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Smoothness in Fractional Evolution Equations and Conservation Laws

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Abstract. The regularity of solutions of the equation

\[ \left( D_\alpha^\sigma (u - u_0) \right) (t, x) + \sigma (u)_x (t, x) = f (t, x), \quad t, x \geq 0, \]

where \( D_\alpha^\sigma \) denotes the fractional derivative, is studied in the case where \( \sigma' > 0 \). It is also shown that the solution to the Riemann problem for the fractional Burgers equation (where \( \sigma (t) = \frac{1}{2} t^2 \)) is continuous and has compact support (in the \( x \)-direction). A result on the continuity of the interface is established. In order to prove these results it is first shown that if \( A \) is an \( m \)-accretive operator in a Banach space, \( k \) is log-convex with \( \lim_{t \to 0} k(t) = +\infty \), and if \( u \) is the solution of

\[ \frac{d}{dt} \int_0^t k(t-s)(u(s) - y) \, ds + A(u(t)) \geq f(t), \quad t > 0, \quad u(0) = y, \]

then \( A(u(t)) \) is continuous when \( t > 0 \).

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1. – Introduction

Recently a new type of approximation of scalar conservation laws in several variables has been introduced in [3]. Rather than adding a viscosity term (for this approach see, e.g., [8]), the order of derivation with respect to time is lowered, that is, the derivative is replaced by a fractional derivative of order \( \alpha \in (0, 1) \). Furthermore, instead of using the Crandall-Liggett theorem as is done in [4], another abstract result, [10], is employed to establish the existence of a strong solution. In [3] the convergence of these strong solutions as \( \alpha \uparrow 1 \) to the entropy solution of \( u_t + \text{div} g(u) = 0 \) is proven and some estimates for the speed of convergence are established.

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The aim of this paper is to investigate further these solutions in the one-dimensional case, i.e., we analyze the regularity of solutions of the nonlinear fractional conservation law

\[ D_t^\alpha (u - u_0) + \sigma (u)_x = f. \]

Here \( D_t^\alpha \) denotes the fractional derivative of order \( \alpha \in (0, 1) \), see [15, p. 133], i.e.,

\[
(D_t^\alpha v)(t) \overset{\text{def}}{=} \frac{d}{dt} \int_0^t g_{1-\alpha}(t-s)v(s) \, ds, \quad t > 0,
\]

\[
(D_t^\alpha v)(0) \overset{\text{def}}{=} \lim_{h \to 0} \frac{1}{h} \int_0^h g_{1-\alpha}(h-s)v(s) \, ds,
\]

where

\[ g_\beta(t) \overset{\text{def}}{=} \frac{1}{\Gamma(\beta)} t^{\beta-1}, \quad t > 0, \quad \beta > 0, \]

and where \( v \) is (at least) continuous and satisfies \( v(0) = 0 \).

As an important tool for studying this equation we consider the abstract fractional nonlinear evolution equation

\[ \frac{d}{dt} \int_0^t k(t-s)(u(s) - y) \, ds + A(u(t)) \ni f(t), \quad t > 0, \quad u(0) = y. \]

In (2), \( u \) is the unknown function with range in a Banach space \( X \), \( y \in X \) and \( f : \mathbb{R}^+ \to X \) are given, and \( k \) is a locally integrable real-valued function with a singularity at the origin. The nonlinear operator \( A \) may be multivalued and maps \( D(A) \subset X \) into (subsets of) \( X \). Our primary current interest concerns the continuity and boundedness of the function \( A(u(t)) \).

In [10], the existence of a strong solution \( u \) of (2), satisfying \( A(u) \in L^1_{\text{loc}}(\mathbb{R}^+; X) \), was obtained. Conditions implying that the solution \( u \) is continuous were given in [3].

In this paper we demonstrate that under rather weak hypotheses one has \( A(u) \in C((0, \infty); X) \). In addition this function is uniformly bounded on \((0, T]\) for each \( T > 0 \). Subsequently, these facts are applied to examine the regularity of the solution of (1).

As a first application we get the continuity of the solution of the Riemann problem for the fractional Burgers equation, i.e., for equation (1) with \( \sigma (u) = \frac{1}{2} u^2 \) and \( f = 0 \). This improves on a result of [11] concerning (1). (In [11] it was assumed that \( \sigma'(u) \geq c_0 > 0 \); an assumption not satisfied by \( \sigma(u) = \frac{1}{2} u^2 \).)

The special structure of the fractional Burgers equation implies that the solution vanishes when \( x \geq \Gamma(1 - \alpha)t^\alpha \), in contrast to the linear case where there is an infinite speed of propagation. We also establish a result on the continuity of the interface. Recall that the entropy solution to the Riemann problem for the (nonfractional) Burgers equation is 1 when \( x < \frac{1}{2} \) and 0 when \( x > \frac{1}{2} \).

A motivation for studying the Riemann problem is, of course, that it is the simplest case where one has a discontinuity. Recall also that many numerical
methods use the solution to the Riemann problem (with other constant states than just 1 and 0) and that this problem provides all solutions to the Cauchy problem \( u_t + \sigma(u)_x = 0 \) which are invariant under the group of homotheties \((t, x) \mapsto (at, ax)\). This group leaves first order conservation laws invariant, see [14, p. 43].

Furthermore, in Theorem 3 the results obtained on (2) are combined with earlier Schauder estimates on linear equations, [2], to establish results on the smoothness of solutions of (1).

The regularity, both temporal and spatial, of solutions of equations involving fractional derivatives of order \( \alpha \in (1, 2) \) have been studied in several papers; [5], [6], and [7]. See also the monograph [12] for further results and references.

2. – Statement of results

Our result on (2) is the following.

**Theorem 1.** Assume that \( X \) is a real Banach space and that

(i) \( k \in L^1_{loc}(\mathbb{R}^+; \mathbb{R}) \) is positive and nonincreasing, \( \lim_{t \to 0} k(t) = +\infty \), and \( \log(k(t)) \) is convex;

(ii) \( A \) is an \( m \)-accretive operator on \( X \);

(iii) \( y \in D(A) \), i.e., \( y \in X \) and \( \sup_{\lambda > 0} \| A_\lambda y \|_X < \infty \);

(iv) \( f \in C(\mathbb{R}^+; X) \) is such that \( \int_0^T \omega_{f,t}(s)|k'(s)| \, ds < \infty \) for each \( T > 0 \) where \( \omega_{f,t} \)

is the modulus of continuity of \( f \), i.e., \( \omega_{f,t}(s) \) is nonincreasing, \( \omega_{f,t}(s) \) is convex;

Then there is a unique strong solution \( u \) of (2) such that \( u(0) = y \), and there is a function \( w \in C((0, \infty); X) \) such that \( \sup_{0 < t < T} \| w(t) \|_X < \infty \) for each \( T > 0 \), \( w(t) \in A(u(t)) \) for all \( t > 0 \) and

\[
\frac{d}{dt} \int_0^t k(t-s)(u(s) - y) \, ds + w(t) = f(t), \quad t > 0.
\]

Moreover, if \( 0 \leq t < t + h \leq \tau \) then

\[
\| u(t + h) - u(t) \|_X \leq \int_0^t \| f(t + h - s) - f(t - s) \|_X r(s) \, ds
\]

\[
+ \left( \sup_{\tau \in [0,h]} \| f(\tau) \|_X \right) \sup_{\lambda > 0} \| A_\lambda (y) \|_X \int_t^{t+h} r(s) \, ds,
\]

where \( r \) is the first kind resolvent of \( k \), i.e.,

\[
\int_{[0,t]} k(t-s) r(s) \, ds = 1, \quad t \in (0, \tau].
\]
Here \( A_\lambda \) denotes the Yosida approximation of \( A \), i.e., \( A_\lambda \equiv \frac{1}{\lambda} (I - J_\lambda) \) where \( J_\lambda = (I + \lambda A)^{-1} \).

