| < C_{R_0} e^{\sqrt{\lambda_k} R_0}$ when $\zeta \in \mathbb{U}(\varrho_0)$.*
- d) *There exists $R > 0$ such that for every $V \in H_{2,R}$, the function ψ_V extends holomorphically to a neighborhood of 0.*
- e) *The function U has strong boundary values.*

PROOF. First, that e) implies a) and a) implies d) is evident.

Next, d) implies c). Under the assumption d), the Baire Category theorem provides a $\varrho > 0$ so small that for each $V \in \mathcal{H}_{2,R}$, the function ψ_V extends as a holomorphic function into the disc $\mathbb{U}(2\varrho)$. We shall use ψ_V to denote this extension.

The Closed Graph theorem applied to the map $V \mapsto \psi_V|_{\mathbb{U}(\varrho)}$ from $\mathcal{H}_{2,R}$ to $H^\infty(\mathbb{U}(\varrho))$ implies the existence of a constant C such that for $V \in \mathcal{H}_{2,R}$,

$$(13) \quad \|\psi_V\|_{\mathbb{U}(\varrho)} \leq C \|V\|_{2,R}.$$

(In this H^∞ denotes the space of bounded holomorphic functions and $\|\cdot\|_{\mathbb{U}(\varrho)}$ the supremum norm.) Apply this with $V = \Phi_k^+$. As

$$\psi_{\Phi_k^+}(s) = \int_{\mathcal{M}} U(s, x) e^{\sqrt{\lambda_k} s} \phi_k(x) \mathbf{d}\mathbf{x} = e^{\sqrt{\lambda_k} s} u_k(s),$$

it follows that

$$\sup_{\zeta \in \mathbb{U}(\varrho)} |e^{\sqrt{\lambda_k} \zeta} u_k(\zeta)| \leq C \|\Phi_k^+\|_{2,R} \leq C e^{\sqrt{\lambda_k} R}.$$

For $\zeta \in \mathbb{U}(\varrho)$, we have $|e^{\sqrt{\lambda_k} \zeta}| \geq e^{-\sqrt{\lambda_k} \varrho}$, so for $\zeta \in \mathbb{U}(\varrho)$

$$|u_k(\zeta)| \leq C e^{\sqrt{\lambda_k}(R+\varrho)}.$$

To conclude, take $R_0 = R + \varrho$ and $\varrho_0 = \varrho$.

Now c) implies b). This is correct because of subharmonicity. Given c), fix a $t_0 \in (\varrho_0, 0)$. Let $\varepsilon \in (t_0, 0)$. There is a constant K_ε such that for each $t \in (t_0, \varepsilon)$ we have $|u_k(t)| \leq K_\varepsilon$ for all $k = 1, \dots$. Indeed, for each such t , we have

$$|u_k(t)| = \left| \int_{\mathcal{M}} U(t, x) \phi_k(x) \mathbf{d}\mathbf{x} \right| \leq \sup_{s \in (t_0, \varepsilon)} \left\{ \int_{\mathcal{M}} |U(s, x)|^2 \mathbf{d}\mathbf{x} \right\}^{\frac{1}{2}}.$$

The subharmonicity of $\log |u_k|$ now implies b) as follows.

Fix ρ_0 as in condition c). For $\epsilon < 0$, let \mathbf{D}_ϵ be the disk with a slit

$$\mathbf{D}_\epsilon = \{\zeta \in \mathbf{C} : |\zeta| < \rho_0, \zeta \notin (-\rho_0, \epsilon]\}.$$

For any given $\alpha > 0$, we can choose $|\epsilon|$ small enough so that the harmonic measure representing 0 on the boundary of \mathbf{D}_ϵ has a mass $> 1 - \alpha$ on $(-\rho_0, \epsilon]$. Since $\log |u_k|$ is subharmonic, for every ζ close enough to 0 we have:

$$\log |u_k(\zeta)| \leq (1 - \alpha) \log K_\epsilon + \alpha (\log C_{R_0} + \sqrt{\lambda_k} R_0).$$

Taking α small enough, b) follows.

Finally, b) implies e). Suppose that the function $U \in \mathcal{C}(\mathcal{M}_\times)$ has the expansion (5) in which the coefficients u_k extend to be holomorphic in the disc $\mathbb{U}(\rho)$ and that there they satisfy the exponential inequalities of b) of the proposition. Let V be an arbitrary function defined and real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$ in \mathcal{M}_\times . It has the expansion (3), and the coefficients extend to be holomorphic in a disc about the origin and admit the bounds (6). For $t < 0$

$$\psi_V(t) = \sum_{k=1, \dots} u_k(t)v_k(t),$$

so the exponential bounds on the u 's and the v 's, and Weyl's asymptotic formula, imply that this series of holomorphic functions converges uniformly on a neighborhood of the origin. Thus, ψ_V extends holomorphically to a neighborhood of the origin, and the proof is complete.

