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Semiregularity, Obstructions and Deformations of Hodge Classes

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Abstract. We show that the deformation theory of a pair (X, η), where X is a compact Kähler manifold and η is a (p, p) class on X, is controlled by a certain sheaf $\mathcal{L}_\eta$ of differential graded Lie algebra on X; consequently, we show that relative obstructions to deforming a pair (X, Y), where Y is a codimension-p submanifold of X, relative to deforming X so that the fundamental class of Y remains of type (p, p), (in particular, deformations of Y fixing X) lie in the kernel of the semiregularity map $\pi_1 : H^1(N_{Y/X}) \to H^{p+1,p-1}(X)$ of Bloch et al. We also give a number of extensions and applications of this result.

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To a codimension-p embedding $Y \subset X$ of compact complex manifolds one may associate at least 3 deformation problems: deforming Y, fixing X (the local Hilbert scheme); deforming the pair (Y, X); deforming X so that the cohomology class $\eta = [Y] \in H^{2p}(X)$ maintains a given Hodge level $q \leq p$. These problems- obviously interrelated- are all influenced by Hodge theory, via the so-called semiregularity map (which has antecedents in Severi and was more recently considered by Kodaira-Spencer, Mumford, Bloch ...)

$$\pi_1 : H^1(N) \to H^{p+1,p-1}(X), N = \text{normal bundle}.$$

Roughly speaking obstructions which a priori lie in $H^1(N)$ are actually in $\ker(\pi_1)$. Thus, e.g. a lower bound on the rank of π_1 yields estimates on the dimension of deformation spaces etc. The precise statement is as follows.

THEOREM 0. *Let X be a compact complex manifold and $Y \subset X$ a connected submanifold of codimension p, with normal bundle N, and fundamental class $\eta = [Y] \in H^p(\Omega_X^p)$. Let $\pi_1 : H^1(N) \to H^{p+1}(\Omega_X^{p-1})$ be the semi-regularity map (reviewed below). Then*

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(i) *obstructions to deforming Y in X lie in* $\ker \pi_1$*;*
(ii) *if moreover X is Kählerian then obstructions to deforming the pair* (X, Y)*, relative to deforming X so that* $\eta \in H^{2p}(X)$ *remains of type* (p, p)*, lie in* $\ker \pi_1$.

[In more detail, (ii) means: given an artin local **C**-algebra (S, m), an ideal $I < S$ with $mI = 0$, a deformation α of (X, Y) over S/I, a deformation α' of X over S, which induces the same deformation as α over S/I and in which the (Gauss-Manin) flat lift of η has Hodge level p, obstructions to lifting α over S lie in $\ker(\pi_1) \otimes I$.]

Theorem 0 was in essence proven by Bloch [B] for the case of deformations over an artin ring of the form $\mathbb{C}[\epsilon]/(\epsilon^n)$, however neither the result nor the proof yield the general artin local case. In the present generality Theorem 0 was first stated in [R0] where the argument was based on the notion of "canonical element" controlling a deformation (see [R3] for a development of the theory and required properties of canonical elements).

The main purpose of this paper is to develop some methods pertaining to the interplay of "canonical" or "Lie-theoretic" deformation theory and Hodge theory and apply them to a proof of Theorem 0; the proof of part (i) in particular is short and essentially self-contained. A central role in these methods is played by a certain differential graded Lie algebra $\mathcal{L} = \mathcal{L}_\eta$ which, as we prove with the method of [R3], controls deformations of X in which a given class η maintains a given Hodge level. Modulo this fact (which moreover is unnecessary for part (i)), the proof of Theorem 0 is quite simple and conceptual: indeed it boils down to constructing a Lie homomorphism $\pi : N[-1] \to \mathcal{L}_\eta$ ("sheaf-theoretic semiregularity") and realizing π_1 as the cohomology map induced by π. Following the proof we present some applications to deformations of maps and integral curves on K3 surfaces. See [R0] for other applications.