A function \( u : \mathbb{R}^+ \to X \) is a strong solution of (2) if there exists a function \( w \in L^1_{\text{loc}}(\mathbb{R}^+; X) \) such that \( w(t) \in A(u(t)) \) a.e. on \( \mathbb{R}^+ \) and
\[
\int_0^t k(t-s)(u(s) - y) \, ds = \int_0^t (f(s) - w(s)) \, ds
\]
for every \( t \geq 0 \).

Our next result concerns the homogeneous version of (1) with, essentially \( \sigma(r) = cr^\gamma, \gamma > 1 \). In particular, this includes the fractional Burgers equation.

**Theorem 2.** Assume that

(i) \( k \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R}) \) is positive and nonincreasing, \( \lim_{t \to 0} k(t) = +\infty \), and \( \log(k(t)) \) is convex;

(ii) \( \sigma \in C^1(\mathbb{R}; \mathbb{R}) \) is strictly increasing on \( (0, 1) \) and there are constants \( C \) and \( \gamma > 1 \) such that
\[
\frac{1}{C} r^\gamma \leq \sigma(r) \leq C r^\gamma, \quad r \in [0, 1].
\]

Then there is a solution \( u \) of the Riemann problem

\[
\frac{d}{dt} \int_0^t k(t-s)(u(s,x)-\chi_{(-\infty,0)}(x)) \, ds + \sigma(u)_x(t,x) = 0, \quad t > 0, \quad x \in \mathbb{R},
\]

\[
u(0,x) = \chi_{(-\infty,0)}(x), \quad x \in \mathbb{R},
\]

which is continuous for \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\} \) and is such that for each \( t > 0 \) the function \( x \mapsto u(t,x) \) is absolutely continuous and nonincreasing, for each \( x \in \mathbb{R} \) the function \( t \mapsto u(t,x) \) is nondecreasing (so that the function \( t \mapsto \int_0^t k(t-s)(u(s,x)-\chi_{(-\infty,0)}(x)) \, ds \) is locally absolutely continuous), and equation (6) holds a.e. on \( \mathbb{R}^+ \times \mathbb{R} \). Moreover,

\[
u(t,x) = 0 \text{ when } x \geq \frac{1}{k(t)} \int_0^1 \frac{\sigma'(r)}{r} \, dr, \quad t > 0,
\]

and the function
\[
\varphi(t) \equiv \inf \{ x > 0 \mid u(t,x) = 0 \}
\]
is continuous and strictly increasing.

Let \( X \) be a (complex) Banach space and let \( I \) be an interval. The Hölder spaces \( C^{(\gamma)}(I; X), \gamma \in [0, 1] \), are defined by

\[
C^{(\gamma)}(I; X) \equiv \left\{ f : I \to X \left| \sup_{s,t \in I, s \neq t} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma} < \infty \right. \right\}.
\]

with norm
\[
\|f\|_{C^{(\gamma)}(I; X)} \equiv \sup_{t \in I} \|f(t)\|_X + \sup_{s,t \in I, s \neq t} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma}.
\]
If $\gamma \in (1, 2]$, then $C^{(\gamma)}(I; X)$ is defined as $\{ f \in C^1(I; X) \mid f' \in C^{(\gamma-1)}(I; X) \}$ with norm $\| f \|_{C^{(\gamma)}(I; X)} \equiv \sup_{t \in I} \| f(t) \|_X + \| f' \|_{C^{(\gamma-1)}(I; X)}$. Observe that $C^{(0)} \neq C$ and $C^{(1)} \neq C^1$.

We consider a function of two variables to be a function of the first variable with values in a function space, that is, $f(t, x)$ is the function $t \mapsto (x \mapsto f(t, x))$.

**Theorem 3.** Assume that $\alpha \in (0, 1)$, $\tau > 0$, $\xi > 0$, $u \in (0, \alpha)$, and that

(i) $\sigma \in C^{(1)}(\mathbb{R}; \mathbb{R})$ and $\sigma'(x) > 0$;

(ii) $u_0 \in C^{(1+\frac{\sigma}{\xi})}([0, \xi]; \mathbb{R})$ and $u_0(0) = u_0'(0) = 0$;

(iii) $f \in C([0, \tau], C([0, \xi]; \mathbb{R})) \cap C^{(\alpha+\delta)}([0, \tau], L^1([0, \xi]; \mathbb{R}))$ where $\delta > 0$, and $f(t, 0) = 0$ and $f(0, x) \in C^{(\alpha)}(\mathbb{R}; \mathbb{R})$.

Then there is a unique solution $u$ of (1) on $(0, \tau] \times (0, \xi]$ with $u(t, 0) = 0$ and $u(0, x) = u_0(x)$ such that $u \in C([0, \tau]; C([0, \xi]; \mathbb{R}))$.

### 3. Proofs

**Proof of Theorem 1.** Let $(k_n)_{n=1}^\infty$ be a sequence of functions that satisfy the assumption (i), except that $\lim_{t \to 0} k_n(t) = \infty$, and are such that $\lim_{n \to \infty} \int_0^t k_n(s) \, ds = \int_0^t k(s) \, ds$, $\lim_{n \to \infty} k'_n(t) = k(t)$, $\lim_{n \to \infty} k'_n(t) = k'(t)$, and $|k'_n(t)| \leq |k'(t)|$ for all $t > 0$. We let $\rho_n$ be the first kind resolvent associated with $k_n$ (cf. (5)); thus $\rho_n$ satisfies

$$\int_{[0, t]} k_n(t-s) \rho_n(ds) = 1, \quad t \geq 0.$$  

The measure $\rho_n$ then has the pointmass $1/k_n(0)$ at 0 and is otherwise induced by an integrable function, that is

$$\rho_n([0, t]) = \frac{1}{k_n(0)} + \int_0^t r_n(s) \, ds, \quad t \geq 0,$n

where $r_n$ is nonnegative and nonincreasing, because $k_n$ is log-convex, see [9, Lemma 2.1]. When $k$ is replaced by $k_n$ one can use (ii) and a standard fixed-point argument to show that there is a unique solution of (2); we denote this solution by $u_n$. It is a consequence of [3, Theorem 1] that $u_n$ converges uniformly on compact subsets of $\mathbb{R}^+$ to a continuous function $u$. However, we need to know more. In particular our next purpose is to show that $w \in C((0, \infty); X)$ where $w(t) \in A(u(t))$ is defined by (14).
By [3, formula (24)] we have for $0 \leq t < t + h$

$$
\|u_n(t + h) - u_n(t)\|_X \leq \int_{[0,t]} \|f(t + h - s) - f(t - s)\|_X \rho_n(ds)
$$

(8) $$
+ \left( \sup_{\tau \in [0,h]} \|f(\tau)\|_X + \|A_{1/k_n}(0)\|_X \right)
\int_{[0,t]} \left( \int_{[0,h]} (k_n(t - s) - k_n(t - s + h - \sigma)) \rho_n(d\sigma) \right) \rho_n(ds).
$$

Now a straightforward calculation using (5) (with $k$ and $r$ replaced by $k_n$ and $\rho_n$, respectively) shows that

$$
\int_{[0,t]} \left( \int_{[0,h]} (k_n(t - s) - k_n(t - s + h - \sigma)) \rho_n(d\sigma) \right) \rho_n(ds)
$$

(9) $$
= \rho_n((t, t + h)) = \int_t^{t + h} r_n(s) \, ds.
$$

By [3, Theorem 1], (8), (9), and by the fact that $\lim_{n \to \infty} \rho_n([0, t]) = \int_0^t r(s) \, ds$, we get (4).