In order to establish our main result, we need the following estimates.

LEMMA 6. *Let $D = \sum_{|\alpha| \leq p} c_\alpha(z) \partial^\alpha$ be a p^{th} order holomorphic differential operator defined on the unit polydisc $\mathbb{U}^N \subset \mathbb{C}^N$, and let $f \in \mathcal{O}(\mathbb{U}^N)$ satisfy $|f| \leq M$. If K is a compact subset of \mathbb{U}^N , then there are bounds on the iterates of D :*

$$(14) \quad \|D^k f\|_K \leq [C(N, K, D)]^k k^{kp} M$$

for a suitable constant $C(N, K, D)$.

PROOF. This lemma is quite crude but sufficient for our purposes. It is a repeated application of the Cauchy inequalities together with a very rough counting estimate.

Fix an $s \in (0, 1)$ small enough that for each $w \in K$, the polydisc $\mathbb{U}^N(w, 2s)$ centered at w and of polyradius $2s$ is contained in \mathbb{U}^N . For all positive integers k we are to establish a bound on $D^k f$ on the set K . By hypothesis, $|f| \leq M$ on \mathbb{U}^N . Let C_0 be a constant such that for each α , $|c_\alpha(z)| \leq C_0$ when $z \in \mathbb{U}^N$.

Let $z \in \mathbb{U}^N(w, 2s - \frac{s}{k})$. There is the estimate that

$$|Df(z)| \leq C_0 v(p) \max_{\alpha} |\partial^\alpha f(z)| \quad (|\alpha| \leq p)$$

if by $v(p)$ we understand the number of summands in the sum that defines the operator D , that is the number of multiindices $\alpha = (\alpha_1, \dots, \alpha_N)$ with $|\alpha| = \alpha_1 + \dots + \alpha_N \leq p$. This number is easily estimated: There is at most one term of order zero, there are at most N terms of first order, there are not more than N^2 terms of second order, \dots , and there are not more than N^p terms of order p . Thus, $v(p) \leq 1 + N + N^2 + \dots + N^p \leq (p + 1)N^p$.

As for the term $\partial^\alpha f(z)$, the Cauchy estimates give that for $z \in \mathbb{U}^N(w, 2s - \frac{s}{k})$

$$|\partial^\alpha f(z)| < \frac{\alpha! M}{\left(\frac{s}{k}\right)^{|\alpha|}} \leq p! k^p s^{-p} M,$$

because $\mathbb{U}^N(z, \frac{s}{k}) \subset \mathbb{U}^N$. Thus, for $z \in \mathbb{U}^N(w, 2s - \frac{s}{k})$,

$$|Df(z)| \leq C_0(p+1)p!N^ps^{-p}k^pM.$$

Apply this with f replaced by Df and the polydisc $\mathbb{U}^N(w, 2s)$ by the polydisc $\mathbb{U}^N(w, 2s - \frac{s}{k})$. On $\mathbb{U}^N(w, 2s - \frac{2s}{k})$ we have

$$|D^2f| \leq (C_0(p+1)p!N^ps^{-p}k^p)^2M.$$

We iterate this process k times to find that in $\mathbb{U}^N(w, s - \frac{s}{k})$ the estimate

$$|D^k f| \leq (C_0(p+1)p!N^ps^{-p})^k k^{kp} M$$

holds. Observe that in this estimate N is the dimension of the ambient space, $(p+1)p!$ is determined by the order of the operator, C_0 is determined by the size of the coefficients of D , and, finally, s is determined by the position of the compact set K within \mathbb{U}^N .

3. – The main result

Finally, consider a function R defined and real-analytic on a neighborhood of \mathcal{M} in \mathcal{M}_\times , say in the domain $A(t_0)$ for some $t_0 > 0$. Concerning R we assume that $R(0, x) = 0$, and that $\frac{\partial R(0, x)}{\partial t} > 0$. Consequently, having fixed R , if we shrink t_0 , we can suppose that $R(t, x) < 0$ when $t \in (-t_0, 0)$ and that $R(t, x) > 0$ when $t \in (0, t_0)$.

Let $U \in \mathcal{C}(\mathcal{M}_\times^-)$, and let the functions ψ_V^R be defined for $t \in (-t_0, 0)$ by the integral (2).

THEOREM 7. *If the function U has strong boundary values, then for every V defined and real-analytic on an neighborhood of $\{0\} \times \mathcal{M}$ in \mathcal{M}_\times , the function ψ_V^R extends to be holomorphic in a disc centered at the origin in the complex plane, and $\psi_V^R(0) = \psi_V(0)$.*

PROOF. The function U has the expansion (5). As U is assumed to have strong boundary values, the coefficients u_k all extend to be holomorphic in a neighborhood of 0 and to satisfy the inequalities in b) of Proposition 5.

