ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 30, nº 2 (2001), p. 405-414
<http://www.numdam.org/item?id=ASNSP_2001_4_30_2_405_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXX (2001), pp. 405-414

Reinhardt Domains and the Gleason Problem

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Abstract. As usual, let $A(\Omega)$ be the uniform algebra consisting of the functions which are holomorphic on Ω, and continuous on $\overline{\Omega}$, and let $H^\infty(\Omega)$ be the set of bounded holomorphic functions on Ω. Throughout this paper Ω will be a bounded Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary.

We show that the maximal ideal (both in $A(\Omega)$ and $H^\infty(\Omega)$), consisting of functions vanishing at $p \in \Omega$, is generated by the functions $(z_1 - p_1)$, $(z_2 - p_2)$, at first for the case that Ω is pseudoconvex, then without this condition.

Mathematics Subject Classification (2000): 32A07 (primary), 46J15 (secondary).

1. – Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $R(\Omega)$ (usually $A(\Omega)$ or $H^\infty(\Omega)$) be a ring of holomorphic functions that contains the polynomials, and let $p = (p_1, \dots, p_n)$ a point of Ω. Recall the Gleason problem, cf. [7]: is the maximal ideal in $R(\Omega)$ consisting of functions vanishing at p, generated by the coordinate functions $(z_1 - p_1), \dots, (z_n - p_n)$?

One says that a domain Ω has the *Gleason R-property* if this is the case for all points $p \in \Omega$. We also say that it has the Gleason-property with respect to $R(\Omega)$.

Gleason mentioned the difficulty of solving this problem even for such a simple domain as the unit ball $B(0, 1)$ in $\mathbb{C}^2$, $p = (0, 0)$, $R(\Omega) = A(\Omega)$. That case was solved by Leibenzon ([10]), who proved that the Gleason problem can be solved on any convex domain in $\mathbb{C}^n$ having a C^2-boundary. Using different techniques, this result was sharpened by Grangé ([8], for $H^\infty(\Omega)$), and by Backlund and Fällström ([1] and [2], for $H^\infty(\Omega)$ and $A(\Omega)$ respectively), for convex domains in $\mathbb{C}^n$ having only a $C^{1+\epsilon}$-boundary.

Using his theorem on solvability of the $\overline{\partial}$-problem ([13]), Øvrelid proved in [14] that a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2-boundary has the Gleason A-property. The following results also use this important theorem.

Pervenuto alla Redazione il 12 giugno 2000 e in forma definitiva il 10 novembre 2000.

Weakly pseudoconvex domains Ω in $\mathbb{C}^2$ with C^∞-boundary, having the property that through every point $p \in \Omega$ there is a complex line which intersects $\partial\Omega$ only in strictly pseudoconvex points, have the Gleason A-property, as Beatrous Jr. proved in [5]. Fornæss and Øvrelid proved in [6] that a pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary has the Gleason A-property. This was extended by Noell ([12]) to pseudoconvex domains in $\mathbb{C}^2$ having a boundary of finite type.

Expanding ideas of [5], Backlund and Fällström proved in [4] that a bounded, pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary and containing the origin, has the Gleason A-property. For every $p \in \Omega$ they constructed a finite open covering of $\overline{\Omega}$, such that the Gleason-problem can be solved easily on each of its open sets; moreover the pairwise intersections of its open sets intersect the boundary only in strictly pseudoconvex points. Then a global solution is obtained by formulating an additive Cousin-problem and again using Øvrelids theorem. By using similar techniques, we prove that the result of Backlund and Fällström also holds without the assumption that the domain Ω contains the origin. In the second part of this paper we show that the condition that Ω needs to be pseudoconvex can be dropped.

Note that the Gleason problem is not always solvable; in fact, Backlund and Fällström showed ([3]) that there even exists an H^∞-domain of holomorphy on which the problem is not solvable.

Main result

Let Ω a bounded Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary. Then Ω has the Gleason-property with respect to both $A(\Omega)$ and $H^\infty(\Omega)$. In other words: given a function f in $A(\Omega)$ that vanishes at $p \in \Omega$, there exist functions f_1, $f_2 \in A(\Omega)$, such that $f = f_1(z_1 - p_1) + f_2(z_2 - p_2)$, and similarly for $H^\infty(\Omega)$.