By a change of variables,

$$
\int_0^h \left( \int_{t-s}^{t} r_n(\sigma) \, d\sigma \right) |k'_n|(s) \, ds = \int_{t-h}^t (k_n(t - \sigma) - k_n(h)) r_n(\sigma) \, d\sigma, \quad 0 < h \leq t.
$$

Since the functions $r_n$ are nonincreasing, it follows that

(10) $$
\lim_{h \downarrow 0} \int_0^h \left( \int_{t-s}^{t} r_n(\sigma) \, d\sigma \right) |k'_n|(s) \, ds = 0,
$$

uniformly for $n \geq 1$ and uniformly for $t$ in a compact subset of $(0, \infty)$. Since $|k'_n(t)| \leq |k'(t)|$ we deduce from (iv) that

(11) $$
\lim_{h \downarrow 0} \int_0^h \omega_{f,t}(s)|k'_n(s)| \, ds = 0 \text{ uniformly in } n.
$$

Use (9) in (8), replace $t + h$ and $t$ by $t$ and $t - s$, respectively, multiply by $|k'_n(s)|$, integrate with respect to $s$ over $[0, h]$ and let $h \downarrow 0$. This gives, by (10) and (11),

(12) $$
\lim_{h \downarrow 0} \int_0^h \|u_n(t - s) - u_n(t)\|_X |k'_n(s)| \, ds = 0,
$$

uniformly for $n \geq 1$ and uniformly for $t$ in a compact subset of $(0, \infty)$. 

Now we can rewrite (2) (with k replaced by $k_n$) for each $t \geq 0$ as

\begin{equation}
  k_n(t)(u_n(t) - y) + \int_0^t (u_n(t) - s) - u_n(t) \rangle_k(s) ds + A(u_n(t)) \ni f(t).
\end{equation}

By (12), and as $u_n$ converges uniformly on compact subsets of $\mathbb{R}^+$ to the continuous function $u$, it follows that $k_n(t)(u_n(t) - y) + \int_0^t (u_n(t) - s) - u_n(t) \rangle_k(s) ds$ converges uniformly on each compact subset of $(0, \infty)$ to $k(t)(u(t) - y) + \int_0^t (u(t) - s) - u(t) \rangle_k'(s) ds$ which must then be a continuous function on $(0, \infty)$. Let

\begin{equation}
  w(t) \overset{\text{def}}{=} f(t) - k(t)(u(t) - y) - \int_0^t (u(t) - s) - u(t) \rangle_k'(s) ds,
\end{equation}

so that (3) holds with $w \in C((0, \infty), X)$. Since $A$ is $m$-accretive it is also closed and therefore we have by (13) and by the convergence results that $w(t) \in A(u(t))$ for all $t > 0$.

It remains to show that $w$ is bounded on $(0, T]$ for each $T > 0$. Since $u(0) = y$ we get from (4), when we take $t = 0$, that

\begin{equation*}
  \|u(h) - y\|_X \leq \left( \sup_{\tau \in [0, h]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right) \int_0^h r(s) ds, \quad h > 0.
\end{equation*}

Because $k$ is nonincreasing there follows by (5) that $\int_0^h r(s) ds \leq 1$ so that

\begin{equation*}
  \|k(t)(u(t) - y)\|_X \leq \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right).
\end{equation*}

Similarly, replace $t$ and $t + h$ in (4) by $t - s$ and $t$, respectively, multiply by $|k'(s)|$ and integrate over $[0, t]$ to obtain

\begin{equation*}
  \left\| \int_0^t (u(t - s) - u(t)) k'(s) ds \right\| \leq \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_X + \sup_{\lambda > 0} \|A_\lambda(y)\|_X \right) \int_0^t \left( \int_0^s r(\sigma) d\sigma \right) |k'(s)| ds
\end{equation*}

Moreover, by (5),

\begin{equation*}
  \int_0^t \left( \int_0^s r(\sigma) d\sigma \right) |k'(s)| ds = \int_0^t (k(t - s) - k(t)) r(\sigma) d\sigma \leq 1,
\end{equation*}

and so by the fact that $k$ and $r$ are nonnegative and by (iii) and (iv) we get the desired conclusion.
PROOF OF THEOREM 2. Since we will show that the solution takes its values in the interval \([0, 1]\) we may without loss of generality assume that \(\sigma \in C^1(\mathbb{R}; \mathbb{R})\) is strictly increasing on \(\mathbb{R}\).

We easily see that by taking \(u(t, x) = 1\) for \(x \leq 0\) and \(t \geq 0\) we have a solution in that region and we are left with the equation

\[
\frac{\partial}{\partial t} \int_0^t k(t-s)u(s, x) \, ds + \sigma(u)_x(t, x) = 0, \quad t > 0, \quad x > 0, \tag{15}
\]

\[u(t, 0) = 1, \quad t > 0, \]

\[u(0, x) = 0, \quad x > 0.\]

In [11, Lemma 3] it is shown that if one lets \(D(A) = \{ u \in L^1(\mathbb{R}^+; \mathbb{R}) \mid \sigma(u) \in AC(\mathbb{R}^+; \mathbb{R}), u(0) = 1, \sigma(u) \in L^1(\mathbb{R}^+; \mathbb{R}) \}, \) and defines \(A(u) = \sigma(u)'\), \(u \in D(A)\), then \(A\) is a closed, \(m\)-accretive operator in \(L^1(\mathbb{R}^+; \mathbb{R})\). By [11, Theorem 5] there exists a solution \(u\) of (15), which is nonincreasing in the \(x\)-variable and nondecreasing in the \(t\)-variable, such that the function \(x \mapsto \sigma(u(t, x))\) is absolutely continuous for almost every \(t > 0\), and such that the function \(t \mapsto \int_0^t k(t-s)u(s, x) \, ds\) is locally absolutely continuous for every \(x \geq 0\), and (15) holds almost everywhere.

By Theorem 1 we know that the function \(t \mapsto \sigma(u(t, x))\) is continuous on \((0, \infty)\) and that (15) holds in \(L^1(\mathbb{R}^+; \mathbb{R})\) for all \(t > 0\). Since \(\sigma(u(t, 0)) = \sigma(1)\) for all \(t > 0\) and \(\sigma(u(t, x)) = \int_0^t \sigma(u(t, y))_x \, dy + \sigma(u(t, 0))\) it follows that \(\sigma(u)\) is continuous in \((0, \infty) \times \mathbb{R}^+\) and since \(\sigma\) is strictly increasing the same result holds for \(u\). By Theorem 1 we also know that \(u(t, x) \rightarrow 0\) in \(L^1(\mathbb{R}^+; \mathbb{R})\) as \(t \downarrow 0\) and from the monotonicity properties of \(u\) we can therefore conclude that \(u\) is continuous in \(\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0, 0)\}\).

Next we derive an inequality that we will use repeatedly below. Assume that \(x_0 \overset{\text{def}}{=} \varphi(t_0) < \infty\) and that \(x_0 < x_1 \leq \varphi(t_1)\) where \(t_1 > t_0 \geq 0\). From the proof of Theorem 1 we know that for each \(t > t_0\) we have

\[
\frac{\partial}{\partial t} \int_0^t k(t-s)u(s, x) \, ds \overset{\text{a.e.}}{=} k(t)u(t, x) + \int_0^t (u(t-s, x) - u(t, x))k'(s) \, ds, \quad x > 0, \]

(where the derivative with respect to \(t\) is a function with values in \(L^1(\mathbb{R}^+; \mathbb{R})\)). Since \(u(s, x) = 0\) when \(s \leq t_0\) and \(x > x_0\) (by the monotonicity properties of \(u\)), we can rewrite this equality for \(t > t_0\) as

\[
\frac{\partial}{\partial t} \int_0^t k(t-s)u(s, x) \, ds \overset{\text{a.e.}}{=} k(t-t_0)u(t, x) + \int_{t-t_0}^t (u(t-s, x) - u(t, x))k'(s) \, ds, \quad x > x_0. \]

Because \(k\) is nonincreasing and \(u\) is nondecreasing in its first variable, it follows from the fact that (1) (or equivalently (3)) holds that for each \(t > t_0\) we get

\[k(t-t_0)u(t, x) + \sigma'(u(t, x))u_x(t, x) \overset{\text{a.e.}}{\leq} 0, \quad x > x_0.\]
In particular, if we choose \( t = t_1 \), then we know that \( u(t, x) > 0 \) for \( x_0 < x < x_1 \) and it follows by the continuity of \( u \) that

\[
(16) \quad k(t_1 - t_0)(x_1 - x_0) \leq \int_{u(t_1, x_1)}^{u(t_0, x_0)} \frac{\sigma'(r)}{r} \, dr.
\]

Since clearly \( \varphi(0) = 0 \) we may take \( t_0 = 0 \). Because the function \( \frac{\sigma'(r)}{r} \) is integrable on \([0, 1]\) and \( k(t) > 0 \), we see from (16) that we have \( \varphi(t_1) < \infty \) and that (7) holds.