The function ψ_V^R is given by

$$\psi_V^R(t) = \sum_{k=1, \dots} \int_{\mathcal{M}} u_k(R(t, x)) \phi_k(x) V(R(t, x), x) \mathbf{d}\mathbf{x}.$$

For $r_1 > 0$, to be fixed later small enough, and for each k , the function u_k is holomorphic in the disc $\mathbb{U}(\varrho(r_1))$ and satisfies in the disc $\mathbb{U}(\varrho(r_1))$ the inequality $|u_k(\zeta)| \leq \text{const } e^{\sqrt{\lambda_k} r_1}$.

Fix a relatively compact neighborhood Ω of \mathcal{M} in its complexification $\mathcal{M}^{\mathbb{C}}$, such that the real analytic functions R and V extend holomorphically to a neighborhood of $\{0\} \times \overline{\Omega}$ in $\mathbb{C} \times \mathcal{M}^{\mathbb{C}}$. Denote their extensions again by R and V .

Since $R(0, x) \equiv 0$, the functions u_k^R defined by

$$u_k^R(t) = \int_{\mathcal{M}} u_k(R(t, x)) \phi_k(x) V(R(t, x), x) \mathbf{d}\mathbf{x}$$

($k \in \mathbb{N}$) extend to be holomorphic in a neighborhood of $0 \in \mathbb{C}$. If η is small enough (depending on r_1), R is holomorphic on $\mathbb{U}(\eta) \times \Omega$ and we shall have $R(\mathbb{U}(\eta) \times \Omega) \subset \mathbb{U}(\varrho(r_1))$, and V will be holomorphic in a neighborhood of the closure of $R(\mathbb{U}(\eta) \times \Omega) \times \Omega$ in $\mathbb{C} \times \mathcal{M}^{\mathbb{C}}$. It is absolutely crucial to notice that Ω does not depend on r_1 . Then the function u_k^R is holomorphic on $\mathbb{U}(\eta)$. For $\zeta \in \mathbb{U}(\eta)$ and $x \in \Omega$, let us set $U_k(\zeta, x) = u_k(R(\zeta, x))$. On $\mathbb{U}(\eta) \times \Omega$, this function satisfies

$$(15) \quad |U_k(\zeta, x)| \leq \text{const } e^{\sqrt{\lambda_k} r_1}.$$

The operator Δ is symmetric, and ϕ_k is an eigenfunction corresponding to the eigenvalue λ_k , so for every positive integer μ

$$(16) \quad \begin{aligned} & \int_{\mathcal{M}} U_k(\zeta, x) \phi_k(x) V(R(t, x), x) \mathbf{d}\mathbf{x} \\ &= \lambda_k^{-\mu} \int_{\mathcal{M}} \Delta^{\mu} (U_k(\zeta, x) V(R(t, x), x)) \phi_k(x) \mathbf{d}\mathbf{x}. \end{aligned}$$

To proceed, we need estimates for the derivatives $\Delta^{\mu} (U_k(\zeta, x) V(R(t, x), x))$.

The operator Δ has real-analytic coefficients, so it extends to a holomorphic differential operator on a neighborhood of \mathcal{M} in its complexification. We can assume that it extends to Ω .

Using a cover of Ω by biholomorphic images of the unit polydisc lets us apply the estimate (14) to this operator (an operator on $\Omega \subset \mathcal{M}^{\mathbb{C}}$) in the equality (16), and simply use for the function $U_k(\zeta, x) V(R(t, x), x)$ the sup norm estimate given by (15) to find that for $\zeta \in \mathbb{U}(\eta)$,

$$(17) \quad |u_k^R(\zeta)| \leq \text{const } \lambda_k^{-\mu} C^{\mu} \mu^{2\mu} e^{\sqrt{\lambda_k} r_1},$$

with C independent of r_1 . This estimate is correct for every choice of k and μ , μ an integer.

In equation (17) take $\mu = \mu_k = \sqrt{\frac{\lambda_k}{4C}} (1 + \vartheta_k)$ with $\vartheta_k \geq 0$ chosen so that μ is the smallest integer greater than or equal $\sqrt{\frac{\lambda_k}{4C}}$. Then

$$\begin{aligned} \lambda_k^{-\mu} C^{\mu} \mu^{2\mu} e^{\sqrt{\lambda_k} r_1} &= 4^{-\mu} (1 + \vartheta_k)^{2\mu} e^{\sqrt{\lambda_k} r_1} \\ &= e^{-\sqrt{\lambda_k} \left(\frac{1 + \vartheta_k}{2\sqrt{C}} \log 4 - \frac{1 + \vartheta_k}{\sqrt{C}} \log(1 + \vartheta_k) - r_1 \right)}. \end{aligned}$$

In this, $\vartheta_k \rightarrow 0$ as $k \rightarrow \infty$, and r_1 can be made arbitrarily small (although Ω has been fixed). As $k \rightarrow \infty$, $\lambda_k \sim \text{const } k^{\frac{2}{d}}$. It follows that the series

$$\sum_{k=1, \dots} \int_{\mathcal{M}} u_k(R(t, x)) \phi_k(x) V(R(t, x), x) \mathbf{d}\mathbf{x}$$

converges uniformly on the disc $\mathbb{U}(\eta)$. The value at $t = 0$ is

$$\sum_{k=1, \dots} \int_{\mathcal{M}} u_k(0) \phi_k(x) V(0, x) \mathbf{d}\mathbf{x} = \psi_V(0).$$

The proof is complete.