Acknowledgements. Part of this paper was written while the first author was visiting Mid Sweden University, Sundsvall. He would like to thank the department of physics and mathematics for its hospitality. The authors are also grateful for the critical remarks of the referee, which led to a better presentation of, in particular, Lemma 1 and Theorem 9.

2. – Some definitions, notations and lemmas

We denote by $S(\Omega)$ the set of strictly pseudoconvex points in the boundary of Ω. For a function h, let $Z_h := \{z : h(z) = 0\}$ be its zero-set. We denote the boundary of a set D by ∂D, and $Co(D)$ will stand for the convex hull of D.

We recall that a domain in $\mathbb{C}^n$ is Reinhardt if it is invariant under the standard $\mathbb{T}^n$ action on $\mathbb{C}^n$ given by $(\Theta_1, \dots, \Theta_n)\cdot(z_1, \dots, z_n) \mapsto (e^{i\Theta_1}z_1, \dots, e^{i\Theta_n}z_n)$.

The map L: $z = (z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$ sends Ω to its logarithmic image $\omega = L(\Omega)$. The logarithmic image of Z_f is denoted by $L(Z_f)$, and $S(\omega)$ will stand for the C^2 strictly convex points of ω.

We recall some basic facts (cf. [11]) about the relation between Ω and ω: Ω is pseudoconvex $\Leftrightarrow$ ω is convex. If Ω has a C^2-boundary, ω will also have a C^2-boundary. Also note that a point z (having no zero coordinate) in $\partial\Omega$ is strictly pseudoconvex $\Leftrightarrow$ $L(z)$ is a strictly convex point of $\partial\omega$.

LEMMA 1. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Let λ be a smooth $\bar\partial$-closed $(0, 1)$-form on Ω, whose coefficients are bounded on Ω. If $\partial\omega\cap\overline{L(\mathrm{supp}\lambda)}$ consists of a finite number of bounded sets containing only C^2 strictly convex points, then there exists a function u in $C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\bar\partial u = \lambda$.*

PROOF. First suppose that Ω does not meet the axes $z_1 = 0$ and $z_2 = 0$. Then the logarithmic image ω of Ω is bounded. The logarithmic image X of $\overline{\mathrm{supp}\lambda} \cap \partial\Omega$ is a closed subset of the strictly convex points of $\partial\omega$. Hence there exists a bounded strictly convex domain $\tilde\omega \subset \mathbb{R}^2$ such that $\omega \subset \tilde\omega$ and $X \subset \partial\tilde\omega$.

Then $\tilde\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : L(z) \in \tilde\omega\}$ is a strictly pseudoconvex domain. The form λ can be trivially extended (by defining it to be 0 at $\tilde\Omega \setminus \Omega$) to a C^∞-form $\tilde\lambda$ on $\tilde\Omega$. Since $\tilde\Omega$ is strictly pseudoconvex, there exists a function $\tilde u \in C(\overline{\tilde\Omega})\cap C^\infty(\tilde\Omega)$ such that $\bar\partial\tilde u = \tilde\lambda$ ([13], p. 158-159). The restriction $u = \tilde u|_{\overline{\Omega}}$ has the desired properties.

Next suppose that Ω meets at least one of the axes. Keeping in mind that Ω is C^2 and pseudoconvex, there are two possibilities:

(1) $0 \in \Omega$. Then Ω meets each axis in a disk about 0.

(2) $0 \notin \Omega$. Then Ω meets only one of the axes, say the z_2-axis, in an annulus.