The monotonicity properties of \( u \) imply that \( \varphi \) is nondecreasing. By the continuity of the function \( u \) it follows that \( \varphi \) is continuous from the left, so in order to establish the claim about continuity we suppose to the contrary that there is a point \( t_0 \geq 0 \) such that \( \lim_{t \downarrow t_0} \varphi(t) = \varphi(t_0) + \delta \) for some \( \delta > 0 \). If we choose \( x_0 \equiv \varphi(t_0) \) and \( x_1 = x_0 + \delta \), then \( x_1 < \varphi(t_1) \) for each \( t_1 > t_0 \) and we get a contradiction from (16) if we let \( t_1 \downarrow t_0 \). Thus we have established the continuity of \( \varphi \).

It remains to prove that \( \varphi \) is strictly increasing. Suppose that this is not the case but that there are two points \( t_1 < t_2 \) such that \( \varphi(t_1) = \varphi(t_2) \). By the continuity of \( u \) we know that \( t_1 > 0 \) and that we can choose \( t_1 \) such that \( \varphi(t_1) \) when \( 0 < t_0 < t_1 \) and \( x_0 = \varphi(t_0) \) we have

\[
k(t_1 - t_0)(x_1 - x_0) \leq \frac{\gamma}{\gamma - 1} C^{\frac{\gamma - 1}{\gamma}} \sigma_u(t_1, x_1).
\]

Since \( \varphi(t) < \varphi(t_1) \) when \( 0 \leq t < t_1 \) it follows that \( t_0 \uparrow t_1 \), and hence \( k(t_1 - t_0) \uparrow \infty \), when \( x_0 \uparrow x_1 \). By the above inequality we therefore obtain (17).

Next, let \( y \) be some small positive number and integrate both sides of equation (15) over \((x_1 - y, x_1)\). Then we get, because \( u(t, x_1) = 0 \) for all \( t \in (0, t_2) \),

\[
(18) \quad \frac{d}{dr} \int_0^r k(t - s) \int_{x_1 - y}^{x_1} u(s, v) \, dv \, ds = \sigma_u(t_1, x_1 - y), \quad t \in (0, t_2).
\]

We let \( r \) be the resolvent of first kind of \( k \), that is, \( r \) satisfies (5). Our assumptions on \( k \) guarantee that such a resolvent exists and that it is positive and nonincreasing, see [9, Lemma 2.1]. Take the convolution (with respect to \( r \)
of both sides of (18) with the function \( p(t) \stackrel{\text{def}}{=} \int_0^t r(t - s) s^\alpha \, ds \) where \( \alpha > \frac{2 - \gamma}{\gamma - 1} \).

By (5),

\[
\int_0^{t_2} (t_2 - s)^\alpha \int_{x_1 - y}^{x_1} u(s, v) \, dv \, ds = \int_0^{t_2} p(t_2 - s) \sigma (u(s, x_1 - y)) \, ds.
\]

Using Hölder’s inequality twice to estimate the left hand side of (19), we obtain

\[
\int_0^{t_2} (t_2 - s)^\alpha \int_{x_1 - y}^{x_1} u(s, v) \, dv \, ds \\
\leq \int_0^{t_2} (t_2 - s)^\alpha \left( \int_{x_1 - y}^{x_1} u(s, v)^\gamma \, dv \right)^{\frac{1}{\gamma}} \, ds \, y^{\frac{\gamma - 1}{\gamma}}
\]

\[
\leq \left( \int_0^{t_2} p(t_2 - s) \int_{x_1 - y}^{x_1} u(s, v)^\gamma \, dv \, ds \right)^{\frac{1}{\gamma}} \, y^{\frac{\gamma - 1}{\gamma}} \left( \int_0^{t_2} \frac{s^{\frac{\alpha\gamma}{\gamma - 1}}}{p(s)^{\frac{1}{\gamma - 1}}} \, ds \right)^{\frac{\gamma - 1}{\gamma}}.
\]

Since \( r \) is nonincreasing and not identically zero there exists a constant \( c_1 \) such that \( p(t) \geq c_1 t^{\alpha + 1} \) when \( t \in [0, t_2] \) and therefore it follows from our choice of \( \alpha \) that

\[
\int_0^{t_2} s^{\frac{\alpha\gamma}{\gamma - 1}} p(s)^{-\frac{1}{\gamma - 1}} \, ds < \infty.
\]

If we now let

\[
w(y) \stackrel{\text{def}}{=} \int_0^{t_2} p(t_2 - s) \int_{x_1 - y}^{x_1} \sigma (u(s, v)) \, dv \, ds,
\]

then the right hand side of (19) equals \( w'(y) \), and so by (ii), (20), and by (21) there is a constant \( c_2 \) such that

\[
w'(y) \leq c_2 y^{\frac{\gamma - 1}{\gamma}} w(y)^{\frac{1}{\gamma}}.
\]

Since \( w(0) = 0 \) and \( w(y) > 0 \) for \( y > 0 \) we get

\[
w(y) \leq \left( c_2 y^{\frac{\gamma - 1}{\gamma}} w(y)^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{2(\gamma - 1)}} y^{\frac{2\gamma - 1}{2(\gamma - 1)}}
\]

and we conclude that there is a constant \( c_3 \) such that

\[
w'(y) \leq c_3 y^\frac{\gamma}{\gamma - 1}.
\]

But from the definition of \( w \), from the fact that \( u \) is nondecreasing in its first variable, and by the monotonicity of \( \sigma \) it follows that

\[
w'(y) \geq \int_0^{t_2 - t_1} p(s) \, ds \, \sigma (u(t_1, x_1 - y))
\]

When this inequality is combined with (17) (where we take \( x = x_1 - y \)) and (22), a contradiction follows. This completes the proof. \( \square \)
PROOF OF THEOREM 3. The idea of the proof is roughly as follows: First we show that if one has a solution for $t$ on some interval $[0, T]$ (one clearly has such a solution when $T = 0$), then it can be extended to a slightly larger interval. From the proof of this fact one sees that if this extension procedure does not give a solution on the entire interval $[0, \tau]$ then there is some maximal interval $[0, \tau')$ on which there is a solution and which is such that $\sup_{T < t} \| \sigma'(u) \|_{C([0, T]; C([0, \xi]))} = \infty$. In order to show that this last fact leads to a contradiction we then apply the same argument as when establishing the existence of a local solution, but we derive estimates for $\|u_x\|_{C([0, T]; L^1([0, \xi]))}$ instead of estimating $\|u_x\|_{C([0, T]; L^1([0, \xi])))}$. It is of crucial importance for this part of the proof that we derive these estimates for all $x \in [0, \xi]$. In this connection, the use of Theorem 1 is essential.

