KUNIHIKO KAJITANI
KAORU YAMAGUTI

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Propagation of Analyticity of the Solutions to the Cauchy Problem for Nonlinear Symmetrizable Systems

KUNIHIKO KAJITANI – KAORU YAMAGUTI

Abstract. This paper is devoted to the study of propagation of analyticity of the solutions to the nonlinear Cauchy problem. Under the assumption that the system is nonsmoothly and uniformly symmetrizable and the initial data are analytic in space variables, one can prove that the solutions belonging to some Gevrey classes become analytic in space variables.

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Introduction

We shall investigate the propagation of analyticity of solutions to the Cauchy problem for nonlinear hyperbolic systems with non-smooth symmetrizer. Consider the following equation in $(0, T) \times R^n$,

$$(1)\quad \frac{\partial u}{\partial t} - \sum_{j=1}^{n} A_j(t, x, u)\frac{\partial u}{\partial x_j} - A_0(t, x, u)u = f(t, x),\ (t, x) \in (0, T) \times R^n\,,$$

$$(2)\qquad u(0, x) = u_0(x),\ x \in R^n\,,$$

where u, f are vector valued functions of N-components and $A_j(t, x, u)$ are $N \times N$ matrices defined in $[0, T] \times R^n \times G$ (G is a bounded open set in R^N). Denote $A(t, x, \xi, u) = \sum_{j=1}^{n} A_j(t, x, u)\xi_j$. We say that $A(t, x, \xi, u)$ is uniformly symmetrizable, if there are a symmetric matrix $R(t, x, \xi, u) \in L^\infty((0, T) \times R^n \times R^n \times G)$ and $c_0 > 0$ such that

$$(3)\qquad R(t, x, \xi, u) \geq c_0 I\,,$$

for a.e.$(t, x, \xi, u) \in [0, T] \times R^n \times R^n \times G$ and $(RA)(t, x, \xi, u)$ is symmetric for a.e. $(t, x, \xi, u) \in [0, T] \times R^n \times R^n \times G$. We remark that $A(t, x, \xi, u)$ is

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uniformly symmetrizable if and only if $A(t,x,\xi,u)$ is a uniformly diagonalizable hyperbolic matrix. This equivalence is proved in [6].

We introduce some function spaces and their norms and semi-norms. For $d \geq 1$ and $B \subset R^n$ we denote by $\gamma^{(d)}(B)$ the set of functions $u(x) \in C^\infty(\overline{B})$ satisfying for all $\rho > 0$

$$|u|_{B,\rho,d} = \sup_{x\in\overline{B},\alpha\in N^n} \frac{|D_x^\alpha u(x)|\rho^{|\alpha|}}{|\alpha|!^d} < \infty .$$

Denote by $\mathcal{A}\ (B)$ the set of real analytic functions $u(x)$ defined in $\overline{B}$ satisfying that there is $\rho > 0$ such that $|u|_{B,\rho,1} < \infty$. For $B \subset R^n, G \subset R^N$ denote by $\gamma^{(d)}(B;\mathcal{A}(G))$ the set of function $f(x,u)$ defined in $\overline{B}\times\overline{G}$ satisfying that for any $\rho > 0$ and for some $\rho_0 > 0$,

$$|f|_{\gamma^{(d)}(B;\mathcal{A}(G)),\rho,\rho_0} = \sup_{x\in\overline{B},u\in\overline{G},\alpha\in N^n,\beta\in N^N} \frac{|D_x^\alpha D_u^\beta f(x,u)|\rho^{|\alpha|}\rho_0^{|\beta|}}{|\alpha|!^d|\beta|!} < \infty .$$

For $\rho > 0, d > 1$ and $m \in R$ define

$$H^m_{\rho,d} = \left\{u \in L^2(R^n);\ \langle\xi\rangle_h^m e^{\rho\langle\xi\rangle_h^{1/d}}\hat{u}(\xi) \in L^2(R^n)\right\},$$

where $\hat{u}(\xi)$ stands for a Fourier transform of u, $\langle\xi\rangle_h = \sqrt{h^2+|\xi|^2}$ and for $\rho < 0$ define $H^m_{\rho,d}$ as the dual space of $H^{-m}_{-\rho,d}$. Denote $L^2_d(R^n) = \cap_{\rho>0}H^0_{\rho,d}(R^n)$. For a topological space X we denote by $C^k([0,T];X)$ the set of functions which are k times differentiable, if k is a nonnegative integer (k-Hölder continuous, if $0 < k < 1$) in X with respect to t in $[0,T]$. We start to state the result of the linear equations. When the coefficients A_j are independent of unkown functions u, we know the following result.

