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Complexification of proper hamiltonian $G$-spaces


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Complexification of Proper Hamiltonian $G$-Spaces

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Abstract. Let $(M, \tau)$ be a symplectic manifold and let $G$ be a Lie group (with finitely many connected components) acting properly by symplectic diffeomorphisms on $M$. Then there is a proper Stein $G$-manifold $X$ with a $G$-invariant Kähler form $\omega$ and a $G$-equivariant totally real embedding of maximal dimension $i : M \hookrightarrow X$ such that $i^*\omega = \tau$. Additionally, if $\tau$ possesses a moment map, this can be extended to a moment map of $\omega$ on $X$. The Kähler form and moment map are unique up to diffeomorphism around $M$ fixing $M$ pointwise.

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1. – Introduction

Let $(M, \tau)$ be a symplectic manifold and let $G$ be a real Lie group acting properly by symplectic automorphisms on $(M, \tau)$. The goal of this paper is to complexify $(M, \tau, G)$. This is of interest since the symplectic reduction of a complex manifold is itself a complex space. This provides a method for analyzing the symplectic reduction of $M$ via its embedding in the symplectic reduction of the complexification of $M$.

Historically, the starting point for complexifications is Whitney’s classical theorem (see e.g. [Hir76]) stating that any smooth paracompact manifold $M$ possesses a real analytic structure. Grauert [Gra58] proved that there is a Stein complexification $X$ of $M$ in the following sense.

There is a real analytic totally real embedding $i : M \hookrightarrow X$, and an anti-holomorphic involution $\sigma : X \to X$ with $\text{Fix} \sigma = M$ such that the manifold $X$ is Stein. In fact, there is a basis of Stein neighborhoods of $M$. Furthermore, $X$ can be chosen so that $M$ is a strong deformation retract of $X$. A Stein complexification satisfying all of the above conditions will be said to be a Stein tube.

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In this context, after an appropriate shrinking, closed 2-forms on $M$ extend to Kähler forms on $X$:

**Theorem 1.1 ([HHL94]).** Let $M$ be a manifold with a closed 2-form $\tau$. Then there is a Stein tube $i : M \hookrightarrow X$ and a Kähler form $\omega$ on $X$, so that $i^* \omega = \tau$.

Now let $G$ act properly and smoothly on $M$. A Stein $G$-tube is a Stein tube $X$ with a Lie group $G$ acting properly on $X$ by holomorphic transformations so that the embedding $i$, the involution $\sigma$ and the strong deformation retract are $G$-equivariant and each $G$-stable neighborhood of $M$ contains a $G$-stable Stein neighborhood.

**Theorem 1.2 ([Ku94], [HHK95], [He93]).** Each proper $G$-manifold $M$ admits a Stein $G$-tube $X$.

The main goal in this paper is to prove Theorem 1.1 under the presence of a proper $G$-action (Chapter 3):

**Theorem 1.3.** Let $G$ be a (real) Lie group with finitely many components acting properly on a manifold $M$ and let $\tau$ be a closed $G$-invariant 2-form on $M$. Then there is a Stein $G$-tube $i : M \hookrightarrow X$ and a $G$-invariant Kähler form $\omega$ on $X$ with $i^* \omega = \tau$.

If $G$ is compact, then Theorem 1.3 is a consequence of Theorems 1.1 and 1.2 by using the averaging process. The case of a non-compact group requires substantially different techniques.

In Chapter 4 it is proved that even moment maps are extendable, i.e. if $\nu$ is a $G$-moment map of $\tau$ on $M$, then there is a $G$-moment map $\mu$ of $\omega$ on $X$ with $i^* \mu = \nu$. In Chapter 5 it is shown that the construction is canonical up to local $G$-equivariant diffeomorphism around $M$.

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### 2. – Preliminaries

A smooth action of a Lie group $G$ on a manifold or complex space $M$ is said to be proper if the mapping $G \times M \to M \times M$, $(g, x) \mapsto (g \cdot x, x)$ is proper. This can be written in terms of sequences: if $x_n \to x$ and $g_n x_n \to y$, then there exists a subsequence $g_{n_k} \to g \in G$ with $gx = y$. Of course, compact groups always act properly. For proper actions, all isotropy groups $G_x = \{ g \in G \mid g \cdot x = x \}$ ($x \in X$) are compact subgroups of $G$, all orbits $G \cdot x = \{ g \cdot x \mid g \in G \} \subset X$ are closed, and moreover the space of orbits $X/G$...
is Hausdorff. Furthermore, there is a (local) slice $S$ through each point $x$ of a proper $G$-manifold $M$, i.e. a locally closed $G_x$-stable submanifold $S \ni x$ such that

$$G \times_{G_x} S \hookrightarrow M, \quad [g, s] \mapsto g \cdot s$$

is a $G$-equivariant open embedding [Pa61], where the $G$-manifold $G \times_{G_x} S$ is the associated bundle over $G/G_x$ to the $G_x$-principal bundle $G \to G/G_x$. The slice $S$ can be chosen $G_x$-equivariantly isomorphic to an open neighborhood of the origin in a $G_x$-representation space, where $x$ is identified with the origin.

2.1. – The moment map

Let $(M, \omega)$ be a symplectic manifold, $G$ a Lie group acting on $M$ and assume that the symplectic form $\omega$ is $G$-invariant. Every $v \in g := \text{Lie}(G)$ induces a fundamental vector field $\tilde{v}$ on $M$ and the contraction $\iota_{\tilde{v}} \omega$ is a closed 1-form. Assume that $\iota_{\tilde{v}} \omega$ is exact, i.e. there is a function $\mu^v$ on $M$ with $d\mu^v = \iota_{\tilde{v}} \omega$. The functions $\mu^v$ define a map $\mu : M \to g^*$ by $\mu(x)(v) := \mu^v(x)$ for all $x \in M$ and $v \in g$.

**Definition 2.1.** Let $M$ be a $G$-manifold, $\omega$ a closed (not necessarily non-degenerate) $G$-invariant 2-form on $M$ and $\mu : M \to g^*$ a $G$-equivariant map satisfying $\iota_{\tilde{v}} \omega = d\mu^v$. Then $\mu$ is said to be an (equivariant) moment map with respect $\omega$ and the $G$-action. If $\omega$ is symplectic, $(M, G, \omega, \mu)$ is called a Hamiltonian space and the quotient $\mu^{-1}(0)/G$ its symplectic reduction.

If $M$ has a complex structure and the form $\omega$ is a Kähler form arising from a $G$-invariant strictly plurisubharmonic function $\rho$, i.e. $\omega = 2i\partial \bar{\partial} \rho$, there is a moment map, namely

$$\mu^v(x) := d\rho(J\tilde{v}_x) \quad \text{for} \quad v \in g.$$

**Remark.** From the point of view of classical mechanics the components of the moment map, i.e. the functions $\mu^v$, are constants of motion with respect to any $G$-invariant Hamiltonian. So the flow of any such Hamiltonian stays on the simultaneous level set of these constants of motion, i.e. the $\mu$-fibers. The observables on the level set are restrictions of global observables.

2.2. – Moment maps on Kähler manifolds

Let $X$ be a proper Hamiltonian Kähler $G$-manifold with invariant Kähler form $\omega$, moment map $\mu$ and $R := \mu^{-1}(0)$. Let $J$ denote the almost complex structure of $X$. The induced vector field of $v \in g$ on $X$ is denoted by $\tilde{v}$.

**Lemma 2.1.** The moment map has the following properties:

(i) $\ker(d\mu)_x = (T_x(G \cdot x))^{\perp_{\omega}}$ for all $x \in X$.