As our foundational reference for (Lie algebra-controlled) deformation theory, we shall use [R2], [R3]; however for the proof of part (i) (which is already sufficient for most applications), essentially any reference, e.g. [GM], will do.

Proof of Theorem. Let $T = T_X$ and $T' \subset T$ be the subsheaf of vector fields tangent to Y along Y, i.e. preserving the ideal sheaf $\mathcal{I}_Y$. We identify the normal sheaf N with the complex in degrees $-1, 0$

$$T' \xrightarrow{id} T$$

and endow $N[-1]$ with a structure of DGLA sheaf given by

$$[\cdot] : T' \times T' \to T', \frac{1}{2}[\cdot] : T' \times T \to T.$$

(the 1/2 factor is needed to make id a Lie derivation). We thus have an exact triangle of DGLA's

$$N[-1] \to T' \to T \to$$

(i.e. $N[-1]$ is a Lie ideal in T'), and these control, respectively, the deformations of Y fixing X, of the pair (X, Y), of X (see e.g. [R2], [R3] for more details on this).

On the other hand, to any class $\eta \in H^q(\Omega^p)$ we may associate a DGLA $\mathcal{L} = \mathcal{L}_\eta$ as follows:

$$\mathcal{L}^{-q} = \Omega^{p-1} \to \mathcal{L}^0 = T,$$

differential = interior multiplication by η, bracket = usual one on T, Lie derivative $T \times \Omega^{p-1} \to \Omega^{p-1}$, zero otherwise.

More concretely, we may represent $\mathcal{L}^0$ (resp. $\mathcal{L}^{-q}$) by the Cech complex of T (resp. of Ω^{p-1} shifted q places to the left). Thus we have an exact triangle of DGLA's

$$\Omega^{p-1}[q-1] \to \mathcal{L} \to T \to$$

with $\Omega^{p-1}[q-1]$ an abelian ideal in $\mathcal{L}$.

By the local cohomology description of $\eta = [Y]$ given, e.g. in [B] it follows directly that interior multiplication by η vanishes (in the derived category) on $T' \subset T$, and consequently we have a commutative diagram of exact triangles

$$\begin{array}{ccccc} N[-1] & \to & T' & \to T \to \\ \pi \downarrow & & \pi' \downarrow & \parallel \\ \Omega^{p-1}[p-1] & \to & \mathcal{L} & \to T \to \end{array} \qquad (1)$$

$\pi_1 = H^2(\pi)$ may be taken as the definition of π_1 but it is immediate that this definition coincides with the one given in in [B]. It may be noted that $H^1(\pi)$ is none other than the infinitesimal Abel-Jacobi map associated to Y. Now we shall prove below that, for X Kählerian, $\mathcal{L}$ controls precisely the deformations of X where η remains of type (p, p). Given this, the Theorem follows immediately from (1): indeed by any general theory (e.g. [GM], [R2]), obstructions are induced by Lie bracket and lie in H^2 of the controlling Lie algebra and thus relative obstructions as in the Theorem lie in $\ker H^2(\pi') = \ker(\pi_1)$. □

Note that for the purpose of part (i) the interpretation of $\mathcal{L}$ is irrelevant, so this part does not require the Kählerian hypothesis (nor for that matter any of the rest of the paper).

It remains to establish the deformation-theoretic significance of $\mathcal{L}$. Precisely, we will show the $\mathcal{L}$ controls deformations $\tilde{X}/S$ plus Čech cochains

$$\begin{gathered} \omega \in \check{C}^q(\Omega^{p-1}_{\tilde{X}/S}), \\ \delta(\omega) + \tilde{\eta} \in \check{C}^q(F^p\Omega_{\tilde{X}/S}), \end{gathered} \qquad (2)$$

where $\tilde{\eta}$ = constant lift of η, modulo coboundaries $\omega = \delta(\tau)$.