THEOREM 1. *Assume* $A(t,x,\xi) = \sum_{j=1}^n A_j(t,x)\xi_j$ *is uniformly symmetrizable and the coefficients* $A_j(t,x) \in C^\mu([0,T];\gamma^{(d)}(R^n))$ *for* $j = 1,\dots,n$ *and* $A_0(t,x) \in C^0([0,T];\gamma^{(d)}(R^n))$. *If* $0 < \mu \leq 1$ *and* $1 < d \leq 1+\mu$, *then for any* $u_0 \in L^2_d(R^n)$ *and* $f \in C^0([0,T];L^2_d(R^n))$ *there exists a solution* $u(t,x) \in C^1([0,T];L^2_d(R^n))$ *of the Cauchy problem* (1)-(2).

Moreover if $Lu = 0$ *in* $\Gamma(\hat{t},\hat{x})$ *and* $u = 0$ *on* $\Gamma(\hat{t},\hat{x}) \cap \{t = 0\}$, *then* $u = 0$ *in* $\Gamma(\hat{t},\hat{x})$, *where we denote by* $\Gamma(\hat{t},\hat{x})$ *the set* $\{(t,x) \in [0,\hat{t}]\times R^n; |x-\hat{x}| \leq \lambda_{\max}(\hat{t}-t), 0 \leq t \leq \hat{t}\}$ ($\lambda_{\max}$ *stands for the maximum of* $\lambda_j(t,x,\xi)|\xi|^{-1}$ *with respect to* (j,t,x,ξ) *and* $\lambda_j(t,x,\xi)$ *are the eigen values of* $A(t,x,\xi)$).

The proof of this theorem can be seen in [7].

Next we consider the nonlinear Cauchy problem (1)-(2). Then we get the following local existence theorem.

THEOREM 2. *Assume that* $A(t, x, \xi, u)$ *is uniformly symmetrizable and* $A_j(t, x, u)$ $(j = 1, \dots, n)$ *belong to* $C^\mu([0, T]; \gamma^{(d)}(R^n; \mathcal{A}(G))$, $A_0(t, x, u)$ *is in* $C^0([0, T]; \gamma^{(d)}(R^n; \mathcal{A}(G))$. *If* $0 < \mu \le 1$ *and* $1 < d \le 1 + \mu$, *for any* $u_0 \in L^2_d(R^n)$ *and* $f \in C^0([0, T]; L^2_d(R^n))$, *then there is* $T_0 \in (0, T)$ *such that there exists a solution* $u \in C^1([0, T_0); L^2_d(R^n))$ *of the Cauchy problem* (1)-(2) *with* $T = T_0$.

Moreover if we take $f \equiv 0$ *and the initial data* $u_0 = \varepsilon\varphi$, *where* $\varepsilon > 0$ *and* $\varphi \in L^2_d(R^n)$, *the time of existence of solution* $T_0 = T_0(\varepsilon)$ *tends to the infinity for* $\varepsilon \to 0$.

We remark that when the coefficients are smooth in the time variable, Theorem 2 proved in [5] in the case of general hyperbolic systems (not necessarily symmetrizable).

Moreover we can investigate the propagation of analyticity of solutions to the Cauchy problem (1)-(2), when the initial data are analytic.