(ii) For $x \in R$ the tangent space $T_x(G \cdot x)$ to the orbit in $x$ is isotropic, i.e. $\omega|_{T_x(G \cdot x)} = 0$. Furthermore $T_x(G \cdot x) \cap J T_x(G \cdot x) = \{0\}$.

(iii) For $v \in g$ let $\gamma$ be the flow curve of $J\tilde{v}$ with $\gamma(0) = x$. Then the curve $\alpha = \mu^v \circ \gamma$ is strictly increasing in a neighborhood of 0 or $\tilde{v}_x = 0$. 

For the proof confer [GuSt84].

One motivation for complexifying Hamiltonian spaces arises from the fact that the symplectic reduction of a complex proper Hamiltonian space with respect to a proper action is a complex space and henceforth carries a much richer structure. The goal is therefore to understand the structure of the symplectic reduction of the real object via its embedding into the complex one.

Let $X$ be a complex proper $G$-manifold with $G$-invariant Kähler form $\omega$ with moment map $\mu$. Define a sheaf on $R$ by

$$\mathcal{O}_R(U) := \{ f : U \to \mathbb{C} \mid \exists V \subset X \text{ open}, V \cap R = U \text{ and } \exists \tilde{f} \in \mathcal{O}(V), \tilde{f} \mid_U = f \}$$

and the structure sheaf on $R/G$ by

$$\mathcal{O}_{R/G}(U) := \mathcal{O}_R^G(\pi^{-1}(U)).$$

The exponent "$G$" denotes the $G$-invariant functions and $\pi : R \to R/G$ the projection.

**Theorem 2.2.** There is a complex structure on $R/G$ making $(R/G, \mathcal{O}_{R/G})$ a complex space.

A proof is given in [AHH98] (see [Amm97] for the case of semi-simple groups). In the present paper only the case of proper free actions on manifolds will be used:

**Proposition 2.3.** Let $G$ act freely and properly on the Kähler manifold $X$ by holomorphic Kähler isometries. Then the quotient $(R/G, \mathcal{O}_{R/G})$ is in a canonical way a complex manifold and the projection map is holomorphic.

For the proof confer [HH00] or [OrigDiss].

This proposition provides the following

**Lemma 2.4.** Let $X$ be a proper Hamiltonian complex $G$-manifold with $G$ acting freely and $R := \mu^{-1}(0)$. Then to each holomorphic $G$-invariant map $\theta : X \to Y$ there is an induced holomorphic map

$$\theta_{\text{ind}} : R/G \to Y.$$

It is important to observe that the zero moment level possesses a particular geometry. Restrictions of invariant Kähler forms to $R$ induce Kähler forms on the quotient $Y$. For the case of an action of a compact group see e.g. [HHL94]. Using the local normal form for Hamiltonian manifolds there is an induced symplectic form on the quotient $R/G$. For this construction, known as the Marsden-Weinstein reduction, confer [GuSt84].

**Lemma 2.5.** Let $\omega$ be a $G$-invariant Kähler form on the proper complex $G$-manifold $X$ with $G$ acting freely, $\mu$ a moment map and $R := \mu^{-1}(0)$. Then there is a natural Kähler form $\omega_{\text{red}}$ on the symplectic reduction $R/G$. 
PROOF. Let \( i_R : R \hookrightarrow X \) denote the embedding. Set \( Q := TR \cap JTR \) and \( F \) the vector bundle spanned by the \( G \)-vector fields. The Kähler form \( \omega \) respects the bundle splitting \( TR = F \oplus \mathbb{R}Q \), i.e. for all \( \eta_i \in F_x \) and \( \kappa_i \in Q_x \) it follows

\[
\omega_x(\eta_1 + \kappa_1, \eta_2 + \kappa_2) = \omega_x(\kappa_1, \kappa_2).
\]

Since \( \omega \) is \( G \)-invariant and \( Q \) is \( G \)-stable, the complex linear vector space isomorphism \((\pi_\ast)_x : Q_x \to T_{\pi(x)}(R/G)\) induces a positive \((1,1)\)-form \( \omega_{\text{red}} \) on \( R/G \) with \( i_R^\ast \omega = \pi^\ast \omega_{\text{red}} \). Hence \( d\pi^\ast \omega_{\text{red}} = 0 \) and by the surjectivity of \( \pi \) the form \( \omega_{\text{red}} \) is closed and therefore Kählerian. \( \square \)

2.3. – Properties of Stein \( G \)-tubes

As the main object of interest we recall the definition of Stein \( G \)-tubes.

**Definition 2.2.** Let \( G \) act properly on a (real) manifold \( M \). A Stein manifold \( X \) with a proper \( G \)-action and a totally real \( G \)-equivariant embedding \( i : M \hookrightarrow X \) is said to be a Stein \( G \)-tube if

1. there is an anti-holomorphic involution \( \sigma : X \to X \) with \( M = \text{Fix} \sigma \).
2. \( M \) is a strong deformation retract of \( X \)
3. Each \( G \)-stable neighborhood of \( M \) can be shrunk to a \( G \)-stable Stein open set in \( X \) which fulfills conditions 1 and 2 as well. (Shrinking Principle)

As mentioned in the introduction (Theorem 1.2), it is of fundamental importance for our considerations that every proper \( G \)-manifold possesses a Stein \( G \)-tube (see [He93], [Kut94], [HHK95]).

Stein \( G \)-tubes possess the following fundamental property.

**Proposition 2.6.** Let \( M \) be a real proper \( G \)-manifold with Stein \( G \)-tube \( X \). Furthermore let \( Z \) be a complex \( G \)-manifold and \( f : M \to Z \) a \( G \)-equivariant real analytic map. Then after shrinking of \( X \) the map \( f \) extends to a \( G \)-equivariant holomorphic map \( \tilde{f} : X \to Z \).

**Proof.** Identify \( G \)-equivariantly a \( G \)-stable neighborhood of \( M \) with a neighborhood \( V \) of the zero section in the normal bundle of \( M \) with convex fibers. Then after shrinking of \( V \) the real analytic function \( f \) extends uniquely to a holomorphic function \( \tilde{f} \) on \( V \). Since \( g^{-1} \circ \tilde{f} \circ g \) is an extension as well, by uniqueness it is equal to \( \tilde{f} \) which is therefore \( G \)-equivariant. \( \square \)

2.3.1. – Embedding of the real symplectic reduction

Let \( M \) be a manifold with proper free \( G \)-action. Let \( i : M \hookrightarrow X \) be a Stein \( G \)-tube of \( M \) with \( G \)-invariant Kähler form \( \omega \) and associated moment map \( \mu \) so that \( i^\ast \omega = 0 \) and \( i^\ast \mu = 0 \). Denote by \( \sigma \) the anti-holomorphic involution on \( X \).

**Lemma 2.7.** There is a \( G \)-invariant Kähler form \( \bar{\omega} \) on \( X \) with an associated moment map \( \bar{\mu} \) with \( i^\ast \bar{\omega} = 0 \) and \( i^\ast \bar{\mu} = 0 \) such that the embedding \( i : M \hookrightarrow X \) induces a totally real embedding \( i_{\text{red}} : M/G \hookrightarrow \bar{\mu}^{-1}(0)/G \) of maximal dimension. The set \( \bar{\mu}^{-1}(0)/G \) can be shrunk to a Stein tube.
PROOF. The involution $\sigma$ is $G$-equivariant and fixes $M$ pointwise. Thus the form $\tilde{\omega} := \omega - \sigma^*\omega$ is a Kähler form with $i^*\tilde{\omega} = 0$ and $\tilde{\mu} := \mu - \sigma^*\mu$ is an associated moment map with $i^*\tilde{\mu} = 0$. Set $R := \tilde{\mu}^{-1}(0)$ and notice that $\sigma$ stabilizes $R$ with $\text{Fix}(\sigma|_R : R \to R) = M$. Hence there is an induced anti-holomorphic involution $\sigma_{\text{ind}} : R/G \to R/G$ whose fixed point set is exactly the image of the induced embedding $i_{\text{ind}} : M/G \hookrightarrow R/G$. A calculation of the dimensions

$$\dim \mathbb{R} R/G = \dim \mathbb{R} X - 2 \dim G = 2(\dim M - \dim G)$$

shows that $M/G$ is of half real dimension of $R/G$, hence totally real of maximal dimension.