To this end we first review the universal variation of Hodge structure associated to X, as developed in [R3], which will provide us with an explicit representative for the GM-constant lift of a cohomology class on X. Consider the following double complex $J_m(T,\Omega)$ on $X<m>\times X$ in bidegrees $[0,n]\times[-m,0]$:

$$\begin{array}{ccccc}
\mathcal{O}\to\dots & \to\Omega^p\to & \dots\to & \Omega^n \\
 & \uparrow & & \uparrow \\
 & \to T\times\Omega^p\to & \dots & \vdots \\
 & \uparrow & & \\
 & \vdots & & \\
\lambda^m T\otimes\mathcal{O}\to\dots & \lambda^m T\otimes\Omega^p & \dots & \lambda^m T\otimes\Omega^n
\end{array}$$

with horizontal arrows induced by exterior derivative and vertical arrows of the form

$$\begin{aligned} v_1\times\dots\times v_k\times\omega\mapsto &\pm\sum_{i=1}^{k} v_1\times\dots\times\hat{v}_i\times\dots\times v_k\times L_{v_i}(\omega)\\ &\pm\sum_{i<j}(-1)^{i+j}v_1\times\dots.\times\hat{v}_i\\ &\times\dots\times\hat{v}_j\times\dots\times v_k\times[v_i,v_j]\times\omega\end{aligned}$$

(i.e. $J_m(T,\Omega)$ is just the standard complex for $\Omega^\bullet$ as T-module, with variables separated.) As explained in [R3], the De Rham cohomology $H^r_{DR}(X_m/R_m)$ of the universal m-th order deformation X_m/R_m of X, together with its Hodge filtration (i.e. the universal m-th order VHS associated to X) is obtained by applying a pure linear algebra construction to a suitable Kunneth component $\mathbf{H}^{0,r}(J_m(T,\Omega^\bullet))$ of the cohomology of $J_m(T,\Omega)$, (i.e. the one mapping to $(\mathbf{H}^0(J_m(T))\oplus\mathbf{C})\otimes H^r(X)$ under the quasi-isomorphism $\mathbf{C}\to\Omega^\bullet$), so one might as well work with the latter group directly. Thanks to Cartan's formula for Lie derivative, the complex $J_m(T,\Omega^\bullet)$ is "split", i.e. isomorphic to the complex with the same entries and *trivial* action of T on $\Omega^\bullet$, the isomorphism in question being assembled from $\pm$ interior multiplication maps

$$M_{i,k,p}:\lambda^k T\boxtimes\Omega^p\to\lambda^{k-i}T\boxtimes\Omega^{p-i},\qquad i\geq 0$$

$$\begin{aligned} v_1\times\dots\times v_k\times\omega\mapsto&\sum\pm v_1\times\dots\times\hat{v}_{j_1}\times\dots\times\hat{v}_{j_i}\times\dots\times v_k\\ &\times i(v_{j_1}\wedge\dots\wedge v_{j_k})(\omega)\\ &\mapsto 0,\ \ i>\min(k,p)\end{aligned}$$

The induced map on cohomology is the Gauss-Manin isomorphism

$$G:(\mathbf{C}\oplus\mathbf{H}^0(J_m(T)))\otimes H^r(X)\to\mathbf{H}^{0,r}(J_m(T,\Omega')).$$