We will show how to deal with the second case, the first one being completely similar. Observe that

$$\Omega_0 = \{(z_1, z_2) : \log|z_2|+c|z_1|^2 < k, \log|z_2|-c|z_1|^2 > -k, |z_1| < \epsilon\}, \quad \epsilon, c, k > 0$$

is strictly pseudoconvex at the intersection of its boundary with the z_2-axis. Its logarithmic image is

$$\omega_0 = \{(x, y) : y + ce^{2x} < k, y - ce^{2x} > -k, e^x < \epsilon\}.$$

The logarithmic image ω of Ω is contained in a half-strip: $|y| < N$, $x < N$. Let $Y \subset \partial\omega$ be a (relative) neighborhood of X, contained in the strictly convex boundary points of ω. Let ω' be the intersection of the half-planes that contain ω and are tangent to ω at some point of Y. Then $\omega \subset \omega'$. Now we take $k > N$ and ϵ so small that $\omega_0 \subset \omega'$. As in the case where ω is bounded, we can find a strictly convex domain $\tilde\omega$ (with C^2-boundary), the boundary of which contains Y and the part of the boundary of ω_0 where x is sufficiently small. Now $\tilde\Omega := (\overline{L^{-1}(\tilde\omega)})^\circ$ has C^2-boundary, is strictly pseudoconvex, and we proceed as in the previous case. □

LEMMA 2. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary. Let $p \in \Omega$. Then there exist analytic polynomials g, h, open sets U_0, U_1, U_2 and a constant $\epsilon > 0$ such that:*

- $Z_g \cap Z_h \cap \overline{\Omega} = \{p\}$,
- U_0 *is strictly pseudoconvex, and* $p \in U_0 \subset\subset \Omega$,
- $|g| > \epsilon$ *on* U_1, $|h| > \epsilon$ *on* U_2,
- $\overline{\Omega} \subset \cup_i U_i$,
- $U_1 \cap U_2 \cap \partial\Omega \subseteq S(\Omega)$.

PROOF. First, we will construct the analytic polynomials g and h, then we construct the open sets U_i. We start with the case that Ω does not contain points with a zero coordinate, using the following elementary fact:
let ω a bounded, convex domain in $\mathbb{R}^2$, having C^2-boundary. Let $q \in \omega$. Then $\partial\omega$ contains 3 strictly convex points, u, v and w, such that q lies in the interior of the triangle uvw. Of course one can choose u, v and w such that the slope of the lines qu and vw are rational numbers.

Given a line l in $\mathbb{R}^2$ passing through $q = L(p_1, p_2)$ with rational slope $\pm\frac{m}{n}$, we construct a polynomial f in $\mathbb{C}^2$ such that $L(Z_f) = l$:
If $\frac{-m}{n} < 0$, $m, n > 0$, we take $f(z) = z_1^m z_2^n - p_1^m p_2^n$.
If $\frac{m}{n} > 0$, $m, n > 0$, we take $f(z) = z_2^n p_1^m - z_1^m p_2^n$.
(Just as in [4].) Choose $u, v, w \in \omega$ as above. Let g be a polynomial on $\mathbb{C}^2$ such that g vanishes at p and the logarithmic image of Z_g is a line in $\mathbb{R}^2$ passing through u. Similarly, let h be a polynomial on $\mathbb{C}^2$ such that h vanishes at p and the logarithmic image of Z_h is a line in $\mathbb{R}^2$ parallel to vw.

Now we are ready to construct the open covering $U_0 \cup U_1 \cup U_2$ of $\overline{\Omega}$.

$\partial\omega$ consists of 3 arcs, namely J_1 (from u to v), J_2 (from v to w), and J_3 (from w to u). Let S_1, S_2, S_3, S_4 be open (in the usual topology on $\partial\omega$) neighborhoods of u, u, v and w respectively, consisting only of strictly convex points, such that $\overline{S_1} \subset S_2$.

It is then possible to choose open sets $V_i \subset \mathbb{R}^2$ as follows: let V_1 such that $d(V_1, L(Z_h)) > \epsilon$, and $V_1 \cap \partial\omega = S_2 \cup (S_3 \cup J_2 \cup S_4)$. V_2 is chosen such that $d(V_2, L(Z_g)) > \epsilon$, and $V_2 \cap \partial\omega = (S_3 \cup J_1 \setminus \overline{S_1}) \cup (S_4 \cup J_3 \setminus \overline{S_1})$.

For sufficiently small ϵ there is a strictly convex set $V_0 \subset\subset \omega$ such that $\overline{\omega} \subset \cup V_i$ and $\partial\omega \subset V_1 \cup V_2$. Then $V_1 \cap V_2 \cap \partial\omega$ contains only strictly convex points. The sets $U_i := (\overline{L^{-1}(V_i)})^\circ$ fulfill the requirements of Lemma 2.