THEOREM 3. *Let* $0 < \hat{t} \le T$. *Assume that* $A(t, x, \xi, u)$ *is uniformly symmetrizable and* $A_j(t, x, u) (j = 1, \dots, n)$ *belong to* $C^\mu([0, T]; \mathcal{A}(R^n \times B))$, $A_0(t, x, u)$ *is in* $C^0([0, T]; \mathcal{A}(R^n \times B))$ *and besides assume that the coefficients* $A_j(t, x, u)$ $(j = 0, 1, \dots, n)$ *are real analytic with respect to* $(x, u) \in (\Gamma(\hat{t}, \hat{x}) \cap \{t = t\}) \times G$. *Let* u *in* $C^1([0, T]; L^2_d(R^n))$ *be a solution of the Cauchy problem* (1)-(2). *If* $0 < \mu \le 1$ *and* $1 < d \le 1 + \mu$ *and* $u_0(x)$ *is analytic in* $\Gamma(\hat{t}, \hat{x}) \cap \{t = 0\}$ *and* $f(t, x)$ *is analytic with respect to* x *in* $\Gamma(\hat{t}, \hat{x}) \cap \{t = t\}$ *for* $t \in [0, \hat{t}]$, *then the solution* $u(t, x)$ *is analytic with respect to* x *in* $\Gamma(\hat{t}, \hat{x}) \cap \{t = t\}$ *for any* $t \in [0, \hat{t}]$.

When L is strictly hyperbolic, Theorem 3 has proved by Mizohata in [10] in the linear case and by Alnihac and Métivier in [1] in the nonlinear case respectively. When L is a second order degenerate hyperbolic and a higher order hyperbolic operator with constant multiplicity, Spagnolo in [11] and Cicognani and Zangihrati in [3] proved Theorem 3 in Gevrey classes respectively. Cicognani in [2] treated this problem when L is strictly hyperbolic and the coefficients are Hölder continuous in the time variable. This result asserts that if the symmetrizer is smooth, the above Theorem 3 holds for $d \le (1 - \mu)^{-1}$. Recently D'Ancona and Spagnolo in [4] investigate the propagation of analyticity for non-uniformly symmetrizable systems.

1. – A priori estimate

We shall derive a priori estimate in Gevrey class $H^m_{\rho,d}(R^n)$ for uniformly symmetrizable linear systems. For $m, \rho, \delta \in R (0 \le \delta \le \rho \le 1)$ denote by $S^m_{\rho,\delta}$ the usual symbol class of oder m of ρ, δ type. For simplicity we write $S^m = S^m_{1,0}$ and introduce the seminorms as follows,

$$|a|^{(m)}_\ell = \sup_{|\alpha+\beta| \le \ell, x, \xi \in R^n} \frac{|a^{(\alpha)}_{(\beta)}(x, \xi)|}{\langle \xi \rangle_h^{m-|\alpha|}}, \tag{1.1}$$

where $\langle\xi\rangle_h = (h^2 + |\xi|^2)^{\frac{1}{2}} (h > 0$ is a large parameter).

Next we define the symbols of Gevrey class in R^n. For $d \geq 1$ denote by $\gamma^d S^m$ the set of symbols $a \in S^m$ satisfying for any $\rho > 0$,

$$|a|^{(m)}_{\rho,d.\ell} = \sup_{x,\xi\in R^n, |\alpha+\beta|\leq \ell, \delta,\gamma\in N^n} \frac{|a^{(\alpha+\delta)}_{(\beta+\gamma)}(x,\xi)|\rho^{|\delta+\gamma|}}{\langle\xi\rangle_h^{m-|\alpha+\delta|}|\delta+\gamma|!^d} < \infty .$$

Define for $\rho \in R$

$$e^{\rho\langle D\rangle_h^{1/d}} u(x) = (2\pi)^{-n} \int e^{ix\xi + \rho\langle\xi\rangle_h^{1/d}} \hat{u}(\xi) d\xi ,$$

for $u \in H^m_\rho$. Then $e^{\rho\langle D\rangle_h^{1/d}}$ maps from $H^m_{\rho_1,d}$ to $H^m_{\rho_1-\rho,d}$ continuously. Moreover for $a \in \gamma^d S^{m_1}$ we can see that $a(x,D)$ maps from $H^m_{\rho,d}$ to $H^{m-m_1}_{\rho,d}$ continuously and consequently

$$a(\rho,x,D) = e^{-\rho\langle D\rangle_h^{1/d}} a(x,D) e^{\rho\langle D\rangle_h^{1/d}} \tag{1.2}$$

maps H^m to H^{m-m_1} continuously, where $H^m = \{u \in \mathcal{S}'(R^n); \langle\xi\rangle_h^m \hat{u}(\xi) \in L^2(R^n)\}$ is the usual Sobolev space. Moreover we can prove the following proposition.