The "constant lift" of a class $\eta \in H^r(X)$ is simply the map

$$G_\eta : \mathbf{C} \oplus \mathbf{H}^0(J_m(T)) \to \mathbf{H}^{0,r}(J_m(T, \Omega))$$

given by $G(\cdot \otimes \eta)$. More explicitly on Čech cohomology, G_η on $\mathbf{H}^0(J_m(T))$ may be described as follows. We may represent an element $v \in \mathbf{H}^0(J_m(T))$ by $(v_1, \dots, v_m)$ where

$$\begin{aligned}
&v_m \in S^m(\check{Z}^1(T)) \subset \check{Z}^m(\lambda^m(T)) \\
&v_i \in S^i(\check{C}^1(T)) \subset \check{C}^i(\lambda^i(T)), \quad 1 \le i < m, \\
&\delta(v_i) = \pm b(v_{i+1}), \qquad\qquad\quad 1 \le i < m \\
&\delta(v_m) = 0,
\end{aligned}$$

b being the map induced by bracket. On the other hand X being Kähler $\eta \in H^{p,q}(X)$, say, may be represented by a Čech cocycle with values in the sheaf $\hat{\Omega}^p$ of *closed* p-forms (which in effect means choosing a lift of η to $F^p H^{p+q}(X)$), and $G_\eta(v)$ may be represented by

$$\begin{array}{lllll|l}
p-m & & & & p & \\
\hline
M_{m,m,p}(v_m \times \eta) & & & & 0 & 0 \\
 & M_{m-1,m,p}(v_m \times \eta) & & & v_1 \times \eta & \\
 & & \vdots & \vdots & \vdots & \\
 & \ddots & & & & \\
 & & M_{2,m,p}(v_m \times \eta) & M_{1,m-1,p}(v_{m-1} \times \eta) & v_{m-2} \times \eta & \\
 & & & M_{1,m,p}(v_m \times \eta) & v_{m-1} \times \eta & \\
 & & & & v_m \times \eta & -m
\end{array}$$

We are now in position to consider the obstruction to the constant lift $G_\eta(v)$ having Hodge level p (in cohomology). Thus consider what hypercoboundary would push $G_\eta(v)$ into $J_m(T, F^p\Omega^\bullet)$, i.e. kill all terms off the p-th column. Working from the bottom up, starting in position $(p-1, -m+1)$ we require first a cochain

$$\begin{aligned}
&\omega_{m-1} \in S^{m-1}(\check{Z}^1(T)) \otimes \check{C}^q(\Omega^{p-1}), \\
&\delta(\omega_{m-1}) = M_{1,m,p}(v_m \times \eta).
\end{aligned} \tag{3}$$

Clearly the latter right-hand side is a cocycle, so the obstruction to ω_{m-1} existing is in $S^{m-1}(H^1(T)) \otimes H^{q+1}(\Omega^{p-1})$. Note that once ω_{m-1} exists, we have

$$\delta(M_{m,m-1,p-1}(\omega_{m-1})) = M_{k+1,m,p}(v_m \times \eta),$$

so all other terms along the bottom diagonal, i.e. in position $(p-k-1, -m+k+1)$, $k = 0, \ldots, m-1$, can be killed too. Next, to kill off the term in position $(p-1, -m+2)$ requires a cochain

$$\begin{aligned} &\omega_{m-2} \in S^{m-2}(\check{Z}^1(T)) \otimes \check{C}^q(\Omega^{p-1}), \\ &\delta(\omega_{m-2}) = M_{1,m,p}(v_{m-1} \times \eta) + d(M_{1,m-1,p-1}(\omega_{m-1})) + L(\omega_{m-1}) \end{aligned} \tag{4}$$

where d and L denote the horizontal and vertical differentials in the complex $J_m(T, \Omega^\bullet)$. Again it is easy to see the latter right-hand side is a cocycle. So the obstruction to ω_{m-2} existing (provided ω_{m-1} does) is in $S^{m-2}H^1(T) \otimes H^{q+1}(\Omega^{p-1})$; and again once ω_{m-2} exists all elements in the diagonal $\{(a, b), a+b = p-m+1, a < p\}$ may be killed too. We continue in this way up to the 0-th row where what is required is

$$\begin{aligned} &\omega_0 \in C^q(\Omega^{p-1}), \\ &\delta(\omega_0) = M_{1,1,p}(v_1 \times \eta) + d(M_{1,1,p-1}(\omega_1)) + L(\omega_1). \end{aligned} \tag{5}$$