Suppose Ω meets only one of the axes, say $z_1 = 0$ (see [4] for the case $0 \in \Omega$). Let $p = (p_1, p_2) \in \Omega$. If $p_1 = 0$ one defines $g(z) := z_1$, $h(z) := z_2 - p_2$, and the rest is easy. Otherwise, the logarithmic image of $z_2 = p_2$ intersects $\partial\omega$ in only one point, a. Now draw a line through $L(p)$ parallel to the tangent line in a. It intersects $\partial\omega$ at two points, say b and c. Since the boundary of $\partial\omega$ between a and b, a and c are not straight lines, and ω is convex, there must be an extreme point d on the arc ab, and one, e, on the arc ac. These points d and e can be chosen such that they have neighborhoods of strictly convex points in $\partial\omega$, and that the line de has rational slope (since ω is convex with

C^2-boundary). Now we choose $g(z) = z_2 - p_2$, and h a polynomial such that h vanishes at p and the logarithmic image of Z_h is parallel to de. The sets U_i can be constructed as above. □

3. – The Gleason problem for pseudoconvex Reinhardt domains

The following result was obtained by Backlund and Fällström ([4]) under the extra assumption that Ω contains the origin.

THEOREM 3. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$, having C^2-boundary. Then Ω has the Gleason A-property.*

PROOF. We solve the Gleason-problem locally and patch the solutions together to a global solution using Lemma 1. Let $p \in \Omega$. Choose g, h, U_0, U_1 and U_2 as Lemma 2. Choose functions $\phi_k \in C_0^\infty(U_k)$, $k = 0, 1, 2$, such that $0 \leq \phi_k \leq 1$ and $\sum_{k=0}^2 \phi_k \equiv 1$ on $\overline{\Omega}$. Let f be a function in $A(\Omega)$, vanishing at p. Since f is holomorphic on Ω, $U_0 \subset\subset \Omega$, the lemma of Oka-Hefer (cf. [11]) implies that there exist functions f_1^0, f_2^0 in $A(U_0)$ such that

$$f = f_1^0(z_1 - p_1) + f_2^0(z_2 - p_2) \quad \text{on} \quad U_0 .$$

Let $F_1^1 = \frac{f}{g}$, $F_2^1 = 0$, $F_1^2 = 0$, $F_2^2 = \frac{f}{h}$. Then $F_i^k \in A(U_k \cap \Omega)$, and

$$(*) \qquad f = F_1^k g + F_2^k h \quad \text{on} \quad \overline{U_k} \cap \overline{\Omega} .$$

Since g is an analytic polynomial, vanishing at p, there are functions g_1, $g_2 \in H(\mathbb{C}^2)$ such that $g = g_1(z_1 - p_1) + g_2(z_2 - p_2)$ on $\mathbb{C}^2$. A similar formula holds for h. Substituting this in $(*)$, we obtain the existence of functions $f_i^k \in A(U_k \cap \Omega)$, $k = 1, 2$, such that

$$f = f_1^k(z_1 - p_1) + f_2^k(z_2 - p_2) \quad \text{on} \quad \overline{U_k} \cap \overline{\Omega}, \quad k = 0, 1, 2 ,$$

(with $f_i^k = F_1^k g_i + F_2^k h_i$.) So

$$j_1 := \sum \phi_k f_1^k \quad \text{and} \quad j_2 := \sum \phi_k f_2^k$$

give a smooth solution of our problem. We want to find u such that

$$(**) \qquad f_1 = j_1 + u(z_2 - p_2) \quad \text{and} \quad f_2 = j_2 - u(z_1 - p_1)$$

are in $A(\Omega)$. Define a form λ as follows:

$$\lambda := \frac{-\overline{\partial} j_1}{z_2 - p_2} = \frac{\overline{\partial} j_2}{z_1 - p_1} .$$