4. – An example

Following the definition of strong boundary values for real-analytic boundaries and considering the classical results for functions with polynomial growth, i.e., functions growing slower than some negative power of the distance to the boundary, one would be tempted:

- to allow smooth boundaries and smooth test functions φ , which no longer need be real-analytic,
- but to restrict to functions u with polynomial growth,
- and finally to require that the map $t \mapsto \int_{\mathcal{M}} u(t, x) \varphi(t, x) \mathbf{d}\mathbf{x}$ (notation as in (1)) extend smoothly (no longer holomorphically) to a neighborhood of 0.

Proposition 8, which gives examples with bounded functions, shows that such a simple approach fails completely.

PROPOSITION 8. *Let γ be a real-valued smooth function defined on $\mathbb{R}/2\pi\mathbb{Z}$, i.e., γ is a 2π periodic function on \mathbb{R} , $0 < \gamma \leq 1$. If γ is not constant, there exists a bounded smooth function u defined on $(0, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ such that:*

- a) *For every $\varphi \in C^\infty[[0, 1) \times \mathbb{R}/2\pi\mathbb{Z}]$, the function $t \mapsto \int_{-\pi}^{+\pi} u(t, \theta) \varphi(t, \theta) d\theta$, defined for $0 < t < 1$, extends smoothly at $t = 0$,*
- b) *but $\int_{-\pi}^{+\pi} u(t\gamma(\theta), \theta) d\theta$ has no limit as $t \rightarrow 0^+$.*

The integral in b) is to be considered as the pairing of u with the test function 1 (using the measure $d\theta$), along the curve $\theta \rightarrow (t\gamma(\theta), \theta)$, instead of along the vertical line $\theta \mapsto (t, \theta)$.

We start with a lemma.

LEMMA 9. Let $\Gamma : [0, 1] \rightarrow (0, 1)$ be a smooth nonconstant function. Let J be a closed interval contained in $(0, 1)$ on which $\Gamma' > 0$. Let $\kappa \in \mathbb{N}$ and $\varepsilon > 0$. There exists $v \in C_0^\infty([0, 1] \times (0, 1))$ such that

- a) $0 \leq v \leq 1$, and $v(\Gamma(y), y) \equiv 1$ for $y \in J$.
- b) If Φ is the map from $C^\infty([0, 1] \times [0, 1])$ to $C_0^\infty((0, 1))$ defined by

$$[\Phi(\varphi)](x) = \int_0^1 v(x, y)\varphi(x, y)dy$$

then, with $\|\cdot\|_\kappa$ denoting the respective C^κ norms:

$$\|\Phi(\varphi)\|_\kappa \leq \varepsilon\|\varphi\|_\kappa.$$

PROOF. Let J_1 be a closed interval contained in $(0, 1)$ containing J in its interior and on which $\Gamma' > 0$. Let $\chi \in C_0^\infty(0, 1)$, $0 \leq \chi \leq 1$, be such that

$$\begin{aligned} \chi(x) &\equiv 1 && \text{for } x \in \Gamma(J) \\ \chi &\equiv 0 && \text{off } \Gamma(J_1). \end{aligned}$$

Denote by Γ^{-1} the inverse map of the restriction of Γ to J_1 , $\Gamma^{-1} : \Gamma(J_1) \rightarrow J_1$. Let $\psi \in C_0^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$ with $\psi(0) = 1$ and with $\psi(y) = 0$ when $|y| > \frac{1}{2}$. For $N \in \mathbb{N}$ we set

$$w_N(x, y) = \chi(x)\psi(N(y - \Gamma^{-1}(x)))$$

if $x \in \Gamma(J_1)$, and $w_N(x, y) = 0$ otherwise.