First we show that we may, without loss of generality, assume that there are positive constants $c_0, c_1$, and $c_2$ such that

$$0 < c_0 \leq \sigma'(r) \leq c_1 < \infty \quad \text{and} \quad \sup_{r \neq s} \frac{|\sigma'(r) - \sigma'(s)|}{|r - s|} \leq c_2 < \infty. \quad (23)$$

By (i) it is sufficient to show that there is an apriori bound for the solution. In analogy with the proof of Theorem 2 we let

$$\mathcal{D}(A) = \{ v \in L^1([0, \xi]; \mathbb{R}) \mid \sigma(v) \in AC([0, \xi]; \mathbb{R}), \quad v(0) = 0 \}, \quad (24)$$

and

$$A(v) = \sigma(v)', \quad v \in \mathcal{D}(A). \quad (25)$$

Then one can easily show (cf. the proof of [11, Lemma 3]) that $A$ is a closed, $m$-accretive operator in $L^1([0, \xi]; \mathbb{R})$ and that $\| (I + \lambda A)^{-1} v \|_{L^\infty([0, \xi])} \leq \| v \|_{L^\infty([0, \xi])}$ for all $v \in L^\infty([0, \xi]; \mathbb{R})$ when $\lambda > 0$. Then it follows from [3, Theorem 4,(a), Prop. 5] that if we find a solution $u$ of (1), then it must satisfy $\sup_{x \in [0, \xi]} |u(t, x)| \leq \sup_{x \in [0, \xi]} |u_0(x)| + \int_0^t g_s(t - s) \sup_{x \in [0, \xi]} |f(s, x)| \, ds$ and this is the desired apriori bound. Thus we shall for the rest of the proof assume that (23) holds.

Suppose next that there is a number $T \in [0, \tau)$ such that there is a solution $u \in C^\alpha([0, T] \times [0, \xi]; \mathbb{R})$ of (1) on $(0, T] \times (0, \xi]$ such that $u_\tau \in C([0, T]; C([0, \xi]; \mathbb{R})), u(0, x) = u_0(x)$ and $u(t, 0) = 0$; if $T = 0$ this solution is taken to be $u(0, x) = u_0(x)$ (so that this hypothesis holds at least with $T = 0$).

We intend to show that this solution can be continued to $[0, \hat{T}] \times [0, \xi]$ where $\hat{T} > T$ and $\hat{T} - T$ is sufficiently small. We do this in two steps. In the first step we solve (27) with $c$ given; in the second step we find a fixed-point for the map $c \mapsto \sigma'(v)$ (where $v$ is the solution of (27) obtained in the first step). This continuation procedure is concluded by formula (41).
Thus we first show (using the same argument as in the proof of [2, Theorem 1]) that there are constants \( \delta \) and \( M_1 \) depending on \( \alpha, \mu, \tau, \xi, c_0, \) and \( c_1 \) such that if \( \hat{T} \in (T, \tau] \) and \( c \in C^{(\mu)}([0, \hat{T}]; C([0, \xi]; \mathbb{R})) \) satisfy
\[
\begin{align*}
    c_0 &\leq c(t, x) \leq c_1, \\
    c(t, x) &= \sigma'(u(t, x)), \quad (t, x) \in [0, T] \times [0, \xi], \\
    (\hat{T} - T)^{\mu} \|c\|_{C^{(\mu)}([0, \hat{T}]; C([0, \xi]))} &\leq \delta,
\end{align*}
\]
then there exists a unique solution \( v \) of the equation
\[
\begin{align*}
    (D_{\tau}^\mu(v - u_0))(t, x) + c(t, x) v_x(t, x) &= f(t, x), \quad (t, x) \in (0, \hat{T}] \times (0, \xi], \\
v(0, x) &= u_0(x), \quad x \in [0, \xi], \\
v(t, 0) &= 0, \quad t \in [0, \hat{T}],
\end{align*}
\]
such that (clearly \( v(t, x) = u(t, x) \) for \( (t, x) \in [0, T] \times [0, \xi] \))
\[
\begin{align*}
    \|v_x\|_{C^{(\mu)}([0, \hat{T}]; C([0, \xi]))} &\leq M_1 \|u_x\|_{C^{(\mu)}([0, T]; C([0, \xi]))} \\
    &\quad + M_1 \|f\|_{C^{(\mu)}([0, T]; C([0, \xi]))} + M_1 \|\sigma'(u_0(x))u_0'(x) - f(0, x)\|_{C^{(\mu)}([0, \xi])} \\
    &\quad + M_1 \|u_x\|_{C^{(\mu)}([0, T]; C([0, \xi]))} \|c\|_{C^{(\mu)}([0, \hat{T}]; C([0, \xi]))}, \quad x \in [0, \xi].
\end{align*}
\]
Observe that the first and last term of this inequality are written in terms of the space variable \( x \in [0, \xi] \). The proof of the existence of \( v \) satisfying (27) will be completed by the paragraph containing formula (39).

To solve (27), we begin by studying the following equation:
\[
(\partial_{\tau}^\mu(v - u_0))(t, x) + b(x)v_x(t, x) = g(t, x), \quad t \in (0, \tau], \quad x \in (0, \xi],
\]
with boundary condition \( v(t, 0) = 0 \) and initial condition \( v(0, x) = u_0(x) \) under the following assumption on the function \( b \):
\[
b \in C(\mathbb{R}^+; \mathbb{R}) \text{ and } 0 < c_0 \leq b(x) \leq c_1 < \infty.
\]
We denote by \( B_b \) the linear operator in \( C_{\theta \to 0}([0, \xi]; \mathbb{C}) \) def \{ q \in C([0, \xi]; \mathbb{C}) \mid q(0) = 0 \} with domain
\[
\mathcal{D}(B_b) = \{ q \in C^1([0, \xi]; \mathbb{C}) \mid q(0) = q'(0) = 0 \}
\]
and defined by
\[
(B_bq)(x) = b(x)q'(x), \quad x \in [0, \xi], \quad q \in \mathcal{D}(B_b).
\]
We denote by \( B \) the corresponding operator with \( b(x) = 1 \) and \( \xi \) replaced by \( \xi_0 = \xi/c_0 \).
Thus (29) can be written as

\begin{equation}
D_t^\alpha (v - u_0) + B_b v = g.
\end{equation}

Next, perform a change of variable so that equation (31) is replaced by

\begin{equation}
D_t^\alpha (v^b - u_0^b) + B v^b = g^b,
\end{equation}

where

\[
\begin{align*}
g^b(t, y) &= g(t, \rho(y)), \\
u_0^b(y) &= u_0(\rho(y)), & y &\in [0, \xi_b]
\end{align*}
\]

and

\[
\begin{align*}
g^b(t, y) &= g(t, \xi), \\
u_0^b(y) &= u_0(\xi) + b(\xi)u_0'(\xi)(y - \xi_b), & y &\in (\xi_b, \xi_0].
\end{align*}
\]

Here \(\xi_b = \int_0^\xi \frac{1}{b(s)}\,ds\) and \(\rho\) is the inverse of the function \(x \mapsto \int_0^x \frac{1}{b(s)}\,ds\). By [1, Theorem 6(a)] equation (32) has a unique solution \(v^b\) which satisfies the bound

\[
\|B v^b(t) - g^b(0)\|_{C^\alpha([0,\tau];C([0,\xi]))} \\
\leq M_2 \left( \|B u_0^b - g^b(0)\|_{C^\alpha([0,\xi_b])} + \|g^b(t) - g^b(0)\|_{C^\alpha([0,\tau];C([0,\xi]))} \right),
\]

where \(M_2\) depends on \(\alpha, \mu, \tau\) and \(\xi_0\). Now we change variables back again, that is, we define

\begin{equation}
v(t, x) = v^b \left( t, \int_0^x \frac{1}{b(s)}\,ds \right), \quad \text{for } x \in [0, \xi].
\end{equation}

We can therefore conclude that there is a unique solution \(v\) of (29) such that

\begin{equation}
\|v\|_{C^\alpha([0,\tau];C([0,\xi]))} \leq M_3 \left( \|b(x)u_0'(x) - g(0, x)\|_{C^\alpha([0,\xi])} \\
+ \|g\|_{C^\alpha([0,\tau];C([0,\xi]))} \right),
\end{equation}

where (with some crude estimates) \(M_3 = \frac{1}{c_0} (M_2 \max\{2, c_1^{-\frac{\delta}{\mu}}\} + 1)\).