This form λ is a $\bar{\partial}$-closed $(0, 1)$-form on Ω, and can be continuously continued to $\overline{\Omega}$. Hence its coefficients are bounded on Ω. The support of λ is contained in $\overline{U_i} \cap \overline{U_j}$, $i \neq j$. Hence we have that $\overline{\text{supp}\lambda} \cap \partial\Omega \subseteq S(\Omega)$. Lemma 1 gives the existence of a function $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\bar{\partial}u = \lambda$. With this u, f_1, f_2 as defined at $(**)$,

$$f = f_1(z_1 - p_1) + f_2(z_2 - p_2) \text{ on } \overline{\Omega},$$

and f_1, f_2 both belong to $A(\Omega)$. This proves that the maximal ideal consisting of functions vanishing at p is generated by $(z_1 - p_1)$ and $(z_2 - p_2)$. □

4. – The Gleason problem for non pseudoconvex Reinhardt domains

To prove that a bounded Reinhardt domain in $\mathbb{C}^2$, with C^2-boundary, has the Gleason-property with respect to $A(\Omega)$ and H^∞, even if it is not pseudoconvex, we need some more machinery. This is developed in the following propositions and corollaries.

Definition. Given a set $V \subseteq \mathbb{R}^n$ and a point $v \in \partial V$, we say v is an extreme point of V if there do not exist r, $s \in \partial V$, distinct from v, and $\lambda \in (0, 1)$ such that $v = \lambda r + (1 - \lambda)s$. In other words: if v is an extreme point of $Co(\omega)$.

Note that V may be strictly convex at a point v without v being an extreme point of V.

Lemma 4. *Let g be a convex C^1-function such that $g(x) = g(0) + xg'(0) + o(x^2)$ at 0. Then g'' exists at 0 and equals 0.*

Proof. Without loss of generality, we take $g(0) = g'(0) = 0$. Since g' is increasing, we find for $y > 0$:

$$g(2y) = \int_0^{2y} g'(x)dx \geq \int_y^{2y} g'(x)dx \geq yg'(y),$$

so that $\frac{g'(y)}{y} \leq \frac{g(2y)}{y^2} = o(1)$. Therefore the second right derivative of g at 0 exists and equals 0. Similarly for the second left derivative. □

Proposition 5. *Let ω a domain in $\mathbb{R}^2$, with C^2-boundary. Denote by E the set of extreme points of ω. Then $\overline{E^\circ} = E$.*

Proof. We endow $\partial Co(\omega)$ with the relative topology. As $E^c \subseteq \partial Co(\omega)$ is clearly open, E is closed in $\partial Co(\omega)$ and $\overline{E^\circ} \subseteq E$. The complement of $\overline{E^\circ}$ in $\partial Co(\omega)$ is a union of disjoint open arcs. We will show that these arcs are in fact straight line segments. Take p in such an arc $U \subseteq \partial Co(\omega)$. If $p \notin E$, then p obviously lies on a straight line segment. So let $p \in E$. Then

$p \in \partial Co(\omega) \cap \partial\omega$. Since ω has C^2-boundary, $Co(\omega)$ has C^1-boundary (in fact it even has $C^{1,1}$-boundary, cf. [9]). After rotating and scaling we can assume that there exists $f \in C^2[-1,1]$ and $g \in C^1[-1,1]$ with the following properties:

- $p = (0, f(0)) = (0, g(0))$
- $X = \{(x, f(x)) : x \in [-1,1]\} \subseteq \partial\omega \cap [-1,1] \times [\min f, \max f]$
- $Y = \{(x, g(x)) : x \in [-1,1]\} \subseteq \partial Co(\omega) \cap [-1,1] \times [\min g, \max g]$
- g is convex
- $f \geq g$ on $[-1,1]$.

Note that $p \in X \cap Y$ and therefore the tangent to $\partial\omega$ at p equals the tangent to $\partial Co(\omega)$ at p: $g'(0) = f'(0)$. Furthermore, since $p \in E$, $f''(0) \geq 0$. But if $f''(0) > 0$, then $p \in \overline{E^\circ}$. Hence $f''(0) = 0$. It follows that

$$g(0) + xg'(0) \leq g(x) \leq f(x) = f(0) + xf'(0) + o(x^2) = g(0) + xg'(0) + o(x^2).$$