PROPOSITION 1.1. *Let $d > 1$, $\rho \in R$, $a \in \gamma^d S^m$. Then the symbol of $a(\rho,x,D)$ given by* (1.2) *belongs to S^m and satisfies*

$$a(\rho,x,\xi) = a(x,\xi) + \rho r(a)(\rho,x,\xi) , \tag{1.3}$$

where the remainder term $r(a)(\rho,x,\xi)$ belongs to $S^{m-1+1/d}$ and moreover there are C independent of h, ρ and $h_0(\rho) > 0$ such that for $h \geq h_0(\rho)$

$$\begin{aligned} &|r(a)|^{(m-1+1/d)}_\ell \leq C|a|^{(m)}_{\rho_0,d,\ell_0} , \\ &|\rho| = 24^{-1} n^{-1/d} \rho_0^{1/d}, \ell_0 = [\ell(1-1/d)^{-1}] + 2\ell + [n/2] + 2 . \end{aligned} \tag{1.4}$$

The proof of this proposition is given in Lemma 1.2 of [7].

We shall derive a priori estimate for the linear operator L,

$$L[u] = \frac{\partial u}{\partial t} - \sum_{j=1}^n A_j(t,x) \frac{\partial u}{\partial x_j} - A_0(t,x)u . \tag{1.5}$$

We assume that $A(t,x,\xi) = \sum_{j=1}^n A_j(t,x)\xi_j$ is uniformly symmetrizable. Let $\chi_0(t), \chi(t)$ be functions in $C_0^\infty(R)$ such that $\chi(t), \chi_0(t) \geq 0$, $\int_{-\infty}^\infty \chi(t)dt = 1$ and $\int_0^T \chi_0(t)dt = 1$ and $\operatorname{supp}\chi(t) \subset (-1,1)$, $\operatorname{supp}\chi_0(t) \subset (0,T)$. For the symmetrizer $R(t,x,\xi)$ of $A(t,x,\xi)$ we define an approximate smooth symbol of R following the idea of Kumanogo-Nagase in [8],

$$\begin{aligned} \widetilde{R}(t,x,\xi) = \int_0^T \int\int &\chi_0((t-s)\langle\xi\rangle_h^{\delta_0})\chi(|\xi-\eta|\langle\xi\rangle_h^{-\rho}) \\ &\times \chi(|x-y|\langle\xi\rangle_h^{\delta}) R(s,y,\eta) ds\, dy\, d\eta \langle\xi\rangle_h^{\delta_0+(\delta-\rho)n} . \end{aligned} \tag{1.6}$$

Then we can prove easily the following lemma.

LEMMA 1.2. *Assume that $A(t,x,\xi)$ is unifomly symmetrizable and belongs to $C^{\mu}([0,T];S^1)$, where $0<\mu\le 1$. Let $0<\delta_0,\delta,\rho<1$ be chosen such that $\epsilon_0=\min\{\delta,1-\rho,\delta_0\mu\}>0$. Then $\widetilde{R}(t,x,\xi)$ given in (1.6) belongs to $S^0_{\rho,\delta}$, $\frac{\partial}{\partial t}\widetilde{R}(t,x,\xi)$ is in $S^{\delta_0}_{\rho,\delta}$ and moreover there is $h\gg 1$ such that $\widetilde{R}(t,x,\xi)$ satisfies,*

$$\text{(i)}\qquad \operatorname{Re}(\widetilde{R}(t,x,D)u,u)_{L^2}\ge c_0\|u\|^2_{L^2},$$

for $u\in L^2$ and

$$\text{(ii)}\qquad \sigma((\widetilde{R}(t,x,D)A(t,x,D)-(A^*(t,x,D)\widetilde{R}^*(t,x,D))(x,\xi)\in S^{1-\epsilon_0}_{\rho,\delta},$$

uniformly in $t\in[0,T]$.

We shall now prove the following result.