Because of the cut-off function χ , $w_N \in C_0^\infty([0, 1] \times (0, 1))$. We take $v = w_N$, for N a sufficiently large positive integer. The only nontrivial statement in the lemma is the estimate of the C^κ norm. We have to estimate, on $\Gamma(J_1)$, the C^κ norm of the function h given by

$$h(x) = \int w_N(x, y)\varphi(x, y)dy = \int \chi(x)\varphi(x, y)\psi[N(y - \Gamma^{-1}(x))]dy.$$

Set $y' = y - \Gamma^{-1}(x)$:

$$h(x) = \int \chi(x)\varphi(x, y' + \Gamma^{-1}(x))\psi(Ny')dy'.$$

There is a constant C such that for each y' fixed (small) the C^κ norm of the function $x \mapsto \chi(x)\varphi(x, y' + \Gamma^{-1}(x))$ is bounded by $C\|\varphi\|_\kappa$. If ψ_N is defined by $\psi_N(y) = \psi(Ny)$ we then have that

$$\|h\|_\kappa \leq C\|\varphi\|_\kappa\|\psi_N\|_{L^1} = \frac{C}{N}\|\psi\|_{L^1}\|\varphi\|_\kappa.$$

Taking N large enough, so that $\frac{C}{N}\|\psi\|_{L^1} \leq \varepsilon$, yields the lemma.

PROOF OF PROPOSITION 8. Fix a nontrivial closed interval J on which $\gamma' > 0$. Choose sequences $\{a_n\}$, $\{b_n\}$, $\{t_n\}$ tending to 0 such that

$$b_{n+1} < a_n < b_n < a_{n-1},$$

and for every $\theta \in \mathbb{R}/2\pi\mathbb{Z} : a_n < t_n\gamma(\theta) < b_n$. Using a trivial change of variable, Lemma 9 give us for each $n \in \mathbb{N}$ a function $u_n \in C_0^\infty((0, 1) \times \mathbb{R}/2\pi\mathbb{Z})$ such that,

$$\begin{aligned} 0 \leq u_n \leq 1, \quad u_n(t, \theta) &= 0 \quad \text{if } t \notin (a_{2n}, b_{2n}) \\ u_n(t_{2n}\gamma(\theta), \theta) &= 1 \quad \text{if } \theta \in J, \end{aligned}$$

and if we set

$$[\Phi_n(\varphi)](t) = \int_0^1 u_n(t, \theta)\varphi(t, \theta)d\theta,$$

for $\varphi \in C^\infty([0, 1] \times \mathbb{R}/2\pi\mathbb{Z})$, then

$$\|\Phi_n(\varphi)\|_n \leq 2^{-n}\|\varphi\|_n$$

($\|\cdot\|_n$ denoting C^n norms, which satisfy $\|\cdot\|_{n+1} \geq \|\cdot\|_n$).

Finally take $u = \Sigma u_n$. Since the supports are disjoint, $0 \leq u \leq 1$. The map

$$t \mapsto \int_{-\pi}^{+\pi} u(t, \theta)\varphi(t, \theta)d\theta = \sum_n \int_{-\pi}^{+\pi} u_n(t, \theta)\varphi(t, \theta)d\theta$$

extends smoothly to a neighborhood of $t = 0$, for the series converges in every C^k norm. Notice that $\int_{-\pi}^{+\pi} u(t_{2n}\gamma(\theta), \theta)d\theta \geq |J|$, but $\int_{-\pi}^{+\pi} u(t_{2n+1}\gamma(\theta), \theta)d\theta = 0$ since $a_{2n+1} \leq t_{2n+1}\gamma(\theta) \leq b_{2n+1}$ and $u(t, \theta) = 0$ for $a_{2n+1} \leq t \leq b_{2n+1}$.

This ends the proof.

Proposition 8 raises the question of what may be a satisfactory definition of boundary values for smooth boundaries and functions with polynomial growth, in the spirit of our definition of strong boundary values, i.e., independently of any partial differential equation that the function may satisfy. Condition a) in the statement of Proposition 8 was proposed because it seemed to be a reasonable analog of the corresponding condition for real analytic boundary values and functions of unrestricted growth. But it is insufficient.

5. – Spaces of test functions

Resume the notation of the first part of the paper: \mathcal{M} is a real-analytic manifold of dimension d endowed with an analytic Riemannian metric g , \mathbf{dx} is

the element of volume on \mathcal{M} induced from the metric g , \mathcal{M}_\times is the product manifold $\mathcal{M}_\times = \mathbb{R} \times \mathcal{M}$, and \mathcal{M}_\times^+ and \mathcal{M}_\times^- are half spaces in \mathcal{M}_\times .

By definition, the function $U \in \mathcal{C}(\mathcal{M}_\times^-)$ has strong boundary values if the function ψ_V extends to be analytic near 0 for every choice of function V defined and real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$. Here ψ_V is the function given in equation (1). Proposition 5 shows that we need not consider *all* real-analytic functions; it suffices to consider only those that are harmonic near $\{0\} \times \mathcal{M}$. Recall also that in [6], it was shown that if $\Omega \subset \mathbb{R}^N$ is a bounded domain with real analytic boundary and with real-analytic defining function ϱ , then $U \in \mathcal{C}(\Omega)$ has strong boundary values if and only if for every entire function F on \mathbb{C}^N , the function ψ_F given for small $t < 0$ by

$$\psi_F(t) = \int_{\{\varrho=t\}} U(x)F(x)dS(x)$$

extends analytically to a neighborhood of 0.