So $g(x) = g(0) + xg'(0) + o(x^2)$. Application of the previous lemma gives that $g''(0) = 0$. Since we can repeat the argument for every point of Y, it follows that g'' is identically 0 on $[-1,1]$, meaning that Y is a straight line segment. Of course U is a straight line segment too. This yields that U (and any other subset of $\overline{E^\circ}^c$) does not contain extreme points, hence $E \subseteq \overline{E^\circ}$. □

LEMMA 6. *Let ω and E be as above, let $e \in E$. There exists a point $b \in E$ arbitrary close to e such that $\partial\omega$ and $\partial Co(\omega)$ coincide on a neighborhood B of b. Furthermore, this neighborhood can be chosen such that it (as part of $\partial Co(\omega)$) consists only of strictly convex points.*

PROOF. $\overline{E^\circ} = E$, hence one can choose a point $a \in E$ arbitrary close to e, such that there is a neighborhood A of a containing only extreme points of ω. Since the extreme points of ω and $Co(\omega)$ are the same, $\partial\omega$ and $\partial Co(\omega)$ coincide on A. Hence the defining function ρ for $\partial Co(\omega)$ can be chosen such that it is a C^2-function around A. There is a point $b \in A$ for which $\rho''(b) > 0$. Then there is a neighborhood $B \subseteq A$ of b on which ρ'' is strictly positive. □

THEOREM 7. *A bounded Reinhardt domain Ω in $\mathbb{C}^2$, with C^2-boundary, has the Gleason-property with respect to both $A(\Omega)$ and $H^\infty(\Omega)$.*

PROOF. First let $f \in A(\Omega)$, $p \in \Omega$ such that $f(p) = 0$. Note that f extends to $\tilde{\Omega}$, the holomorphic hull of Ω, and that $L(\tilde{\Omega}) = Co(\omega)$.

Suppose ω is bounded. There are extreme points e_1, e_2, $e_3 \in \partial\omega$ with the property $L(p) \in Co(e_1, e_2, e_3)$. According to the previous lemma there exist points a, b, c, arbitrarily close to e_1, e_2, e_3, having neighborhoods A, B, C respectively, containing only strictly convex points such that A, B, $C \subset \partial\omega \cap \partial Co(\omega)$. These a, b and c could have been chosen such that the slopes of the lines ab and $L(p)c$ are rational. Just as in Lemma 2, we construct polynomials g and h that vanish at p, such that $L(Z_g)$ is a line through $L(p)$ and c, and $L(Z_h)$ is a line through $L(p)$ parallel to ab. Then one can construct the appropriate covering of $Co(\omega)$, and simply copy the proof of Theorem 3.

Now suppose ω is not bounded. We only consider the case that Ω contains points of the form $(0, a)$; the other cases can be solved similarly. Applying the ideas of the second part of Lemma 2 (to $Co(\omega)$ instead of to ω) yields the appropriate polynomials g, h and sets U_i. Repeating the proof of Theorem 3 proves the assertion.

Next let $f \in H^\infty(\Omega)$, $p \in \Omega$ such that $f(p) = 0$. Like above, we obtain an open covering $\{U_i\}$ of $\overline{\Omega}$, and matching functions ϕ_i. As in the proof of Theorem 3, we obtain a $(0, 1)$-form λ:

$$\frac{-\overline{\partial}\left(\sum \phi_k f_1^k\right)}{z_2 - p_2} = \lambda = \frac{\overline{\partial}\left(\sum \phi_k f_2^k\right)}{z_1 - p_1}.$$

The functions f_i^k are bounded and holomorphic. $\phi_k \in C_0^\infty(U_k)$, so $\overline{\partial}\phi_k$ is bounded. The function $\min(|\frac{1}{z_1-p_1}|, |\frac{1}{z_2-p_2}|)$ is bounded on suppλ, since $d(p, U_i \cap U_j \cap \Omega) > \delta$. Hence the form λ is bounded on Ω, and we can apply lemma 1 to find a function $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ with $\overline{\partial}u = \lambda$. Now copy the proof of Theorem 3. □

REMARK. The crux of this approach is to formulate a $\overline{\partial}$-problem ($\overline{\partial}u = \lambda$) on Ω suitable for solving the Gleason problem, in such a way that λ can be extended by 0 to a larger domain $\widehat{\Omega}$ where $\overline{\partial}u = \lambda$ has a good solution. While $\widehat{\Omega}$ is strictly pseudoconvex in our situation, the method will give results in some other cases, e.g. when $\widehat{\Omega}$ is an analytic polyhedron in $\mathbb{C}^2$.