Turning to the algebra $\mathcal{L}$ and its deformation theory, we claim that the obstructions are the same as for keeping $G_\eta(v)$ of level p, which it suffices to show in the universal situation. For the first-order case this is clear: given $v \in H^1(T)$, the data required to lift v to $\mathbf{H}^1(\mathcal{L})$ is precisely a cochain $\omega_1 \in \check{C}^q(\Omega^{p-1})$ with $\delta(\omega_1) = i(v)(\eta)$, same obstruction as for $G_\eta(v)$ to be of level p. Next we turn to the second-order case. The complex $J_2(\mathcal{L})$ takes the form

$$\begin{array}{ccccc} T & \to & \Omega^{p-1}[q] & & \\ \uparrow & & \uparrow & & \\ \lambda^2 T & \to & T \otimes \Omega^{p-1}[q] & \to & \sigma^2\Omega^{p-1}[q] \end{array}$$

Given $v \in \mathbf{H}^0(J_2(T))$, the assumption $G_\eta(v)$ is of level p to first order means that writing

$$v = (v_2, v_1), \ v_i \in S^i(\check{Z}^1(T)), \ b(v_2) = \delta(v_1),$$

we have some $\omega_1' \in C^q(\Omega^{p-1}) \otimes \check{Z}^1(T)$ with

$$M_{1,2,p}(v_2 \otimes \eta) = \delta(\omega_1').$$

To lift this data to $\mathbf{H}^0(J_2(\mathcal{L}))$ requires precisely

$$\begin{aligned} &\omega_0' \in \check{C}^q(\Omega^{p-1}), \\ &\delta(\omega_0') = L(\omega_1) + i(v_1)(\eta). \end{aligned}$$

In other words, the obstruction is $[L(\omega_1) + i(v_1)(\eta)] \in H^{q+1}(\Omega^{p-1})$. On the other hand as we saw above (5) the obstruction to $G_\eta(v)$ being of level p is

$$[L(\omega_1) + i(v_1)(\eta) + d(M_{1,1,p-1}(\omega_1))] \in H^{q+1}(\Omega^{p-1}).$$

Now X being Kähler, we have $[d(M_{1,1,p-1}(\omega_1))] = 0$, so the two obstructions coincide.

In the general m-th order case, the situation is similar: given $v \in \mathbf{H}^0(J_m(T))$ plus data $\omega_{m-1}, \dots, \omega_0$ making $G_\eta(v)$ of level p, the data required to lift v to $\mathbf{H}^0(J_m(\mathcal{L}))$ consists of cochains $\omega'_{m-1}, \dots, \omega'_0$ with

$$\delta(\omega'_i) = \delta(\omega_i) + (d - \text{exact cocycle}),$$

so again the obstructions are the same. □

We conclude with a brief partial treatment of semiregularity for maps, insofar as results follow from the above. Let

$$f : Y \to X$$

be a generically finite map of compact Kähler manifolds of dimensions $n - p$, n, and let $\tilde{Y} \subset Y \times X$ be the graph of f. Assuming, say, that

$$h^1(\mathcal{O}_X) = h^0(T_X) = 0, \tag{6}$$

it is well known that deformations of $Y \times X$ are of the form $Y' \times X'$ with Y', X' deformations of Y, X, respectively, and it follows easily that deformations of the triple (f, Y, X) correspond bijectively with deformations of the pair $(Y \times X, \tilde{Y})$, hence the above results apply. Note that

$$N_{\tilde{Y}} \simeq f^* T_X,$$

while $\tilde{\eta} = [\tilde{Y}] \in H^n(\Omega^n_{Y \times X}) \subset H^{2n}(Y \times X)$ is "the same" as the pullback map

$$\tilde{\eta}^* = f^* : H^\bullet(X) \to H^\bullet(Y)$$

or its dual, the Gysin map

$$\tilde{\eta}_* = f_* H^\bullet(Y) \to H^{\bullet + p}(X),$$

and $\tilde{\eta}$ being of type (n, n) means $\tilde{\eta}^*$ preserves Hodge level or $\tilde{\eta}_*$ raises Hodge level by $\leq p$, so we conclude that