THEOREM 1.3. *Assume that $A(t,x,\xi)$ is uniformly symmetrizable and the coefficients $A_j(t,x)(j=1,\dots,n)$ are in $C^{\mu}([0,T];\gamma^{(d)}(R^n))$ and $A_0(t,x)$ in $C^0([0,T];\gamma^{(d)}(R^n))$. If $0<\mu\le 1$ and $1<d\le 1+\mu$ and $t_0\in[0,T]$, then there is a positive function $\rho(t)\in C^1([t_0,T])$ such that for any $u(t,x)\in C^1([t_0,T];L^2_d)$ we have,*

$$\text{(1.7)}\qquad \|\langle D\rangle_h^{\ell}e^{\rho(t)\langle D\rangle_h^{1/d}}u(t,\cdot)\|_{L^2(R^n_x)}\le C_\ell\left\{\|\langle D\rangle_h^{\ell}e^{\rho(t_0)\langle D\rangle_h^{1/d}}u(t_0,\cdot)\|_{L^2(R^n)}+\int_{t_0}^t\|\langle D\rangle_h^{\ell}e^{\rho(s)\langle D\rangle_h^{1/d}}Lu(s,\cdot)\|_{L^2(R^n_x)}ds\right\}$$

for $t\in[t_0,T]$.

PROOF. Put $v(t,x)=e^{\rho(t)\langle D\rangle_h^{1/d}}u(t,x)$. Then it follows from (1.3) and (1.4) of Proposition 1.1 that we have

$$\text{(1.8)}\qquad e^{\rho(t)\langle D\rangle_h^{1/d}}L[u(t,x)]=(L-\rho'(t)\langle D\rangle_h^{1/d}-\rho(t)K(t,x,D))v(t,x)=:\tilde{L}[v],$$

where

$$\text{(1.9)}\qquad K(t,x,\xi)=i\sum_{j=1}^{n}r(A_j)(t,x,\xi)\xi_j+r(A_0)(t,x,\xi),$$

belongs to $C^0([0,T];S^{1/d})$. Hence to derive (1.7) it suffices to prove the following estimate

$$\text{(1.10)}\qquad \|v(t,\cdot)\|_\ell\le C_\ell\left(\|v(t_0)\|_\ell+\int_{t_0}^t\|\tilde{L}v(s)\|_\ell\right),$$

for $t \in [t_0, T]$, where we denote by $\|\cdot\|_\ell$ the norm of Sobolev space H^ℓ. For simplicity we prove (1.10) with $\ell = 0$. Define for $v(t) \in C^1([t_0, T]; L^2) \cap C^0([t_0, T]; H^1)$,

$$e(t) = \frac{1}{2}\mathrm{Re}(\widetilde{R}(t, x, D)v(t), v(t)).$$

Then it follows from (i) in Lemma 1.2 that there are $h > 0, c_0 > 0$ and $c_1 > 0$ such that we have for $t \in [0, T]$

$$c_0\|v(t)\|_{L^2} \le e(t) \le c_1\|v(t)\|_{L^2}. \tag{1.11}$$

Differentiating $e(t)$ with respect to t, we get

$$\begin{aligned} e'(t) &= \frac{1}{2}\mathrm{Re}(\widetilde{R}'(t)v(t), v(t)) + \mathrm{Re}(\widetilde{R}(t)v'(t), v(t)) \\ &= \frac{1}{2}\mathrm{Re}(\widetilde{R}'(t, x, D)v(t), v(t)) + \rho'(t)(\widetilde{R}(t)\langle D\rangle_h^{1/d} v(t), v(t)) \\ &\quad + \mathrm{Re}(\widetilde{R}(t, x, D)(A(t, x, D) + A_0(t, x, D) + \rho(t)K(t, x, D))v(t), v(t)) \\ &\quad + \mathrm{Re}(\widetilde{R}(t, x, D)\tilde{L}v(t), v(t)). \end{aligned}$$

Taking account of Lemma 1.2 we can see easily

$$\begin{aligned} e'(t) \le &\left(\left\{\rho'(t)\langle D\rangle_h^{1/d} + C(\langle D\rangle_h^{\delta_0} + \langle D\rangle_h^{1-\epsilon_0} + \rho(t)\langle D\rangle_h^{1/d})\right\}v(t), v(t)\right) \\ &+ \|\tilde{L}v(t)\|\|v(t)\|. \end{aligned} \tag{1.12}$$

We can take δ, δ_0, ρ such that

$$\delta_0 \le 1/d, 1 - \epsilon_0 \le 1/d$$

if $d \le 1 + \mu$ where ϵ_0 is given in Lemma 1.2. Then we can choose $\rho(t)$ as follows,

$$\begin{gathered} \rho'(t) + C\rho(t) + C \le 0, t \in [t_0, T], \\ \rho(t) > 0, t \in [t_0, T]. \end{gathered}$$