In order to see that $R/G$ can be shrunk to a Stein tube we use the fact that $M/G$ possesses a Stein tube $Y$ since $M/G$ is a real manifold. Shrinking $Y$ sufficiently, the embedding $i_{\text{ind}} : M/G \hookrightarrow R/G$ extends to a holomorphic map $j : Y \to R/G$. This map $j$ is biholomorphic in a neighborhood of $M/G$ onto its image. Shrinking this neighborhood to a Stein neighborhood, the image is a Stein tube of $M/G$ embedded in $R/G$. \qed

3. – Proof of the main theorem

Let $M$ be a real proper $G$-manifold with a $G$-invariant closed 2-form $\tau$.

For the reader’s convenience, we sum up the main steps of the proof. We start with the case where $M$ is the acting group $G$ itself, realize the $G$-equivariant complexification of the space and construct an invariant Kähler form on this complexification. The next case treated is to suppose that $M$ is a product $G \times S$ with $G$ acting by multiplication on the first factor and $S$ is an arbitrary real manifold. Here we split the given 2-form $\tau$ into a part $\tau_G$ arising from a 2-form on $G$, a part $\tau_S$ arising from a 2-form on $S$ and the rest, namely $\tau_M$, containing the “mixed terms”. Then we construct the corresponding Kähler forms separately. For this, the form $\tau_M$ has to be split again. Finally for the general case, we use the fact that $M$ can be realized as a $G$-equivariant quotient $G \times K S$ of the product $G \times S$ by a compact subgroup $K$ of $G$. The situation is lifted to $G \times S$ where the previous case solves the problem. Averaging over $K$ and Kähler reduction of the complexification of $G \times S$ due to a moment map with respect to the $K$-action are the essential tools in the last step in order to push down the solution on the complexification of $G \times K S$.

NOTATION. Let $J$ be the almost complex structure of a complex manifold $X$ and $\eta$ a $k$-form on $X$. Define the $k$-form $J\eta$ by $J\eta(v_1, \ldots, v_k) := \eta(Jv_1, \ldots, Jv_k)$ for all vector fields $v_1, \ldots, v_k$ and for a 0-form $f$, i.e. a function, $Jf := f$. Furthermore, $d^c \eta := i(\partial - \overline{\partial})\eta$. 
3.1. – The group case

In the first step let $M$ be the group $G$ itself and let the $G$-action be defined by left multiplication.

**Notation.** Throughout this section we will let $e$ denote both the neutral element in the group $G$ and its image in an associated Stein $G$-tube $G^*$.

**Proposition 3.1** [Wi93]. Let $G$ be a real Lie group. Then there is a Stein $G$-tube $i : G \hookrightarrow G^*$ admitting a submanifold $\Sigma$ with $e \in \Sigma$ so that

(i) $T_e \Sigma = JT_eG$
(ii) the map $G \times \Sigma \rightarrow G^*$, $(g, s) \mapsto g \cdot s$ is a $G$-equivariant diffeomorphism.

**Lemma 3.2.** There is a Stein $G$-tube $i : G \hookrightarrow G^*$ and a $G$-invariant strictly plurisubharmonic function $\rho_+ : G^* \rightarrow \mathbb{R}^{\geq 0}$ with $\{ \rho_+ = 0 \} = G$ and $i^*d\rho_+ = 0$.

**Proof.** Let $\Sigma$ be the slice cited in Proposition 3.1. For sufficiently small $\Sigma$ an open neighborhood of $0 \in T_e \Sigma$ can be identified with a neighborhood of $e \in \Sigma$ with 0 corresponding to $e$. Consider the square of the norm function on $T_e \Sigma$ pulled back to $\Sigma$ via this identification. Extend this function $G$-invariantly to $G^* \cong G \times \Sigma$ and denote it $\rho_+$. Shrinking $\Sigma$ and thereby $G^*$ again, $\rho_+$ is strictly plurisubharmonic and $i^*d\rho_+ = 0$. □

3.1.1. – The 2-form $\tau$ is "$G$-exact"

The following lemma will be used in the case in which the $G$-invariant 2-form $\tau$ on $G$ is equal to $d\alpha$ for some $G$-invariant 1-form $\alpha$ on the group. ($\tau$ is "$G$-exact").

**Lemma 3.3.** Let $\alpha$ be a $G$-invariant 1-form on $G$. Then there is a Stein $G$-tube $i : G \hookrightarrow G^*$ and a $G$-invariant function $\rho : G^* \rightarrow \mathbb{R}$ with $i^*d\rho = \alpha$.

**Proof.** The slice $\Sigma$ used in Proposition 3.1 satisfies $T_e \Sigma = JT_eG$. So there is a function $\rho$ on $\Sigma$ regarded as being $G$-invariantly extended to $G \times \Sigma \cong G^*$ and which satisfies

$$\alpha_*(\xi) = (d^c\rho)_*(\xi) = (d\rho)_*(J\xi) \quad \forall \xi \in \mathfrak{g}.$$ 

By the $G$-invariance of both $d^c\rho$ and $\alpha$ we obtain

$$i^*d^c\rho = \alpha.$$ □

**Corollary 3.4.** Let $\alpha$ be a $G$-invariant 1-form on $G$. Then there is a Stein $G$-tube $i : G \hookrightarrow G^*$ and a $G$-invariant strictly plurisubharmonic function $\rho : G^* \rightarrow \mathbb{R}$ so that

$$i^*dd^c\rho = d\alpha.$$ 

**Proof.** By Lemma 3.3 there is a $G$-invariant function $\rho_0$ on some Stein $G$-tube $G^*$ so that $i^*d^c\rho_0 = \alpha$. Choosing $G^*$ sufficiently small, there is a $G$-invariant strictly plurisubharmonic function $\rho_+$ with $i^*d^c\rho_+ = 0$. Scaling $\rho_+$ with a sufficiently large factor $\lambda \in \mathbb{R}^{>0}$ the bilinear form

$$(dd^c\rho_0 + dd^c(\lambda \cdot \rho_+))_e \in (\wedge^2 \mathfrak{g})^*$$
is non degenerate. Hence by $G$-invariance there is a $G$-stable neighborhood of $G \subset G^*$, so that $\rho := \rho_0 + \lambda \cdot \rho_+ \in \mathfrak{g}$ is strictly plurisubharmonic and still holds $i^*d^c\rho = \alpha$. The proof is completed by shrinking this set to a $G$-stable Stein neighborhood of $G$. \hfill \square

3.1.2. – The 2-form $\tau$ is arbitrary on the group

The next step is to consider an arbitrary closed $G$-invariant 2-form $\tau$ on $G$.