It is the purpose of the present section to exhibit some other spaces of test functions that suffice to establish the existence of strong boundary values.

The first result shows that it is sufficient on \mathcal{M}_\times^- to consider functions that are independent of t .

PROPOSITION 10. *If $U \in \mathcal{C}(\mathcal{M}_\times^-)$, then the following conditions are equivalent:*

- a) *For every $V \in \mathcal{A}(\mathcal{M})$ the function ψ_V defined for $t < 0$ by*

$$(18) \quad \psi_V(t) = \int_{\mathcal{M}} U(t, x)V(x)d\mathbf{x}$$

continues holomorphically to a neighborhood of $0 \in \mathbb{C}$.

- b) *For every function V real-analytic on a neighborhood of $\{0\} \times \mathcal{M}$ in \mathcal{M}_\times , the function ψ_V defined for sufficiently small $t < 0$ by*

$$(19) \quad \psi_V(t) = \int_{\mathcal{M}} U(t, x)V(t, x)d\mathbf{x}$$

continues holomorphically to a neighborhood of $0 \in \mathbb{C}$.

PROOF. We have to show that a) implies b). Let Ω be a neighborhood of \mathcal{M} in a Stein complexification of \mathcal{M} . By the Baire Category theorem, there exists $\varrho > 0$ such that for every $f \in \mathcal{O}(\Omega)$ the function ψ_f has a holomorphic extension to the disc $\mathbb{U}(\varrho)$. Denote this extension by $\chi(f)$. The map $\chi : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\mathbb{U}(\varrho))$ is linear and is continuous, as follows from the Closed Graph theorem. Fix an $\varepsilon > 0$ and consider functions V that are holomorphic on $\mathbb{U}(\varepsilon) \times \Omega$. For small $t < 0$ and $\zeta \in \mathbb{U}(\varepsilon)$, set

$$\Phi(t, \zeta) = \int_{\mathcal{M}} U(t, x)V(\zeta, x)d\mathbf{x}.$$

For each fixed $\zeta \in \mathbb{U}(\varepsilon)$, the partial map $t \mapsto \Phi(t, \zeta)$ extends holomorphically to $\mathbb{U}(\varrho)$.

This provides an extension of Φ to a function $\tilde{\Phi}$ given by $\tilde{\Phi}(t, \zeta) = [\chi(V_\zeta)](t)$, where V_ζ is the function defined on Ω by $V_\zeta(x) = V(\zeta, x)$.

We will denote by $\partial_\zeta V_\zeta$ the function on Ω defined by $\partial_\zeta V_\zeta(x) = \frac{\partial}{\partial \zeta} V(\zeta, x)$. It follows from the continuity of χ that $\tilde{\Phi}$ is a continuous function on $\mathbf{U}(\rho) \times \mathbf{U}(\varepsilon)$.

The quantity $\tilde{\Phi}(t, \zeta)$ depends holomorphically on t . It is also holomorphic in ζ , for, as $\Delta\zeta$ tends to 0:

$$\frac{\tilde{\Phi}(t, \zeta + \Delta\zeta) - \tilde{\Phi}(t, \zeta)}{\Delta\zeta} = \chi\left(\frac{V_{\zeta+\Delta\zeta} - V_\zeta}{\Delta\zeta}\right)(t) \rightarrow \chi(\partial_\zeta V_\zeta)(t),$$

again by the continuity of χ .

Therefore $\tilde{\Phi}$ is a holomorphic function on $\mathbb{U}(\varrho) \times \mathbb{U}(\varepsilon)$. Thus the map $t \mapsto \Phi(t, t)$ extends holomorphically to the disc of radius $\min\{\varrho, \varepsilon\}$.

The proposition is proved.

The next result is in the direction of Proposition 3.8 of [6].

PROPOSITION 11. *Denote by $\mathcal{M}_\times^{\mathbb{C}}$ a complexification of \mathcal{M}_\times that is a Stein manifold and that has the property that compacta in \mathcal{M}_\times are $\mathcal{O}(\mathcal{M}_\times^{\mathbb{C}})$ -convex. The function $U \in \mathcal{C}(\mathcal{M}_\times)$ has strong boundary values if and only if for each $F \in \mathcal{O}(\mathcal{M}_\times^{\mathbb{C}})$ the function ψ_F given for small $t < 0$ by*

$$(20) \quad \psi_F(t) = \int_{\mathcal{M}} U(t, x) F(t, x) dx$$

extends as an analytic function in a neighborhood of 0.