Our next claim is that (34) holds with \(\tau\) replaced by an arbitrary \(\hat{T} \in (0, \tau]\), \(\xi\) replaced by an arbitrary \(\hat{X} \in [0, \xi]\), and with \(M_3\) unchanged. To see this, choose \(\hat{T} \in (0, \tau]\), \(\hat{X} \in [0, \xi]\), and redefine \(b, u_0\) and \(g\) as \(b(x) = b(\hat{X})\), \(u_0(x) = u_0(\hat{X}) + u_0'(\hat{X})(x - \hat{X})\), and \(g(t, x) = g(t, \hat{X})\) for \(x \in (\hat{X}, \xi]\) and \(t \in [0, \hat{T}]\) and \(g(t, x) = g(\hat{T}, x)\) for \(x \in [0, \xi]\) and \(t \in (\hat{T}, \tau]\). Then we can
use the uniqueness of the solution and the definition of the Hölder norms to conclude that we in fact have our claim, i.e.,
\[
\|u_x\|_{C([0,\hat{T}];C([0,x];\mathbb{C}))} \leq M_3 \left( \|b(x)u_0'(x) - g(0, x)\|_{C([0,\hat{T}];C([0,x];\mathbb{C})))} + \|g\|_{C([0,\hat{T}];C([0,x];\mathbb{C})))} \right), \quad \hat{T} \in (0, \tau], \quad x \in [0, \xi].
\]

Choose
\[
\delta = \frac{1}{4M_3},
\]
and \(\hat{T} \in (T, \tau]\) such that the last part of (26) holds. Having a solution of (29) satisfying (35) and having chosen \(\hat{T}\), we proceed to find a solution of (27).

Let \(P\) denote the set
\[
P \overset{\text{def}}{=} \{ p \in C([0, \hat{T}]; C([0, \xi]; \mathbb{C})) \mid p(t, x) = u_x(t, x), \quad 0 \leq t \leq T \}.
\]

For each \(p \in P\) we have to find a solution \(w\) of the equation
\[
(D^0_p (w - u_0)(t, x) + c(T, x)w_x(t, x) = f(t, x) + (c(T, x) - c(T, \hat{x}))p(t, \hat{x}),
\]
on \([0, \hat{T}] \times [0, \xi]\) with boundary condition \(w(t, 0) = 0\) (and initial condition \(w(0, \hat{x}) = u_0(\hat{x})\)) and \(c\) as in (26). Note that this equation is of type (29). Observe also that the right-hand side of (37) evaluated at \(t = 0\) is
\[
f(0, x) + (c(T, x) - c(0, x))u_0'(x),
\]
and therefore the term \(b(x)u_0'(x) - g(0, x)\) appearing in (35) is now, when \(b(x) = c(T, x)\), equal to \(c(0, x)u_0'(x) - f(0, x)\). Thus we conclude from (ii) and from the results above on (29) that we can find a solution \(w\) of (37) such that \(w_x \in C([0, \hat{T}]; C([0, \xi]; \mathbb{C}))\). Moreover, the uniqueness guarantees that we have \(w_x \in P\).

Let us denote the mapping \(p \to w_x\) by \(w_x = G(p)\). Using the linearity of equation (37), and (35) with \(b(x) = c(T, x)\) once more, we conclude that
\[
\|(G(p_1) - G(p_2))(t, x)\|_{C([0, \hat{T}]; C([0, x]; \mathbb{C}))} \leq M_3 \|(c(T, x) - c(t, x))(p_1 - p_2)(t, x)\|_{C([0, \hat{T}]; C([0, x]; \mathbb{C}))}, \quad x \in [0, \xi].
\]

Let \(p_\Delta = p_1 - p_2\) and \(c_\Delta(t, x) = c(T, x) - c(t, x)\). Since \(p_1\) and \(p_2\) \(\in P\) it follows that \(p_\Delta(t, x) = 0\) for \(t \in [0, T]\) and therefore we can, when analyzing the term \((c(T, x) - c(t, x))(p_1 - p_2)(t, x)\), assume that \(c(t, x) = c(T, x)\) for \(t \in [0, T]\). Thus we conclude from the last part of (26) and from (36) that
\[
\sup_{t \in [0, \hat{T}]\atop x \in [0, x]} |c_\Delta(t, x)p_\Delta(t, x)| \leq \frac{1}{4M_3} \sup_{t \in [0, \hat{T}]\atop x \in [0, x]} |p_\Delta(t, x)|, \quad x \in [0, \xi].
\]
Furthermore, if we write \( c_\Delta (t, x) p_\Delta (t, x) - c_\Delta (s, x) p_\Delta (s, x) = c_\Delta (t, x) (p_\Delta (t, x) - p_\Delta (s, x)) + (c_\Delta (t, x) - c_\Delta (s, x)) (p_\Delta (s, x) - p_\Delta (T, x)) \) using the fact that \( p_\Delta (T, x) = 0 \), and use (26) once again, then we conclude that

\[
\| (c(T, x) - c(t, x)) (p_1 - p_2)(t, x) \|_{C^\mu([0, \hat{T}; C([0, x]))} \\
\leq \frac{1}{2M_3} \| (p_1 - p_2)(t, x) \|_{C^\mu([0, \hat{T}; C([0, x]))}, \quad \forall x \in [0, \xi].
\]

Hence we have, using (38), for every \( x \in [0, \xi] \),

\[
(39) \quad \| (G(p_1) - G(p_2))(t, x) \|_{C^\mu([0, \hat{T}; C([0, x]))} \leq \frac{1}{2} \| (p_1 - p_2)(t, x) \|_{C^\mu([0, \hat{T}; C([0, x]))},
\]

and we see that the mapping \( G \) is a contraction and that there is a unique fixed-point, i.e., a function \( v \) such that \( v_x = G(v_x) \). Thus we get a solution of (27) on the interval \([0, \hat{T}]\).

If we take \( p_0 \in P \) to be such that \( p_0(t, x) = u_x(T, x) \) for \( t \in [T, \hat{T}] \) then \( \| p_0 \|_{C^\mu([0, \hat{T}; C([0, x]))} = \| u_x \|_{C^\mu([0, T]; C([0, x]))} \). Using inequality (35) to estimate \( \| G(p_0) \|_{C^\mu([0, \hat{T}; C([0, x]))} \) and then (39) to estimate \( \| G(v_x) - G(p_0) \|_{C^\mu([0, \hat{T}; C([0, x]))} \), we conclude that (28) holds with

\[
M_1 = \max \{ 1 + 4M_3 \| v \|_{C^\mu([0, T]; C([0, x]))}, 2M_3 \}.
\]

With \( c \) fixed, the solution \( v \) of (27) can of course be continued to \([0, \tau] \times [0, \xi] \). However, our goal is to solve (1), i.e., (27) with \( c(t, x) = \sigma'(v(t, x)) \). For this purpose we apply another fixed-point argument on \([0, \hat{T}] \) with \( \hat{T} - T \) sufficiently small (and recall that we have a solution of (1) on \([0, T]\)).

We let \( M_4 \) be the constant

\[
M_4 \overset{\text{def}}{=} c_1 + c_2 \max \{ 1, \xi \} M_1 \| u_x \|_{C^\mu([0, T]; C([0, \xi]))} \\
+ \xi c_2 M_1 \| f \|_{C^\mu([0, T]; C([0, \xi]))} + \xi c_2 M_1 \| \sigma'(u_0(x)) u'_0(x) - f(0, x) \|_{C^\mu([0, \xi])},
\]

and choose \( \hat{T} \in (T, \tau] \) such that

\[
(40) \quad (\hat{T} - T)^\mu \leq \frac{\delta}{M_4 e^{M_4 \xi}}.
\]

For our fixed-point argument we let

\[
V = \left\{ c \in C^\mu([0, \hat{T}]; C([0, \xi]; \mathbb{R})) \left| c(t, x) = \sigma'(u(t, x)), \quad t \in [0, T], \quad x \in [0, \xi], \right. \right. \\
\left. \left. c_0 \leq c(t, x) \leq c_1, \quad t \in [T, \hat{T}], \quad x \in [0, \xi], \right. \right. \\
\left. \left. \| c \|_{C^\mu([0, \hat{T}; C([0, x]))} \leq M_4 e^{M_4 x}, \quad \forall x \in [0, \xi] \right\}.
\]
Note that $V$ is convex and not empty. Now we define the function $F(c)$ for $c \in V$ by

$$F(c)(t, x) \overset{\text{def}}{=} \sigma'(v(t, x)),$$

where $v$ is the solution of (27). (By the definition of $V$ and by (40) condition (26) is satisfied and hence such a (unique) solution exists.)