First assume $G$ to be connected and simply connected. Then $\tau$ defines a central Lie algebra extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by defining the Lie bracket on $\hat{\mathfrak{g}} \cong \mathbb{R} \times \mathfrak{g}$ as $[\zeta, \eta] := (\zeta, \eta)$, $[s, \xi] := (s, \xi)$. Let $\hat{G}$ be the unique, connected, simply connected Lie group associated to $\hat{\mathfrak{g}}$. Associated to the natural projection $\hat{\mathfrak{g}} \to \mathfrak{g}$ there are a surjective group morphism $\pi_{\hat{G}} : \hat{G} \to G$ with kernel $\mathbb{R}$ and the induced $\hat{G}$-equivariant holomorphic map $\pi_{\hat{G}} : \hat{G}^* \to G^*$ on some Stein $\hat{G}$- and $G$-tubes. Here we regard the $G$-action on $G^*$ pulled back via $\pi_{\hat{G}}$ to a $\hat{G}$-action. Note that $\mathbb{R}$ acts on $\hat{G}^*$ as a subgroup of $\hat{G}$. Define $\tau := \pi_{\hat{G}}^* \tau$. The functional

$$\alpha_\tau : \hat{\mathfrak{g}} \to \mathbb{R}, \quad (s, \zeta) \mapsto (-2s)$$

defines a $\hat{G}$-invariant 1-form $\alpha$ on $\hat{G}$. For $(s, \zeta), (t, \xi) \in \hat{\mathfrak{g}}$ it follows that

$$(d\alpha)_\tau((s, \zeta), (t, \xi)) = -\frac{1}{2} \alpha_\tau(\tau_\tau(s, \zeta), (s, \xi)) = \tau_\tau(s, \zeta) = \tau_\tau((s, \zeta), (t, \xi)).$$

Hence, by the $\hat{G}$-invariance of both sides, $d\alpha = \tau$.

**Lemma 3.5.** Let $\tau$ be a closed, $G$-invariant 2-form on $G$. Then there is a Stein $G$-tube $i : G \hookrightarrow G^*$ and a $G$-invariant Kähler form $\omega$ on $G^*$ with $i^*\omega = \tau$.

**Proof.** Suppose first that $G$ is connected and simply connected. By Lemma 3.3 and Corollary 3.4 there is an exact $\hat{G}$-invariant Kähler form $\hat{\omega}$ on some Stein $\hat{G}$-tube $i : \hat{G} \hookrightarrow \hat{G}^*$ with $i^*\hat{\omega} = \hat{\tau}$.

The next step will be to push down $\hat{\omega}$ to a Kähler form $\omega$ on a Stein $G$-tube $G^*$. Let $Z$ denote the vector field induced by the central $\mathbb{R}$-action on $\hat{G}^*$. The 1-form $i_z\hat{\omega}$ is closed and $\mathbb{R}$-invariant. Since $\hat{G}^*$ can be retracted to the simply connected Lie group $\hat{G}$, there is a moment map $\mu : \hat{G}^* \to \mathbb{R} \cong \text{Lie}(\mathbb{R})^*$ defined by

$$\mu(x) := \int_0^x i_z\hat{\omega}.$$ 

Set $R := \mu^{-1}(0)$ and note that $\hat{G} \cdot e \subseteq R$. By Lemma 2.5 there is a Kähler form $\omega$ on a Stein $G$-tube of $G$ and $\omega$ is even $G$-invariant due to the $\hat{G}$-invariance of $\hat{\omega}$. Here we identify $R/\mathbb{R}$ with $G^*$ using the universal property introduced in Lemma 2.4.
Set \( i_R : R \hookrightarrow \hat{G}^* \). The form \( \omega \) fulfills \( i_R^*(\pi_G^*)^*\omega = i_R^*\tilde{\omega} \). Since \( \hat{G} \cdot e \subset R \), it also follows that \( i^*(\pi_G^*)^*\omega = i^*\tilde{\omega} \). Thus

\[
\pi_G^*i^*\omega = i^*(\pi_G^*)^*\omega = i^*\tilde{\omega} = \tilde{\tau} = \pi_G^*\tau
\]

and, by the surjectivity of \( \pi_G \),

\[
i^*\omega = \tau.
\]

Now let \( G \) be arbitrary. There is a Lie group morphism \( p \) of the identity component \( H \) of the universal covering to \( G \). This induces a locally biholomorphic \( H \)-equivariant map \( p^* : H^* \to G^* \) with \( H \) acting on \( G^* \) via \( p \).

Thus there is an \( H \)-invariant Kähler form \( \tilde{\omega} \) on \( H^* \) such that if \( \iota : H \hookrightarrow H^* \) is the canonical embedding, then \( \iota^*\tilde{\omega} = p^*\tau \).

Let \( U \) and \( V \) be open neighborhoods of \( e \in G^* \) and \( e \in H^* \) respectively so that \( p^*|_V : V \to U \) is biholomorphic. We may assume that the intersection of every \( G \)-orbit with \( U \) is connected. Since \( p^* \) is \( H \)-equivariant, \( \omega|_U := ((p^*)^{-1})^*\tilde{\omega} \) defines a Kähler form on \( U \) satisfying

\[
L_\xi\omega = 0 \text{ for all } \xi \in h = \text{Lie}(H) \cong g.
\]

So \( \omega \) can be extended \( G \)-equivariantly on \( G \cdot U \). Finally \( p^* \circ \iota = i \circ p \) implies that on \( \iota^{-1}(V) \)

\[
p^*i^*\omega = i^*(p^*)^*\omega = i^*\tilde{\omega} = p^*\tau.
\]

By \( \hat{G} \)-invariance this holds globally and, by the surjectivity of \( p \),

\[
i^*\omega = \tau. \quad \Box
\]

3.1.3. – A basic property for closed \( G \)-invariant 1-forms

The following lemma will be necessary for a construction in the product case section.

**Lemma 3.6.** Let \( \lambda \) be a closed \( G \)-invariant 1-form on \( G \). Then there is a Stein \( G \)-tube \( i : G \hookrightarrow G^* \) and a pluriharmonic, \( G \)-invariant function \( \theta : G^* \to \mathbb{R} \) with \( i^*d\theta = \lambda \) and \( \theta|_G \equiv 0 \).

**Proof.** Let \( \xi_1, \ldots, \xi_n \) be a basis of \( g := \text{Lie}(G) \) and \( \tilde{\xi}_i \) denote the induced vector field of \( \xi_i, i = 1, \ldots, n \). Since \( \lambda \) is both \( G \)-invariant and closed, \( \lambda(\tilde{\xi}_i) \) is constant. For \( \lambda = 0 \) there is nothing to prove, so we can assume that \( \lambda(\tilde{\xi}_i) = \delta_{ii} \).

Now we choose a Stein \( G \)-tube \( G^* \) so that \( \tilde{\xi}_1(x), \ldots, \tilde{\xi}_n(x), J\tilde{\xi}_1(x), \ldots, J\tilde{\xi}_n(x) \) form a basis of \( T_x(G^*) \) for all \( x \in G^* \). The structure constants \( c_{ij}^k \) of the Lie algebra \( g \) with respect to the fixed basis are defined by

\[
[\xi_i, \xi_j] = \sum_k c_{ij}^k \xi_k.
\]
The closedness of $\lambda$ shows that $c_{ij}^1 = 0$ for all $i, j = 1, \ldots, n$ since

$$0 = (d\lambda)_e(\xi_i, \xi_j) = \frac{1}{2} \lambda_e([\xi_i, \xi_j]) = -\frac{1}{2} c_{ij}^1$$

for all $i, j = 1, \ldots, n$.

Now the pointwise dual $\beta(x) := (J\xi_1(x))^*$ defines a (smooth) 1-form $\beta$ on $G^*$. We will see that $\beta$ is closed, $d^c$-closed and $G^c$-invariant where $G^0$ denotes the component of $G$ containing $e$.