NOTE. The definition of complexification of a real-analytic manifold is not rigidly fixed in the literature. For our purposes, it is sufficient to take a complexification of \mathcal{M}_\times to be any d -dimensional complex manifold that contains \mathcal{M}_\times as a closed, totally real, real-analytic submanifold. For example, it is not necessary here to require that the complexification admit an antiholomorphic involution that leaves \mathcal{M}_\times fixed pointwise. But note also that if the complexification $\mathcal{M}_\times^{\mathbb{C}}$ of \mathcal{M}_\times does admit such an antiholomorphic involution, and if it is a Stein manifold, then compacta in \mathcal{M}_\times are necessarily $\mathcal{O}(\mathcal{M}_\times^{\mathbb{C}})$ -convex: The existence of the antiholomorphic involution implies that $\mathcal{O}(\mathcal{M}_\times^{\mathbb{C}})$ contains many holomorphic functions that are real-valued on \mathcal{M}_\times . There are enough of these that there is an embedding of $\mathcal{M}_\times^{\mathbb{C}}$ into a suitable \mathbb{C}^M that carries \mathcal{M}_\times bianalytically onto a real-analytic submanifold of \mathbb{R}^M , whence the convexity of compacta in \mathcal{M}_\times with respect to the algebra $\mathcal{O}(\mathcal{M}_\times^{\mathbb{C}})$.

PROOF. Without loss of generality, we can suppose that the complexification $\mathcal{M}_\times^{\mathbb{C}}$ is a closed submanifold of \mathbb{C}^M for some sufficiently large M .

With this arrangement, the proof of Proposition 3.8 in [6] translates immediately into the present setting to yield the desired result.

A second result in the same vein is this:

PROPOSITION 12. *The function $U \in \mathcal{C}(\mathcal{M}_x^-)$ has strong boundary values if and only if for every function F harmonic on \mathcal{M}_x , the function ψ_F defined by the equation (20) extends as an analytic function in a neighborhood of 0.*

NOTE. Proposition 12 implies certain cases of Proposition 11. The reason for this is that there is a fixed complexification, say \mathcal{M}_x^* , into which each function harmonic on \mathcal{M}_x extends as a holomorphic function (and into which the Laplace-Beltrami operator Δ_x extends as a holomorphic differential operator)⁽²⁾.

PROOF. It will be convenient to denote by $\text{Har}(\mathcal{M}_x)$ the space of harmonic functions on \mathcal{M}_x . We assume the function $U \in \mathcal{C}(\mathcal{M}_x^-)$ to have the property that for each function $V \in \text{Har}(\mathcal{M}_x)$ the function ψ_V given by (1) extends to be holomorphic in a neighborhood of the origin. It follows by the Baire Category theorem and the Closed Graph theorem that there is an $\eta_0 > 0$ small enough that for each $V \in \text{Har}(\mathcal{M}_x)$ the function ψ_V extends to be holomorphic in the disc $\mathbb{U}(\eta_0)$ and that the map $\text{Har}(\mathcal{M}_x) \rightarrow \mathcal{O}(\mathbb{U}(\eta_0))$ given by these extensions is continuous.

As the map $V \mapsto \psi_V$ is continuous, for each $\eta \in (0, \eta_0)$ there are constants $R = R(\eta)$ and C_R such that

$$\|\psi_V\|_{\mathbb{U}(\eta)} \leq \|V\|_{2,R} = C_R \left\{ \int_{-R}^R \int_{\mathcal{M}} |V(t, x)|^2 dt d\mathbf{x} \right\}^{\frac{1}{2}}.$$

By definition the space $\text{Har}(\mathcal{M}_x)$ is dense in the space $\mathcal{H}_{2,R}$, so the preceding inequality implies that for each $V \in \mathcal{H}_{2,R}$, the function ψ_V extends holomorphically to the disc $\mathbb{U}(\eta)$.

The theorem is proved.

6. – Some functions with strong boundary values

It was shown in [6] that solutions on \mathcal{M}_x^- of partial differential equations for which $\{0\} \times \mathcal{M}$ is noncharacteristic have strong boundary values. In this section we construct by elementary means examples of functions with strong boundary values that are solutions of no such equations.

⁽²⁾Perhaps the simplest example of this is the usual Laplacian on the unit ball \mathbb{B} in \mathbb{R}^N . The Laplacian $\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ on \mathbb{B} extends to the holomorphic differential operator $\sum_{j=1}^N \frac{\partial^2}{\partial z_j^2}$ on \mathbb{C}^N . Every harmonic function on \mathbb{B} extends uniquely as a holomorphic function on the *Lie ball*, which is the domain $\tilde{\mathbb{B}}$ in \mathbb{C}^N defined by

$$\tilde{\mathbb{B}} = \{x + iy \in \mathbb{C}^N : |x|^2 + |y|^2 + 2\sqrt{|x|^2|y|^2 - (x, y)^2} < 1\}.$$

See [1], p. 59.