COROLLARY 2. *Assuming* (6), *obstructions to deforming* f, *relative to deforming* X, Y *so that* $\tilde{\eta}_*$ *raises Hodge level by at most* p, *lie in*

$$\ker \tilde{\pi}_1 : H^1(f^*T_X) \to H^{n+1,n-1}(X \times Y).$$

Consider next the case of deformations of f with X fixed. As is well known [AC], these are controlled by the normal sheaf N_f, which fits in an exact diagram (identifying $T_{X\times Y} = p_1^*T_Y \otimes p_2^*T_X$):

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & p_1^*T_Y(-\tilde{Y}) & \to & T'_{X\times Y} & \to & T'_{X,f} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & p_1^*T_Y & \to & p_1^*T_Y \oplus p_2^*T_X & \to & p_2^*T_X & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & T_Y & \to & f^*T_X & \to & N_f & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \tag{7}$$

Again $N_f[-1]$ forms a DGLA sheaf on Y and obstructions to deforming (Y, f) are in $H^1(N_f)$ and come from the bracket map $N_f \times N_f \to N_f[1]$.

On the other hand, $H^{n-1,n+1}(Y \times X)$ has as one Künneth component

$$H^{n-p,n-p}(Y) \otimes H^{p-1,p+1}(X) \simeq H^{p-1,p+1}(X)$$

and by its definition the semiregularity map for $\tilde{Y}$ factors

$$\begin{array}{ccc} H^1(f^*T_X) & \xrightarrow{\tilde{\pi}_1} & H^{n-1,n+1}(Y \times X) \\ \downarrow & & \downarrow \\ H^1(N_f) & \xrightarrow{\pi_{1,f}} & H^{p-1,p+1}(X). \end{array}$$

where $\pi_{1,f}$ is induced by interior multiplication by the component of $[\tilde{Y}]$ in $H^{n-p,n-p}(Y) \otimes H^{p,p}(X)$ so we conclude (note this does not use assumption (6)):

COROLLARY 3. *Obstructions to deforming* (f, Y), *fixing* X, *relative to deforming* Y *so that* $\tilde{\eta}_*$ *raises Hodge level by* $\leq p$, *lie in* $\ker \pi_{1,f}$.

Note that there are many cases, e.g. Y is a curve and $h^{n-1}(\mathcal{O}_X) = 0$, where the cohomological condition on $\tilde{\eta}$ is vacuous, for then $\tilde{\eta}_*$ is simply given by

$$[Y]_Y \mapsto [Y]_X = \eta$$
$$[pt]_Y \mapsto [pt]_X.$$

In particular, suppose Y is a smooth connected curve of genus g and X is a $K3$ surface. Then from (7) we get a nonzero map

$$N_f \longrightarrow N_f/_{\mathrm{tor}} \longrightarrow K_Y$$

and the semiregularity map factors through $H^1(K_Y) \to H^2(\mathcal{O}_X)$ Serre dual to $H^0(K_X) \to H^0(K_X|_Y) = H^0(\mathcal{O}_Y)$, which map is clearly nonzero, hence so is $\pi_{1,f}$ because $H^1(N_f) \to H^1(K_Y)$ is surjective, Y being a curve. On the other hand $c_1(N_f) \sim K_Y$, so $\chi(N_f) = g - 1$. We conclude then that the deformation space of (f, Y) is at least g-dimensional.