Therefore from (i) of Lemma 1.2, (1.11) and (1.12) we obtain (1.10) for $\ell=0$. □

Let $R > 0$ and $\hat{x} \in R^n$. Denote $B(R) = \{x \in R^n; |x-\hat{x}| < R\}$. We define a Gevrey space over $B(R)$. Denote by $L^2_d(B(R))$ the set $\{u \in \gamma^{(d)}(B(R)); \exists\, \tilde{u} \in L^2_d(R^n)$ s.t.$\tilde{u} = u$ in$B(R)\}$ and for $u \in L^2_d(B(R))$ we introduce a norm as follows,

$$\|u\|_{H^m_{\rho,d}(B(R))} = \inf_{\tilde{u}\in L^2_d(R^n), \tilde{u}|_{B(R)}=u} \|\tilde{u}\|_{H^m_{\rho,d}(R^n)}.$$

Let $(\hat{t}, \hat{x}) \in [0, T] \times R^n$ and $\rho(t)$ a real valud continuous function defined in $[0, T]$. For $\Gamma(\hat{t}, \hat{x})$ defined in Theorem 1 we denote $B_s(\hat{t}, \hat{x}) = \Gamma(\hat{t}, \hat{x}) \cap \{t = s\}$ for $s \in [0, \hat{t}]$. Then we note that if u is in $C^0([0, T]; L^2_d(R^n))$ for any $(\hat{t}, \hat{x}) \in (0, T] \times R^n$, u belongs to $H^m_{\rho(t),d}(B_t(\hat{t}, \hat{x}))$.

THEOREM 1.4. *Assume that the same conditions as in Theorem* 1.3 *are valid and let* $u \in C^1([0,T]; L^2_d(R^n))$, $t_0 \in [0,T)$ *and* $\rho(t)$ *is given in Theorem* 1.3. *Then there is a positive constant* C *such that we have*

$$\|u(t)\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \leq C\left\{ \|u(t_0)\|_{H^m_{\rho(0),d}(B_{t_0}(\hat{t},\hat{x}))} + \int_{t_0}^{t} \|Lu(s)\|_{H^m_{\rho(s),d}(B_s(\hat{t},\hat{x}))} ds \right\}, \tag{1.13}$$

for $t \in [t_0, \hat{t})$.

PROOF. Put $f(t,x) = Lu(t,x)$. Let $\tilde{f} \in C^0([0,T]; L^2_d(R^n))$ such that $\tilde{f} = f$ in $\Gamma(\hat{t},\hat{x}) \cap \{t \geq t_0\}$ and $\tilde{u}_0(x)$ in $L^2_d(R^n)$ such that $\tilde{u}_0(x) = u(t_0,x)$ in $B_{t_0}(\hat{t},\hat{x})$. Then it follows from Theorem 1 in the introduction that there is a solution $\tilde{u}$ of the Cauchy problem $L\tilde{u} = \tilde{f}$, for $t \in (t_0,\hat{t}), x \in R^n$ and $\tilde{u}(t_0) = \tilde{u}_0$ satisfying $\tilde{u}(t,x) = u(t,x)$ in $\Gamma(\hat{t},\hat{x})$ and the estimate (1.10). Noting that $\|u(t)\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \leq \|\tilde{u}(t)\|_{H^m_{\rho(t),d}(R^n))}$, we get the estimate (1.13) taking the infimun of $\tilde{f}$ in (1.10). □

2. – Local existence theorem

In this section we shall prove Theorem 2 in the introduction by use of the standard contruction mapping method. We may assume the initial data $u_0 \equiv 0$ and $f(0,x) \equiv 0$ without loss of generality, if necessary, changing the unknown function $u = w + u_0 + \frac{\partial u}{\partial t}(0)t$. We define for $T_0 \in (0,T]$, $M > 0$,

$$X_m(T_0,M) = \left\{ u \in C^1([0,T]; L^2_d(R^n));\ u \text{ satisfying } |u|_{d,m,T_0} := \sup_{0\leq t\leq T_0} \left(\|u(t)\|_{H^m_{\rho(t),d}(R^n)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H^{m-1}_{\rho(t),d}(R^n)} \right) \leq M \right\}, \tag{2.1}$$