We work on the manifold $\mathbb{R} \times \mathbb{T}$ with \mathbb{T} the unit circle in \mathbb{C} . A complexification of this manifold is $\mathcal{N} = \mathbb{C} \times \{\mathbb{C} \setminus \{0\}\}$. The surface $\mathbb{R} \times \mathbb{T}$ has the property that compacta in it are $\mathcal{O}(\mathcal{N})$ -convex.

Let φ be any smooth function defined on \mathbb{T} . Let P be a polynomial in one variable. Set $U(t, e^{i\theta}) = P(t) \varphi(e^{i\theta})$. For every entire function F holomorphic on \mathcal{N} , the map

$$t \mapsto \psi_F(t) = \int_0^{2\pi} U(t, \theta) F(t, \theta) d\theta$$

is holomorphic in t , $t \in \mathbb{C}$, with the obvious estimate, for $|t| \leq 1$:

$$|\psi_F(t)| \leq C_F \sup_{|s| < 1} |P(s)| \sup_{\theta \in \mathbb{R}} |\varphi(e^{i\theta})|.$$

If we consider sequences of functions φ_n and of polynomials P_n as above, and if we take $\epsilon_n > 0$ small enough, then the series

$$\sum_n \epsilon_n P_n(t) \varphi_n(e^{i\theta})$$

will converge uniformly for $t \in \mathbb{C}$, $|t| < 1$, and $\theta \in \mathbb{R}$, to a function denoted by U . In addition, the map

$$t \mapsto \psi_F(t) = \int_0^{2\pi} U(t, e^{i\theta}) F(t, e^{i\theta}) d\theta$$

is holomorphic in t , $|t| < 1$. That is to say, the restriction of U to $\{(t, e^{i\theta}) : t < 0, \theta \in \mathbb{R}\}$ has strong boundary values!

Finally it is easy to select the functions φ_n and the polynomials P_n so that the resulting function F cannot satisfy any partial differential equation, possibly with variable coefficients, of the type

$$Q(u) = \left[\frac{\partial^k}{\partial t^k} + \sum_{p+q \leq k, p < k} a_{p,q} \frac{\partial^{p+q}}{\partial t^p \partial \theta^q} \right] (u) = 0$$

near $t = 0$. (We do not have to restrict to real analytic coefficients.)

Indeed, take the functions φ_n to have disjoint supports and not to vanish identically, and take $P_n = t^n$. If we consider an operator Q of order k as above, then on the support of φ_k , and for $t = 0$ we have $Q(f) = k! \epsilon_k \varphi_k(\theta) \neq 0$.

Appendix on the distribution of eigenvalues. For the reader unfamiliar with Weyl's asymptotic formula we wish to point out that we need to use only very rough results which are almost immediate, as we now indicate.

On \mathcal{M} , equipped with normalized area measure $d\mu$, we denote by $\|\cdot\|_s$ a Sobolev s -norm. So $\|\cdot\|_0$ is simply the L^2 norm. Let P be a positive elliptic linear differential operator of order m , with smooth coefficients. Positive means that for some $c > 0$, $(Pu, u) = \int \bar{u} P u d\mu \geq c \|u\|_0^2$. An example, with $m = 2$, is $\Delta + 1$, with Δ the Laplace-Beltrami operator. By the elementary spectral theory of self adjoint compact operators applied to the inverse of P , there is an

orthonormal basis of $L^2(\mu)$ consisting of eigenfunctions Φ_1, Φ_2, \dots associated to eigenvalues λ_k with $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$, and λ_k tending to ∞ as k tends to ∞ .

By basic elliptic theory, for any positive integer q : $\|u\|_{qm} \leq C_q \|P^q u\|_0$. Take q such that $qm > \frac{d}{2}$, d the dimension of \mathcal{M} . Then the Sobolev Embedding lemma gives $\sup |u| \leq C \|P^q u\|_0$. Therefore if u is in the span of Φ_1, \dots, Φ_k :

$$\sup |u| \leq C \|P^q u\|_0 \leq C \lambda_k^q \|u\|_0.$$

Finally, we use the very elementary fact that if E is a subspace of $L^\infty(\mu)$ (μ a probability measure), such that for every $u \in E$, $\sup |u| \leq M \|u\|_0$ ($\|\cdot\|_0$ the L^2 norm), then E has dimension at most M^2 -see [7] p. 118.

The above yields $k \leq C^2 \lambda_k^{2q}$; so

$$\lambda_k \geq C^{-\frac{1}{q}} k^{\frac{1}{2q}}.$$

For $m = 2$, we have to take $q > \frac{d}{4}$, giving an estimate not so far from the sharp one with $k^{\frac{2}{d}}$ instead of $k^{\frac{1}{2q}}$ (see [5] Chapter XXIX), while an estimate with k^ϵ , for some $\epsilon > 0$, is all we need.

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