By the uniqueness we know that we have $F(c)(t, x) = \sigma'(u(t, x))$ for $t \in [0, T]$ and $x \in [0, \xi]$ and by (23) we also have $c_0 \leq F(c)(t, x) \leq c_1$. Finally we note that since $v(t, 0) = 0$ we have

$$F(c)(t, x) = \sigma' \left( \int_0^x u_x(t, r) \, dr \right),$$

and it follows that

$$\|F(c)\|_{C([0, \tilde{t}]; C([0, \xi]))} \leq c_1 + c_2 \int_0^\xi \|u_r\|_{C([0, \tilde{t}]; C([0, r]))} \, dr \leq M_4 \int_0^\xi \|c\|_{C([0, \tilde{t}]; C([0, r]))} \, dr \leq M_4 e^{M_4 x}, \quad x \in [0, \xi],$$

where the second inequality is a consequence of (28) and the definition of $M_4$, and where the last inequality follows because $c \in V$. This shows that $F(c) \in V$.

Finally we observe that by [2, Theorem 1 and (4)] the set of solutions of (27) one gets when $c \in V$ is contained in a bounded subset of $C((\mu + \alpha)/2) \times C(0, \xi); R)$ (for example) and therefore this set of solutions, and hence also $F(c) = \sigma'(v)$ for $c \in V$ is contained in a compact subset of $C((\mu + \alpha)/2) \times C(0, \xi); R)$). (Note in particular that since our boundary condition is now $v(t, 0) = 0$ we do not need the assumption that the function $x \mapsto c(t, x)$ is a continuous function with values in $C((\mu + \alpha)/2) \times [0, T); R)$. Therefore the constant $M$ appearing in [2, formula (4)] depends on $\|c\|_{C((\mu + \alpha)/2) \times C(0, \xi); R)}$, $c_0$ and $c_1$, but not otherwise on $c$.) Thus we know by the Schauder fixed-point theorem that there is a function $c \in V$ such that $F(c) = c$ and the corresponding solution of (27) is then the unique solution of (1) on $[0, \tilde{t}] \times [0, \xi]$.

If the claim of the theorem does not hold there is, by the continuation argument above, a maximal number $\hat{t} \in (0, T]$ such that there is a solution of (1) on $(0, \hat{t}) \times (0, \xi)$, and such that $u_x \in C((\mu + \alpha)/2) \times [0, T]; C([0, \xi]; R))$ for all $T \in (0, \hat{t})$. If $\sup_{T < \hat{t}} \|u_x\|_{C((\mu + \alpha)/2) \times [0, T]; C([0, \xi]; R))} < \infty$, then this solution can be continued by the argument used above, and we get a contradiction. Furthermore, it also follows from the argument in the above that if $\sup_{T < \hat{t}} \|\sigma'(u)\|_{C((\mu + \alpha)/2) \times [0, T]; C([0, \xi]; R))} < \infty$, then $\sup_{T < \hat{t}} \|u_x\|_{C((\mu + \alpha)/2) \times [0, T]; C([0, \xi]; R))} < \infty$. Thus we assume that

$$\sup_{T < \hat{t}} \|\sigma'(u)\|_{C((\mu + \alpha)/2) \times [0, T]; C([0, \xi]; R))} = \infty,$$

and we will derive a contradiction from this.
We want to apply Theorem 1 and therefore we define the operator $A$ by (24) and (25). It is straightforward to check that by (ii) $y = u_0$ belongs to $\mathcal{D}(A) \subset \mathcal{D}(A)$ and that by (iii) the function $t \mapsto f(t, x) \in L^1([0, \xi]; \mathbb{R})$ satisfies the assumption (iv) of Theorem 1. Thus Theorem 1 may be applied to (1) and so we obtain the existence of a unique (strong) solution $u \in C([0, \tau]; L^1([0, \xi]; \mathbb{R}))$. By uniqueness, this solution coincides with the one constructed above on $[0, \tau) \times [0, \xi]$.

It follows from Theorem 1, together with the results on the local solution that we already have established, that the function

$$t \mapsto \sigma(u)_x(t, x) \in L^1([0, \xi]; \mathbb{R})$$

is uniformly continuous on $[0, \tau)$.

An immediate consequence of this result, of (23), and of the fact that $u(t, 0) = 0$, is that

$$u \text{ is uniformly continuous on } [0, \tau) \times [0, \xi],$$

and hence we also conclude that

$$t \mapsto u_x(t, x) \in L^1([0, \xi]; \mathbb{R})$$

is uniformly continuous on $[0, \tau)$.

In the above, the results of [1] were applied to the operator $u \mapsto u_x$ in the space of continuous functions. Now we shall do the same thing but with integrable functions instead. We let $\xi_0 = \xi/c_0$ and denote by $B$ the linear operator in $L^1([0, \xi_0]; \mathbb{C})$ with domain

$$\mathcal{D}(B) = \{ v \in AC([0, \xi_0]; \mathbb{C}) \mid v(0) = 0 \}$$

and

$$(Bv)(x) = v'(x), \quad x \in [0, \xi_0], \quad v \in \mathcal{D}(B).$$

As the norm in $\mathcal{D}(B)$ we can take $\|w\|_{\mathcal{D}(B)} = \|w'\|_{L^1([0, \xi_0])}$.

If $b \in C(\mathbb{R}^+; \mathbb{R})$ satisfies $c_0 \leq b(x) \leq c_1$, then we can use an argument similar to the one employed when deriving (35) to conclude that it follows from [1, Theorem 6] that there is a constant $M_5$ (which depends on $\alpha$, $\mu$, $\tau$, $\xi$, $c_0$ and $c_1$) and a unique solution $v$ of (29) such that

$$\|v^\mu\|_{C(\mu)([0, \tilde{T}]; L^1([0, \xi]))} \leq M_5 \left( \|\chi_{[0, \tilde{T}]}(\rho(\zeta))h_0(\rho(\zeta))\|_{\mathcal{D}(B)(\frac{\mu}{\sigma}, \infty)} + \|g\|_{C(\mu)([0, \tilde{T}]; L^1([0, \xi]))} \right),$$

for all $\tilde{T} \in [0, \tau]$ and $\chi \in [0, \xi]$ where $h_0(\chi) = b(x)u^\mu_0(\chi) - g(0, \chi)$, $\rho$ is the inverse of the function $x \mapsto y = \int_0^x \frac{1}{b(s)} \, ds$, and where $\mathcal{D}(B)(\frac{\mu}{\sigma} = \ldots$
In this argument one extends the functions as constants in the $t$-direction and as 0 in the $x$-direction (but $u_0$ is extended as a constant) and changes the $x$-variable to the new variable $y = \int_0^x \frac{1}{1 + s} \, ds$.