Let $\xi_1, \xi_2 \in \{\xi_1, \ldots, \xi_n, J\xi_1, \ldots, J\xi_n\}$ and calculate

$$d\beta(\xi_1, \xi_2) = \frac{1}{2} \xi_1(\beta(x)) - \frac{1}{2} \xi_2(\beta(x)) - \frac{1}{2} [\beta(\xi_1, \xi_2)].$$

The first terms vanish, since $\beta(\xi_1), \beta(\xi_2)$ are constant. Furthermore the term $[\beta(\xi_1, \xi_2)]$ vanishes, because $[\xi_1, \xi_2]$ is a linear combination of the vector fields $\xi_i, J\xi_j$ for $i, j = 2, \ldots, n$, i.e. $i, j \neq 1$ since the constants $c_{ij}^1$ vanish. Thus $d\beta = 0$.

Analogously $d^c\beta = 0$:

$$d^c\beta(\xi_1, \xi_2) = -JdJ\beta(\xi_1, \xi_2) = -dJ\beta(J\xi_1, J\xi_2)$$

$$= -\frac{1}{2} J\xi_1(\beta(x)) + \frac{1}{2} J\xi_2(\beta(x)) - \frac{1}{2} \beta(J[\xi_1, \xi_2]).$$

The individual terms vanish for the same reason as above, because $\beta(\xi_i)$ is constant and $J[\xi_1, \xi_2] = [J\xi_1, \xi_2]$. Finally,

$$\mathcal{L}_{\xi_i} \beta = d(\beta(\xi_i)) = 0,$$

since $\beta(\xi_i)$ is constant. Thus $\beta$ is $G^0$-invariant.

Now there is a contractible open neighborhood $U$ of $e \in G^*$ which intersects each $G$-orbit in a connected set. Define $\theta : U \to \mathbb{R}$ by

$$\theta(x) = \int_x^e \beta.$$

We can consider $\theta$ to be extended $G$-invariantly on $G \cdot U$ since

$$\mathcal{L}_{\xi_i} \theta = d\theta(\xi_i) = \beta(\xi_i) = 0.$$

Furthermore, due to the $G$-invariance of both $\imath^* d^c\theta$ and $\lambda$ and

$$(d^c\theta)_e(\xi_i) = (d\theta)_e(J\xi_i) = \beta(\xi_i) = \delta_{1i} = \lambda_e(\xi_i),$$

it follows that $\imath^* d^c\theta = \lambda$. Finally, $\theta|_G \equiv 0$ follows from $\imath^* \beta = 0$. $\square$
3.2. – The product case

Now we turn to the product case, i.e. $M = G \times S$ and $G$ acts on $M$ by left multiplication on the first factor.

For any Stein tube $S^*$ let $i_S : S \hookrightarrow S^*$ be the totally real embedding and analogously for any Stein $G$-tube $G^*$ set $i_G : G \hookrightarrow G^*$. Let $\pi_G : G \times S \to G$ and $\pi_S : G \times S \to S$ denote the projections and $\pi_G^*$ and $\pi_S^*$ their holomorphic extensions to $G^* \times S^*$ respectively.

Let us first consider a 2-form $\tau$ of a special type. Given a closed 1-form $\eta'$ on $S$ and a $G$-invariant closed 1-form $\lambda'$ on $G$ set $\eta := \pi_S^* \eta'$ and $\lambda := \pi_G^* \lambda'$ and let $\tau := \lambda \otimes \eta$ be the associated closed $G$-invariant 2-form on $G \times S$ seen as a section in the bundle $\pi_G^* T^* G \otimes \pi_S^* T^* S$.

3.2.1. – Extension of $\lambda \otimes \eta$ for $\lambda$ and $\eta$ closed

**Lemma 3.7.** Let $\tau = \lambda \otimes \eta$ be as above. Then there is a Stein $G$-tube $i = i_G \times i_S : G \times S \hookrightarrow G^* \times S^* = X$ and a closed, $G$-invariant $(1, 1)$-form $\omega$ on $X$ with $i^* \omega = \tau$.

**Proof.** Fix a closed 1-form $\eta'$ on $S$ with $\pi_S^* \eta' = \eta$. Set $\eta := (\pi_S^*)^* \eta'$. By Lemma 3.6 there is a $G$-invariant pluriharmonic function $\theta' : G^* \to \mathbb{R}$ on some Stein $G$-tube $i_G : G \hookrightarrow G^*$ with $i_G^* d^c \theta' = \lambda'$ and $\theta'|_G \equiv 0$ and set $\theta := \theta' \circ \pi_G^*$. We define the $G$-invariant 2-form

$$\omega := -d\theta \otimes J\eta + \theta d^c \eta + d\theta \otimes \eta.$$  

Locally there is a function $b$, so that

$$db \text{ locally } = \eta.$$

Thus

$$\omega \text{ locally } = -dd^c(\theta \cdot b),$$

since

$$-dd^c(\theta \cdot b) = -d(\theta d^c b + bd^c \theta)$$
$$= -d\theta \otimes d^c b - \theta dd^c b + d^c \theta \otimes db$$
$$= -d\theta \otimes J\eta + \theta d^c \eta + d\theta \otimes \eta.$$

So $\omega$ is closed, $G$-invariant and of type $(1, 1)$. To show that $i^* \omega = \tau$ note that, since $i^* \theta = 0$, it follows that $i^* d\theta = 0$. By definition $i^* d^c \theta = \lambda$ and $i^* \eta = \eta$. Thus

$$i^* \omega = -i^* d\theta \otimes i^* J\eta + i^* \theta \cdot i^* d^c \eta + i^* d^c \theta \otimes i^* \eta = \lambda \otimes \eta = \tau. \quad \square$$

3.2.2. – The main lemma

Fix an arbitrary point $s_0 \in S$ and the embeddings $i_S : S \hookrightarrow G \times S, s \mapsto (e, s)$ and $i_G : G \hookrightarrow G \times S, g \mapsto (g, s_0)$.

Now the main Lemma can be formulated.
**Lemma 3.8.** Let $\tau$ be a closed, $G$-invariant 2-form on $G \times S$ with $i^*_0 \tau = 0$ and $i^*_\partial \tau = 0$. Then there is a closed $G$-invariant $(1, 1)$-form $\omega$ on $G^* \times S^*$ with $i^* \omega = \tau$.

**Proof.** Fix a basis $\lambda_1', \ldots, \lambda_n'$ of the vector space of $G$-invariant 1-forms on $G$, so that the subsystem $\lambda_1', \ldots, \lambda_n'$ forms a basis of the closed invariant forms. Set $\lambda_i := \pi_G^* \lambda_i'$. The general form of $\tau$ is

$$\tau = \sum_k \eta_k \otimes \lambda_k + \sum_{i,j} f_{ij} \lambda_i \wedge \lambda_j,$$

where $f_{ij} = \pi_G^* f_{ij}'$ and $\eta_k = \pi_S^* \eta_k'$ with $f_{ij}'$ functions and $\eta_k'$ 1-forms on $S$. Note that $f_{ij}(s_0) = 0$, since $i^*_\partial \tau = 0$. Now we decompose $\tau$:

$$\tau_c := \sum_{k=r}^n \eta_k \otimes \lambda_k, \quad \tau_r := \tau - \tau_c = \sum_{k=1}^{r-1} \eta_k \otimes \lambda_k + \sum_{i,j} f_{ij} \lambda_i \wedge \lambda_j.$$

The bundle $\Lambda^3 T^*(G \times S)$ splits canonically into

$$\bigwedge^3 T^*S \oplus \left( T^*G \otimes \bigwedge^2 T^*S \right) \oplus \left( \bigwedge^2 T^*G \otimes T^*S \right) \oplus \bigwedge^3 T^*G.$$

Since the 3-form $d\tau$ vanishes, its $(T^*G \otimes \bigwedge^2 T^*S)$-component vanishes and thus $\sum_{k=1}^n d\eta_k \otimes \lambda_k = 0$. Hence the forms $\eta_k, k = 1, \ldots, n$ and $\tau_c$ are closed. Lemma 3.7 solves the problem for $\tau_c$, i.e. there is $G$-invariant closed $(1, 1)$-form $\omega_c$ on $G^* \times S^*$ with $i^* \omega_c = \tau_c$.