Now suppose in addition that f is of degree 1 to its image $\bar{Y}$. It is then clear that *unobstructed* deformations of (f, Y) must move $\bar{Y}$, hence must project injectively to $H^0(N_f/_{\mathrm{tor}})$, and since $N_f/_{\mathrm{tor}}$ is a subsheaf of K_Y with quotient =tor (supported exactly on the ramification locus of f), its h^0 is $< g$ unless $g = 0$ or tor$= 0$; and if tor$= 0$, i.e. f is unramified, then $N_f \simeq K_Y$ so $\pi_{1,f}$ is injective and (f, Y) is unobstructed. So putting things together we conclude

COROLLARY 4. *On a $K3$ surface, the locus of integral curves of geometric genus $g > 0$ is generically reduced, purely g-dimensional (or empty), and smooth at any immersed curve.*

Appendix: The semiregularity homomorphism

In the course of the proof of Part (i) of the Theorem, we implicitly alluded to the fact that the semiregularity map π is a Lie homomorphism, which implies that so is π'. As this may not be generally known, we include a proof for completeness.

First we recall the local fundamental class and semi-regularity map. Take an acyclic cover $\mathcal{U} = (U_\alpha)$ of an open subset $U \subset X$ and let $Y \cap U_\alpha$ be defined by $F_\alpha = (f_\alpha^1, \dots, f_\alpha^p) = (0)$, and set

$$\mathrm{dlog} F_\alpha = \mathrm{dlog} f_\alpha^1 \wedge \ldots \wedge \mathrm{dlog} f_\alpha^p = \frac{df_\alpha^1 \wedge \ldots \wedge df_\alpha^p}{f_\alpha^1 \cdot \ldots \cdot f_\alpha^p}.$$

This yields a cocycle in $\check{Z}^{p-1}(U_\alpha \backslash Y, \Omega^p)(\Omega^p = \Omega_X^p)$, for the open cover $(D_{i,\alpha} = U_\alpha - (f_\alpha^i = 0))$, whence a class

$$\tilde{\eta}_\alpha \in H^p_{Y \cap U_\alpha}(\Omega^p) = H^0_{Y \cap U_\alpha}(\Omega^p[p]),$$

and these glue together to yield

$$\tilde{\eta}_U \in H^0_{Y \cap U}(U, \Omega^p[p])$$

which maps to the fundamental class

$$\eta_U = [Y \cap U] \in H^0(U, \Omega^p[p]).$$

(More pedantically, one computes $H^{p-1}(U - Y, \Omega^p)$ from the Čech bicomplex $(\check{C}^{\cdot\cdot}, \delta_1, \delta_2)$ associated to the biindexed cover $(D_{i,\alpha})$ of $U - Y$. Writing on $U_\alpha \cap U_\beta$

$$F_\beta = AF_\alpha$$

for a suitable matrix A expressed as a product of elemetary matrices, it is easy to see that

$$\delta_2(\mathrm{dlog} F_\alpha) = \mathrm{dlog} F_\alpha - \mathrm{dlog} F_\beta$$

is a sum of terms with $< p$ distinct f_α^i in the denominator, so by elementary properties of local cohomology there is an explicit $(p-2, 1)$-cochain $G_{\alpha\beta}$ with

$$\delta_1(G_{\alpha\beta}) = \mathrm{dlog} F_\alpha - \mathrm{dlog} F_\beta,$$

and $(\mathrm{dlog} F_\alpha, G_{\alpha\beta})$ is a bicocycle representing a class in $H^{p-1}(U - Y, \Omega^p)$ whose image is $\tilde{\eta}_U$.)