Having (44), our next goal is to estimate the first term on the right hand side. We claim that if $h$ is an arbitrary function in $\mathcal{C}^{(2)}([0, \xi] : \mathbb{R})$, which is extended as zero to $(\xi, \infty)$, then there is a constant $M_6 \overset{\text{def}}{=} 2c_1^\alpha \frac{\xi}{\xi_0} + 4$, such that

(45) \[ \|X_{[0, \xi]}(\rho(y))h(\rho(y))\|_{\mathcal{D}_B(\frac{\mu}{\alpha}, \infty)} \leq M_6 \|h\|_{\mathcal{C}^{(2)}([0, \xi])}, \quad \xi \in [0, \xi]. \]

To see this we argue as follows: Let $w(y) = X_{[0, \xi]}(\rho(y))h(\rho(y))$ and extend this function as 0 on $(-\infty, 0)$ and let $t \in (0,1)$ be arbitrary. Now write $w = w_1 + w_2$ where $w_1(y) = \int_0^\infty \frac{1}{t \gamma} (w(y) - w(y - r)) \, dr$ and where $w_2(y) = \int_0^\infty \frac{1}{t \gamma} (w(y - r)) \, dr$. We note that $w(y) = 0$ when $y < 0$ and when $y > \xi_0$. To see this we argue as follows: Let $w(y) = X_{[0, \xi]}(\rho(y))h(\rho(y))$ and extend this function as 0 on $(-\infty, 0)$ and let $t \in (0,1)$ be arbitrary. Now write $w = w_1 + w_2$ where $w_1(y) = \int_0^\infty \frac{1}{t \gamma} (w(y) - w(y - r)) \, dr$ and where $w_2(y) = \int_0^\infty \frac{1}{t \gamma} (w(y - r)) \, dr$. We note that $w(y) = 0$ when $y < 0$ and when $y > \xi_0$. Furthermore, $w(y) \leq \|h\|_{\mathcal{C}^{(2)}([0, \xi])}$ when $0 \leq y < r$ or $\xi_0 < y \leq \xi_0 + r$ (because then either $w(y)$ or $w(y - r)$ vanishes) and $|w(y) - w(y - r)| = 0$ otherwise. It follows from these inequalities that $\|w_1\|_{L^1([0, \xi])} \leq (t^\frac{\mu}{\alpha} c_1^\alpha \xi_0 \Gamma(1 + \frac{\nu}{\alpha}) + 2r) \|h\|_{\mathcal{C}^{(2)}([0, \xi])}$. Furthermore, $\|w_2\|_{\mathcal{D}_B(\frac{\mu}{\alpha}, \infty)} \leq M_6 \|h\|_{\mathcal{C}^{(2)}([0, \xi])}$ and by the definition of the interpolation space $\mathcal{D}_B(\frac{\mu}{\alpha}, \infty) = (L^1([0, \xi]; \mathbb{C}), \mathcal{D}(B))_{\frac{\mu}{\alpha}, \infty}$ (see e.g., [13, Definition 1.2.2]), this is exactly what we need in order to get (45).

Using (45) we see that (44) implies that the function $v$ that solves (29) satisfies

(46) \[ \|v_{\hat{T}}\|_{\mathcal{C}(\mu, [0, \hat{T}]; L^1([0, x])))} \leq M_5 \left( M_6 \|h\|_{\mathcal{C}^{(2)}([0, \xi])} + \|g\|_{\mathcal{C}(\mu, [0, \hat{T}]; L^1([0, \xi])))} \right), \]

for all $\hat{T} \in [0, \tau]$ and all $\xi \in [0, \xi]$.

Let $c(t, x) \overset{\text{def}}{=} \sigma'(u(t, x))$. By (42) we can choose $T \in (0, \hat{t})$ such that

(47) \[ \sup_{t, x \in [0, \hat{t}], \xi \in [0, \xi]} |c(t, x) - c(s, x)| \leq \frac{1}{2M_5}. \]

Let $\hat{T}$ be some arbitrary number in $(T, \hat{t})$.

Now we rewrite (1) in the form

\[
(D^\alpha_t (u - u_0))(t, x) + c(T, x)u_x(t, x) = f(t, x) + (c(T, x) - c(t, x))u_x(t, x) \\
= g(t, x), \quad t \in [0, \hat{T}], \quad x \in [0, \xi].
\]
Note that this equation is of type (29); hence the estimate (46) may be applied to $u$ with $b$ extended as a constant for $x > \xi$. Also observe that $c(T,x)u_0'(x) - g(0,x) = c(0,x)u_0'(x) - f(0,x) = \sigma'(u_0(x))u_0'(x) - f(0,x)$. Thus we see by (46) that

$$
\begin{align*}
\|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} & \leq M_7 \\
& \quad + M_5 \|X_{[T,\hat{T}]}(t)(c(T,x) - c(t,x))u_x(t,x)\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}},
\end{align*}
$$

where $M_7$ is some constant such that

$$
M_5 \|f\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} + M_5 M_6 \|\sigma'(u_0(x))u_0'(x) - f(0,x)\|_{C^{(2,\mu)}([0,\xi])}
$$

$$
+ M_5 \|X_{[0,T]}(t)\big(c(T,x) - c(t,x)\big)u_x(t,x)\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} \leq M_7,
$$

for all $x \in [0,\xi]$ and for all $T \in (T, \hat{T})$. Now a simple calculation shows that

$$
\begin{align*}
\|X_{[T,\hat{T}]}(t)(c(T,x) - c(t,x))u_x(t,x)\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} & \leq \sup_{t \in [T,\hat{T}]} \sup_{x \in [0,\xi]} |c(T,x) - c(t,x)| \|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} \\
& \quad + \sup_{i \neq x} \int_0^x \frac{|c(t,x) - c(s,x)|}{|t-s|^{\mu}} |u_x(s,x)| \, dx.
\end{align*}
$$

Invoking this inequality together with (47) in (48) we get

$$
\begin{align*}
\|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} & \leq 2M_7 \\
& \quad + 2M_5 \sup_{i \neq x} \int_0^x \frac{|c(t,x) - c(s,x)|}{|t-s|^{\mu}} |u_x(s,x)| \, dx \\
& \quad \leq 2M_7 + 2M_5 \sup_{i \neq x} \int_0^x \|c\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} |u_x(s,x)| \, dx.
\end{align*}
$$

Since $c(t,x) = \sigma' \int_0^x u_x(t,r) \, dr$ it follows from (23) that

$$
\|c\|_{C^{(\mu,1)}([0,\hat{T};\dot{C}([0,x]))}} \leq c_1 + c_2 \|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}}, \quad x \in [0,\xi].
$$

From (49) and (50) it follows that for each $x \in [0,\xi]$ there exists a number $s(x) \in [0,\hat{T})$ such that

$$
\begin{align*}
\|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} & \leq 1 + 2M_7 + 2c_1 M_5 \sup_{s \in [0,\hat{T})} \|u_x(s,x)\|_{\dot{L}^1([0,\xi])} \\
& \quad + 2M_5 c_2 \int_0^x \|u_x\|_{C^{(\mu,1)}([0,\hat{T};\dot{L}^1([0,x]))}} |u_x(s(x),x)| \, dx.
\end{align*}
$$
By (43) there is a finite set of points $\{t_j\}_{j=1}^n \subset [0, \hat{t})$ such that if $s \in [0, \hat{t})$ then there is an index $j(s) \in \{1, \ldots, n\}$ such that

$$\|u_x(s, x) - u_x(t_j(s), x)\|_{L^1([0,\xi])} \leq \frac{1}{4M_8 c_2}. \tag{52}$$

Let $M_8 = \max\{4M_5 c_2, 2 + 4M_7 + 4c_1 M_5 \sup_{x \in [0, \hat{t}]} \|u_x(s, x)\|_{L^1([0, \xi])}\}$, (by (43) $M_8 < \infty$). Then we conclude from (51) and (52) that we in fact have

$$\|u_x\|_{C^\mu([0, \hat{t}); L^1([0, \xi]))} \leq M_8 + M_8 \int_0^\xi \|u_x\|_{C^\mu([0, \hat{t}); L^1([0, \xi]))} |u_x(t_j(s(x)), x)| \, dx$$

$$\leq M_8 + M_8 \int_0^\xi \|u_x\|_{C^\mu([0, \hat{t}); L^1([0, \xi]))} p(x) \, dx,$$

where $p(x) = \max_{1 \leq j \leq n} |u_x(t_j, x)|$ so that we have $p \in L^1([0, \xi]; \mathbb{R})$. But now it follows from Gronwall’s inequality that

$$\|u_x\|_{C^\mu([0, \hat{t}); L^1([0, \xi]))} \leq M_8 e^{M_8 \int_0^\xi p(s) \, ds} \leq M_8 e^{M_8 \|p\|_{L^1([0, \xi])}}.$$

This inequality combined with (50) contradicts (41) and the proof is complete. $\square$

REFERENCES

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