So it remains to construct an extension $\omega_r$ of $\tau_r$. We calculate

$$0 = d\tau_r = \sum_{k=1}^{r-1} d\lambda_k \otimes \eta_k + \sum_{i,j} \lambda_i \wedge \lambda_j \otimes df_{ij}.$$

In order to see that $\eta_k$ is exact for $k = 1, \ldots, r - 1$ notice that $\Lambda_1 := d\lambda_1, \ldots, \Lambda_{r-1} := d\lambda_{r-1}$ are linearly independent in the vector space of $G$-invariant 2-forms independent of $S$. We complete them to a basis $\Lambda_1, \ldots, \Lambda_m$. Now we can apply the dual basis vector $\Lambda_k^*$ to the upper equation and obtain the exactness of $\eta_k$:

$$\eta_k = -\sum_{i,j} \Lambda_k^*(\lambda_i \wedge \lambda_j) df_{ij}.$$

Let $(\frac{\partial}{\partial \lambda_i})_{i=1,\ldots,n}$ be the $G$-vector fields dual to $(\lambda_i)_{i=1,\ldots,n}$, i.e. $\lambda_i(\frac{\partial}{\partial \lambda_j}) = \delta_{ij}$. Since

$$d\lambda_k \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) = -\frac{1}{2} c_{ij}^k,$$

where $c_{ij}^k$ denote the according Lie algebra structure constants, it follows that

$$\frac{1}{2} \sum_k \eta_k c_{ij}^k = df_{ij}.$$
For simplicity we define $G$-invariant functions $b_k : G \times S \to \mathbb{R}$ by $db_k = \eta_k$ and $b_k(s_0) = 0$. Due to $f_{ij}(s_0) = 0$, the equation (*) transforms to

$$\frac{1}{2} \sum_k b_k c^{k}_{ij} = f_{ij}.$$  

We calculate

$$d \left( - \sum_k b_k \lambda_k \right) = \sum_k \lambda_k \otimes \eta_k + \frac{1}{2} \sum_{k,i,j} b_k c^{k}_{ij} \lambda_i \wedge \lambda_j$$

$$= \sum_k \lambda_k \otimes \eta_k + \sum_{i,j} f_{ij} \lambda_i \wedge \lambda_j = \tau_r.$$  

By Lemma 3.3 there are $G$-invariant functions $\rho'_k$ on $G^\ast$ with $i^*d\rho'_k = \lambda'_k$ and $\rho'_k|_G \equiv 0$; set $\rho_k := \rho'_k \circ \pi_G$. Extend the $b_k$ to functions on $G^\ast \times S^\ast$ independent of $G^\ast$, denote these extensions $b_k$ as well, and define

$$\rho := - \sum_k b_k \rho_k.$$  

It follows that $\omega_r := dd^c \rho$ is an exact, $G$-invariant $(1, 1)$-form with

$$i^* \omega_r = - d^c \left( \sum_k (\rho_k d^c b_k + b_k d^c \rho_k) \right)$$

$$= - d \left( \sum_k b_k \lambda_k \right) = \tau_r,$$

since $i^* \rho_k \equiv 0$.  

3.2.3. -- Extension of an arbitrary 2-form $\tau$

The general extension is obtained by decomposing $\tau$ into relevant pieces.

**Lemma 3.9.** Let $\tau$ be a closed, $G$-invariant 2-form on $G \times S$. Then there is a Stein $G$-tube $i : G \times S \to G^\ast \times S^\ast$ and a closed, $G$-invariant $(1, 1)$-form $\omega$ on $G^\ast \times S^\ast$ with $i^* \omega = \tau$.

**Proof.** We decompose $\tau$ into three parts $\tau = \tau_G + \tau_M + \tau_S$, so that each part is still closed and $G$-invariant; $\tau_G$ and $\tau_S$ will be 2-forms arising from 2-forms on $G$ and $S$ respectively while $\tau_M$ contains the "mixed terms". In order to obtain the decomposition define the $G$-invariant closed 2-form $\tau'_G := i^* \tau_G$ on $G$. By Lemma 3.5 there is a $G$-invariant Kähler form $\omega'_G$ on $G^\ast$ with $i^*_G \omega'_G = \tau'_G$. We set $\omega_G := (\pi_G^*)^* \omega'_G$ and obtain $i^* \omega_G = \tau_G$. Analogously, for the closed 2-form $\tau'_M := i^* \tau_M$, by Theorem 1.1 there is a closed $(1, 1)$-form $\omega'_M$ with the desired properties on $S^\ast$ and set $\omega_S := (\pi_S^*)^* \omega'_M$.

The difference $\tau_M := \tau - \tau_G - \tau_S$ is a $G$-invariant closed 2-form containing the "mixed terms". This can be extended to a $G$-invariant closed $(1, 1)$-form $\omega_M$ on $G^\ast \times S^\ast$ by the main Lemma (Lemma 3.8).

Thus by adding the constructed components, i.e. setting $\omega := \omega_G + \omega_M + \omega_S$, a $G$-invariant closed $(1, 1)$-form $\omega$ is obtained with $i^* \omega = \tau$.  

3.2.4. – Extension as a Kähler form

Finally, it is an elementary matter to adjust the above extension to obtain a Kähler form.

**Lemma 3.10.** Let τ be a closed G-invariant 2-form on $G \times S$. Then there is a Stein G-tube $i : G \times S \hookrightarrow X \subset G^* \times S^*$ and a G-invariant Kähler form $\omega$ on $X$ with $i^* \omega = \tau$.

**Proof.** By Lemma 3.9 there is a closed G-invariant $(1,1)$-form $\omega_0$ on some Stein G-tube $X$ with $i^* \omega_0 = \tau$. For $X$ sufficiently small there is a G-invariant strictly plurisubharmonic function $\rho_+ : X \to \mathbb{R}^{\geq 0}$ with

$$\{\rho_+ = 0\} = \{d\rho_+ = 0\} = M = G \times S \subset X.$$  

Fix a G-invariant partition of unity $\{\chi_\alpha\}$ so that $(\text{supp } \chi_\alpha)/G \subset X/G$ is compact and the interiors of supp $\chi_\alpha$ form a locally finite cover of G-stable open sets in $X$. Choose $\varepsilon_\alpha > 0$ so that $V_\alpha := \{x \mid \chi_\alpha(x) > \varepsilon_\alpha\}$ is a cover as well. The conditions $\{\rho_+ = 0\} = M$ and $\{d\rho_+ = 0\} = M$ imply

$$dd^c(\chi_\alpha \rho_+)_{|M} = (\chi_\alpha dd^c \rho_+)_{|M},$$

since the terms $d\chi_\alpha \wedge d^c \rho_+$, $d^c \chi_\alpha \wedge d\rho_+$ and $\rho_+ dd^c \chi_\alpha$ vanish on $M$. The sets $V_\alpha/G$ are relatively compact, so that there are constants $c_\alpha > 0$ such that

$$\omega_0 - c_\alpha dd^c(\chi_\alpha \rho_+)$$

is a Kähler form on a $G$-stable neighborhood of $M \cap V_\alpha$ in $X$ since the form $dd^c(\chi_\alpha \rho_+)$ is a Kähler form in some open neighborhood of $M \cap V_\alpha$. Set $\rho := \sum c_\alpha \chi_\alpha \rho_+$ and note that the sum is locally finite. Thus the form

$$\omega := \omega_0 - dd^c \rho$$

is a $G$-invariant real $(1,1)$-form on a $G$-stable neighborhood of $M$ which is positive on $M$. Thus there is a possibly smaller Stein $G$-tube $G^* \times S^*$, again denoted by $X$, such that $\omega$ is a $G$-invariant Kähler form. The fact $d\rho_+ |_{M} = 0$ yields $i^*d\rho_+ = i^*d^c \rho_+ = 0$, hence $i^*dd^c \rho = 0$ which implies $i^* \omega = \tau$. □