Now by a similar remark about denominators, note that $\tilde{\eta}_\alpha$ is killed by any function vanishing on $Y \cap U_\alpha$; likewise, if $v_\alpha \in \Gamma(U_\alpha, T')$, the interior product

$$i(v_\alpha)(\mathrm{dlog} F_\alpha) = \sum (-1)^i \frac{v_\alpha(f_\alpha^i)}{f_\alpha^i} \mathrm{dlog} f_\alpha^1 \wedge \ldots \wedge \widehat{\mathrm{dlog} f_\alpha^i} \wedge \ldots \wedge \mathrm{dlog} f_\alpha^p$$

has vanishing cohomology class $i(v_\alpha)(\tilde{\eta}_\alpha)$. Using the Čech bicomplex above these statements may be extended to U and hence globalised: thus the arrows given by (interior) multiplication by η

$$\mathcal{I}_Y \to \Omega^p[p]$$

$$T' \to \Omega^{p-1}[p]$$

vanish in the derived category; in particular interior multiplication by η descends to a map ('sheaf-theoretic semi-regularity')

$$\pi : N \to \Omega^{p-1}[p]$$

which induced the (cohomological) semi-regularity

$$\pi_1 = H^1(\pi) : H^1(N) \to H^{p+1}(\Omega^{p-1}),$$

as well as the infinitesimal Abel-Jacobi map

$$H^0(\pi) : H^0(N) \to H^p(\Omega^{p-1}).$$

Now we come to the crux of the (semi-regularity) matter:

LEMMA 1. *The composite*

$$N \times N \xrightarrow{[\,]} N[1] \xrightarrow{\pi} \Omega^{p-1}[p+1]$$

vanishes in the derived category; in other words, π is a Lie homomorphism in the derived category.

PROOF. First a calculus obervation: if ω is a closed $p-$form and x, y vector fields on a manifold then (check!)

$$i([x, y])(\omega) = L_x(i(y)\omega) - L_y(i(x)\omega) - d(i(x \wedge y)\omega)\,.$$

Now take sections $v' = (v'_\alpha)$, $v'' = (v''_\alpha) \in \Gamma(U, N)$. So

$$[v', v''] = ([t'_{\alpha,\beta} v''_\beta] - [t''_{\alpha\beta}, v'_\beta])$$

As $t'_{\alpha\beta}, t"_{\alpha\beta} \in T'(U_\alpha \cap U_\beta)$, note that the cohomology classes corresponding to $i(t'_{\alpha\beta} \wedge v''_\beta)(\mathrm{dlog} F_\alpha)$, $i(t_{\alpha\beta} \wedge v'_\beta)(\mathrm{dlog} F_\alpha)$ vanish, hence $\pi([v', v''])$ is represented by

$$L_{t'_{\alpha\beta}}(i(v''_\beta)\mathrm{dlog} F_\alpha) + L_{t''_{\alpha\beta}}(i(v'_\beta)\mathrm{dlog} F_\alpha) - L_{v''_\beta}(i(t'_{\alpha\beta})\mathrm{dlog} F_\alpha)$$

$$-L_{v'_\beta}(i(t''_{\alpha\beta})\mathrm{dlog} F_\alpha) - L_{t'_{\alpha\beta}}(i(v'_\alpha)\mathrm{dlog} F_\alpha) + L_{t''_{\alpha\beta}}(i(v'_\alpha)\mathrm{dlog} F_\alpha)$$

Now consider the diagram

$$\begin{array}{ccccc} N \times N & \to & T'[1] \otimes \Omega^p[p] \;\oplus\; \Omega^p[p] \otimes T'[1] & \to & T[1] \otimes \Omega^p[p] \oplus \Omega^p[p] \otimes T[1] \\ \downarrow & & \downarrow & & \downarrow \\ N[1] & \to & \Omega^{p-1}[p+1] & & = \Omega^{p-1}[p+1], \end{array}$$

where the top left arrow is given by $\partial \times \pi \oplus \pi \times \partial$, $\partial : N \to T'[1]$ the natural map. We have just proven that the left square commutes while the right one does obviously. Clearly the top arrows compose to zero because $N \to T'[1] \to T[1]$ do. Hence the composite $N \times N \to N[1] \to \Omega^{p-1}[p+1]$ vanishes, as claimed.

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