PROPOSITION 8. *Let $\omega \subset \mathbb{R}^2$. If the set of C^2-boundary points of ω contains a dense subset of E, then $\overline{E^\circ} = E$.*

PROOF. We endow $\partial Co(\omega)$ and $\partial\omega$ with the relative topology. As $E^c \subseteq \partial Co(\omega)$ is clearly open, E is closed in $\partial Co(\omega)$, thus $\overline{E^\circ} \subseteq E$. To prove the other inclusion, suppose $e \in \overline{E^\circ}^c \cap E$. This point e cannot be an isolated point of E; then it would be in this dense subset of E. But such points have a neighborhood consisting of C^2-boundary points in $\partial\omega$, thus a neighborhood consisting of C^1-boundary points in $\partial Co(\omega)$.

Therefore there would be a sequence $\{e_n\}$ of C^2-boundary points in $\overline{E^\circ}^c \cap E$ that converges to e. However, the proof of proposition 5 shows that such points e_n do not exist. This is a contradiction, hence $\overline{E^\circ}^c \cap E = \emptyset$, and $E \subseteq \overline{E^\circ}$. □

THEOREM 9. *Let $\Omega \subset \mathbb{C}^2$ be a bounded Reinhardt domain. Suppose ω is bounded as well, and that the set of C^2-boundary points of ω contains a dense subset of E. Then one can solve the Gleason-problem for both $A(\Omega)$ and $H^\infty(\Omega)$.*

PROOF. Using proposition 8 we can repeat the proof of Theorem 7. □

REMARK. The only thing that matters is that there are enough strictly pseudoconvex points in the boundary of Ω to make a "good" cover of Ω. This can e.g. be done in the setting of theorem 9 if we merely assume that $0 \notin \overline{\Omega}$ instead of ω being bounded. In that case, given a point $p \in \Omega$, one takes

$g(z) = z_2 - p_2$ (if Ω contains points of the form $(0, a)$) or $g(z) = z_1 - p_1$ (if Ω contains points of the form $(a, 0)$), and proceeds like, e.g., in Lemma 2.

We do not know if the Gleason problem can be solved for a bounded domain $\Omega \subset \mathbb{C}^2$ of the form $|z_1|^2 < |z_2|^3 < 2|z_1|^2$ for $|z_1| \leq 1$, that is rounded off in a strictly pseudoconvex way for larger z_1.

5. – An example

Let Ω be a bounded convex domain in $\mathbb{C}^n$. For every $f \in H^\infty(\Omega)$, vanishing at p, the Leibenzon-divisors ψ_i are defined in the following way:

$$\psi_i(z_1, \ldots, z_n) := \int_0^1 \frac{\partial f}{\partial z_i}(p + t(z - p))dt \quad i = 1, \ldots, n.$$

If in addition Ω has C^2-boundary, then

$$\psi_i \in H^\infty(\Omega), \quad f(z) = \sum_{i=1}^n (z_i - p_i)\psi_i(z) \quad \forall z \in \Omega,$$

as Leibenzon proved in [10]. In [8] Grangé was able to show that the functions ψ_i remain in $H^\infty(\Omega)$ if Ω only has $C^{1+\epsilon}$-boundary. There he also gave the following exampl: let $h(x) := \frac{-x}{\log x}$ for $x > 0$, $h(0) := 0$. Let

$$\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1, |z_1|^2 + h(|z_2|) - 1 < 0\}.$$

Ω is convex, $\partial\Omega$ is C^1, even C^∞ and strictly pseudoconvex at the points $(z_1, z_2) \in \Omega$, $z_2 \neq 0$. Then a function $\phi \in H^\infty(\Omega)$ was given for which the Leibenzon-divisor $\psi_2 \notin H^\infty(\Omega)$.

However, Ω satisfies the conditions as described in the remark after Theorem 9 and hence there exist functions f_1 and f_2 in $H^\infty(\Omega)$ such that $\phi(z) = z_1 f_1(z) + z_2 f_2(z)$.

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