3.3. – The general case via Abels’ theorem

The main Theorem will be proved via a real analytic version of Abels’ global Slice Theorem. It is known that for any proper $G$-action on a $C^\infty$-manifold $M$ there is a compatible real analytic structure on $M$ making the action real analytic ([II93]). In fact this structure is unique ([Ku96]).

The following theorem is valid for Lie groups $G$ which admit a maximal compact subgroup $K$ unique up to conjugation. Therefore let us restrict in the sequel to the case where $G$ possesses only finitely many components where such a maximal compact subgroup $K$ exists in general.
Theorem 3.11 [HHK96]. Let $G$ act properly (and real analytically) on a manifold $M$ and let $K$ be a maximal compact subgroup. Then there is a $K$-stable real analytic submanifold $S \subset M$ so that the map

$$G \times_K S \xrightarrow{\sim} M$$

$$[g, s] \mapsto g \cdot s$$

is a $G$-equivariant real analytic bijection with real analytic inverse.

Remark. The theorem is based on Abels’ theorem ([Ab74]) that proves the same statement in the category of smooth manifolds.

The Stein $G$-tube of an Abels representation $M = G \times_K S$ is constructed concretely as the categorical quotient $(G^* \times S^*)//K$ ([HHK96]), i.e. the quotient with respect to the $K$-invariant holomorphic functions. The categorical quotient of a Stein manifold with respect to a compact group is a Stein space ([He91]). This allows us to construct the Kähler extension by pushing down an extension from $G^* \times S^*$ to $(G^* \times S^*)//K$.

Proof of the Main Theorem. Let $M = G \times_K S$ and $\tau$ be a closed $G$-invariant 2-form on $M$. As mentioned above $i : M \hookrightarrow (G^* \times S^*)//K$ is a Stein $G$-tube. We lift the situation to $G \times S$ via the projection $p : G \times S \rightarrow G \times_K S$ which extends to a holomorphic projection $p^* : G^* \times S^* \rightarrow (G^* \times S^*)//K$.

The inclusion $\tilde{i} : G \times S \hookrightarrow G^* \times S^*$ is a Stein $G$-tube as well. Of course, $p^* \circ \tilde{i} = i \circ p$. Note that $G \times S$ and $G^* \times S^*$ are endowed with $(G \times K)$-actions making $\tilde{i}$ equivariant. The 2-form $\tilde{\tau} := p^* \tau$ is $(G \times K)$-invariant. By Lemma 3.10 there is a $G$-invariant Kähler form $\tilde{\omega}$ on $G^* \times S^*$ with $\tilde{i}^* \tilde{\omega} = \tilde{\tau}$.

By the averaging process $\tilde{\omega}$ can be assumed $K$-invariant as well. For $v \in k$ let $\tilde{v}_K$ denote the associated $K$-vector field on $G \times S$ and $G^* \times S^*$ respectively. The 1-form $\tilde{\tau}_v \tilde{\omega}$ vanishes for all $v \in k$. Fix an arbitrary point $x_0 \in G \times S$ and define

$$
\mu^v(x) = \int_{x_0}^{x} \tilde{v}_K \tilde{\omega}
$$

on the $K$-stable Stein $G$-tube $X = G^* \times S^*$. Note that the associated map $\mu : X \rightarrow k^*$ vanishes identically on $G \times S$. Furthermore $k^*\mu^v - \mu^{\text{Ad}(k)v}$ is constant and vanishes on $G \times S$, hence vanishes identically. So $\mu$ is a $K$-moment map with $G \times S \subset R := \mu^{-1}(0)$. By Lemma 2.5 there is an induced Kähler form $\omega$ on $R/K$ satisfying $i^*_k(p^*)^*\omega = i^*_K \omega$ and hence $i^*(p^*)^*\omega = i^* \tilde{\omega}$. Since $TR$ and $JTR$ span $TX|_R$, the image of the map $R \rightarrow X//K$ induced by the embedding of $R$ contains a $G$-stable open neighborhood $V$ of $G \times_K S \subset X//K$. Shrinking $X$ to a $G$-stable Stein neighborhood of $G \times S$ in the $p^*$-preimage of $V$ makes the induced $G$-equivariant map $R/K \rightarrow X//K$ biholomorphic such that we can identify these spaces. Due to the $G$-equivariance of the projection the form $\omega$ is even $G$-invariant. In order to show $i^* \omega = \tau$ calculate

$$p^* i^* \omega = i^*(p^*)^* \omega = i^* \tilde{\omega} = \tilde{\tau} = p^* \tau$$

and by surjectivity of $p$ we obtain finally

$$i^* \omega = \tau.$$
Note that the above proof only requires the existence of an Abels representation. Thus, even if $G$ has infinitely many components, the main theorem holds for $M = G \times K S$ of this type. In particular, we have the following local version.

**Theorem 3.12.** Let $M$ be a manifold with proper $G$-action and $\tau$ a closed $G$-invariant 2-form. For each $x_0 \in M$ there is a $G$-stable neighborhood $U$ of $x_0$ in the Stein $G$-tube $X$ and a $G$-invariant Kähler form $\omega$ on $U$ with $(i_{\mid_{-1(U)}})^* \omega = \tau$.

### 4. Extension of the moment map

Next it will be shown that if the totally real manifold possesses a moment map, then this is extendable to a moment map with respect to the Kähler form on the complexification.

**Theorem 4.1.** Let $\nu : M \to g^*$ be a moment map on $M$ with respect to a closed $G$-invariant 2-form $\tau$ and $\omega$ a closed $G$-invariant 2-form on some Stein $G$-tube $X$ with $i^* \omega = \tau$. Then there is a moment map $\mu : X \to g^*$ with respect to $\omega$ with $i^* \mu = \nu$.

**Proof.** Let $\nu \in g$ and $\tilde{\nu}$ denote the induced vector field on $M$ and $X$ respectively. The 1-form $i_{\nu^*} \tau$ on $M$ is exact by assumption and $M$ is a strong deformation retract of $X$. Thus, fixing $x_0 \in M$, 

$$
\mu^\nu(x) = \int_{x_0}^x i_{\nu^*} \tau + \nu^\nu(x_0)
$$

is well-defined on $X$ and fulfills $i^* \mu = \nu$. Note that the map $g \to C^\infty(X), \nu \mapsto \mu^\nu$, is linear. The associated map $\mu : X \to g^*$ satisfies the moment map condition $i_{\omega^*} \omega = d\mu^\nu$. Thus we must only prove the $G$-equivariance of $\mu$, i.e. 

$$
\mu^\nu(g \cdot x) = \mu^w(x) \text{ for all } x \in X
$$

with $w = \text{Ad}(g)\nu$. Note that $\tilde{w} = g_* \tilde{\nu}$ and thus 

$$
d(\mu^\nu(g \cdot x) - \mu^w(x)) = g^* i_{\nu^*} \tau - i_{w^*} \omega = i_{g_* \tilde{\nu}^*} \omega - i_{\tilde{w}^*} \omega = i_{\tilde{w}^*} \omega - i_{\tilde{w}^*} \omega = 0.
$$

So $g^* \mu^\nu - \mu^w \in g^*$ is constant. But for any $x \in M$ 

$$
g^* \mu^\nu(x) - \mu^w(x) = \nu^\nu(g \cdot x) - \nu^w(x) = 0
$$

by the $G$-equivariance of $\nu$. \qed
5. – Construction is canonical

Stein G-tubes can be considered as “germs”, i.e. two Stein G-tubes of a proper G-manifold M are G-equivariant biholomorphic after sufficient shrinking of both. The following theorem shows that any two G-invariant Kähler extensions of a G-invariant 2-form on X are likewise equivalent.

**THEOREM 5.1.** Let M be a proper G-manifold and i : M \hookrightarrow X an associated Stein G-tube. For a closed G-invariant 2-form \( \tau \) on M suppose that \( \omega_0 \) and \( \omega_1 \) are G-invariant Kähler forms on X with \( i^*\omega_0 = i^*\omega_1 = \tau \). Then there are G-stable neighborhoods \( U_0, U_1 \) of M and a G-equivariant diffeomorphism \( \varphi : U_0 \to U_1 \) with \( \varphi|_M = \text{id}_M \) so that

\[
\varphi^*\omega_1 = \omega_0.
\]

**PROOF.** Using a G-invariant Riemannian metric on X, the exponential map on \( JT_M \) identifies G-equivariantly a G-stable neighborhood \( V \) of the zero section with a G-stable neighborhood \( U \) of \( M \subset X \). We can assume the set \( V_x := V \cap JT_x M \) to be convex for all \( x \in M \), so that, via the identification, the G-equivariant map \( (t, v) \mapsto (1 - t) \cdot v \) can be regarded as a smooth G-equivariant homotopy on \( U \), i.e. a smooth map

\[
\psi : [0, 1] \times U \to U
\]

defining \( \psi_t := \psi(t, \cdot) : U \to U \) with \( \psi_0 = \text{id}_U, \psi_1(U) = M, \psi_1|_M = \text{id}_M \) and \( \psi_t \) is G-equivariant.

Define the sections \( \sigma_t : X \to [0, 1] \times X, x \mapsto (s, x) \) and note that for any k-form \( \eta \) on \( [0, 1] \times X \)

\[
\frac{\partial}{\partial t}(\sigma_t^*\eta) = \sigma_t^*L_{\frac{\partial}{\partial t}}\eta.
\]

Now consider the closed 2-form \( \omega := \omega_1 - \omega_0 \). It follows that \( \psi_t^*\omega = 0 \), since \( i^*\omega = 0 \). Furthermore \( \psi_1^*\omega = \omega \). In order to establish the existence of a G-invariant 1-form \( \beta_0 \) with \( \omega = d\beta_0 \), we will use a slightly modified version of a calculation in [GuSt84].

\[
\omega = \psi_1^*\omega - \psi_0^*\omega = \int_0^1 \frac{d}{dt}_{t=s} [\psi_t^*\omega]ds = \int_0^1 \frac{d}{dt}_{t=s} [\sigma_t^*\psi^*\omega]ds
\]

\[
= \int_0^1 [\sigma_t^*L_{\frac{\partial}{\partial t}}\psi^*\omega]ds = \int_0^1 [\sigma_0^*\omega]_{\frac{\partial}{\partial t}}\psi^*\omega]ds = d \left( \int_0^1 [\sigma_s^*L_{\frac{\partial}{\partial t}}\psi^*\omega]ds \right)
\]

For simplicity, set \( \beta_0 := f_0^1[\sigma_s^*\omega]_{\frac{\partial}{\partial t}}ds \) and notice that \( \beta_0 \) is G-invariant and \( i^*\beta_0 = 0 \). Consider \( \beta_0 \) as a function on \( JT_M \) and pull it back via the exponential map to a G-invariant function \( f : U \to \mathbb{R} \). This function satisfies \( f|_M \equiv 0 \) and \( df|_M = \beta_0|_M \). Thus \( \beta := \beta_0 - df \) is a G-invariant 1-form with \( \beta|_M \equiv 0 \) and \( d\beta = \omega_1 - \omega_0 \).
Thus we can apply Moser's method to the curve $\omega_t := (1-t) \cdot \omega_0 + t \cdot \omega_1$ of $G$-invariant Kähler forms on $U$. For this, define the $G$-invariant time-dependent vector field $\xi_t$ by

$$t_{\xi_t} \omega_t = -\beta.$$  

Since $\xi_t|_M \equiv 0$ there is a $G$-stable neighborhood $U_0$ so that the flow

$$\varphi_t : U_0 \to X$$

is defined for all $t \in [0,1]$ satisfying $\varphi_t|_M = \text{id}_M$. The general formula on time-dependent forms ([MDS95], p. 92) yields

$$\frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* \frac{\partial \omega_t}{\partial t} + \varphi_t^* (t_{\xi_t} d\omega_t) + \varphi_t^* (dt_{\xi_t} \omega_t)$$

$$= \varphi_t^* d\beta + 0 + \varphi_t^* (-d\beta) = 0. $$

Thus, from $\varphi_0^* \omega_0 = \omega_0$ we obtain

$$\varphi_t^* \omega_t = \omega_0.$$  

The map $\varphi := \varphi_1 : U_0 \to \varphi_1(U_0)$ is a $G$-equivariant diffeomorphism with $\varphi|_M = \text{id}_M$ and $\varphi^* \omega_1 = \omega_0$.  

**COROLLARY 5.2.** In addition to the assumptions of Theorem 5.1, let a moment map $\nu : M \to g^*$ on $M$ be given with respect to $\tau$ and $\mu_0$ and $\mu_1$ be moment maps with respect to $\omega_0$ and $\omega_1$ and assume that $i^* \mu_0 = i^* \mu_1 = \nu$, where $\nu$ is a moment map with respect to $\tau$. Then the constructed diffeomorphism $\varphi$ satisfies

$$\varphi^* \mu_1 = \mu_0.$$  

**PROOF.** For $\nu \in g$ the map $\varphi$ stabilizes the induced vector field $\vec{v}$, i.e.

$$\varphi_* \vec{v} = \vec{v}$$

and hence

$$d(\varphi^* \mu_1^\nu - \mu_0^\nu) = \varphi^* d\mu_1^\nu - d\mu_0^\nu$$

$$= \varphi^* t_\nu \omega_1 - t_\nu \omega_0$$

$$= t_\nu \omega_1 - t_\nu \omega_0$$

$$= t_\nu \omega_1 - t_\nu \omega_0 = 0.$$

Thus $\varphi^* \mu_1 - \mu_0 \in g^*$ is constant. But $\varphi(x) = x$ for any $x \in M$ and hence

$$\varphi^* \mu_1(x) - \mu_0(x) = \nu(x) - \nu(x) = 0.$$  

Therefore

$$\varphi^* \mu_1 = \mu_0.$$  

In summary the $G$-invariant Kähler extension (with moment map) is unique as germ up to diffeomorphisms which are the identity on $M$.  

REFERENCES


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