where $\rho(t)$ is given in Theoren 1.3. For $v \in X_m(T_0,M)$ we define an operator Φ from $X_m(T_0,M)$ to $C^1([0,T_0]; L^2_d(R^n))$ by $\Phi(v) = u$ which is a solution of the following Cauchy problem,

$$L(t,x,v)u(t,x) = f(t,x), (t,x) \in (0,T_0) \times R^n, \tag{2.2}$$

$$u(0,x) = 0, x \in R^n. \tag{2.3}$$

Then we can prove that Φ is a mapping from $X_m(T_0, M)$ into itself, if we choose T_0, m, M suitably. First we take $m \in N$ sufficiently large such that for $u, v \in H^{m-1}_{\rho,d}(R^n)$ we have from Lemma 1.2 in [5],

$$\|uv\|_{H^{m-1}_{\rho,d}(R^n)} \leq C_m \|u\|_{H^{m-1}_{\rho,d}(R^n)} \|v\|_{H^{m-1}_{\rho,d}(R^n)}. \tag{2.4}$$

Next we prove that $\Phi(v)$ belongs to $X_m(T_0, M)$. Since v is in $X_m(T_0, M)$, $A_j(t,x) = A_j(t,x,v(t,x))(j = 1,\dots,n)$ are in $C^\mu([0,T_0]; \gamma^{(d)}(R^n))$ and $A_0(t,x) = A_0(t,x,v(t,x))$ is in $C^0([0,T_0]; \gamma^{(d)}(R^n))$. Therefore it follows from Theorem 1 and (1.7) of Theorem 1.3 (with $u_0 \equiv 0$) that by use of (2.4) we can see that the solution $u = \Phi(v)$ in $C^1([0,T_0]; L^2_d(R^n))$ satisfies

$$|\Phi(v)|_{d,m,T_0} \leq C(M)\left(\int_0^{T_0} \|f(s)\|_{H^m_{\rho(s),d}(R^n)}ds + \sup_{0\leq t\leq T_0} \|f(t)\|_{H^m_{\rho(t),d}(R^n)}\right) \leq M,$$

if we choose $T_0 = T_0(M) > 0$ small enough, because of $\|f(0)\|_{H^m_{\rho(0),d}(R^n)} = 0$. Finally we shall prove that if we let $T_0 > 0$ more small (if necessary), we have for $v_1, v_2 \in X_m(T_0, M)$

$$\sup_{0\leq t\leq T_0} \|\Phi(v_1) - \Phi(v_2)\|_{H^{m-1}_{\rho(t),d}(R^n)} \leq \frac{1}{2} \sup_{0\leq t\leq T_0} \|v_1 - v_2\|_{H^{m-1}_{\rho(t),d}(R^n)}. \tag{2.5}$$

Put $w = \Phi(v_1) - \Phi(v_2)$. Then w satisfies

$$L(t,x,v_1)w = (L(t,x,v_2) - L(t,x,v_1))v_1,\ w(0,x) = 0.$$

Hence using again (1.7) and (2.4) we get

$$\begin{aligned}\|w(t)\|_{H^{m-1}_{\rho(t),d}(R^n)} &\leq C(M)\int_0^t \|(L(v_1) - L(v_2))v_1(s)\|_{H^{m-1}_{\rho(s),d}(R^n)}ds \\ &\leq C(M)\int_0^t \|v_1 - v_2\|_{H^{m-1}_{\rho(s),d}(R^n)}ds,\end{aligned}$$

which implies (2.5), if $T_0 > 0$ is sufficiently small. Thus we can obtain a solution in $X_m(T_0, M)$ of the Cauchy problem (1)-(2). Moreover when $f \equiv 0$ and $u_0 = \epsilon\phi$, we can take $f = \epsilon\tilde{\phi}$ in the equation (2.2). Hence we can see easily that we take $T_0 = C(M)^{-1}\epsilon^{-1}$. Thus we have proved Theorem 2.

3. – Propagation of analyticity

To investigate the analyticity of solutions to the Cauchy problem (1)-(2), we introduce a convinient norm in $C^0([0,T]; L^2_d(R^n))$ following the idea of Lax [9], Mizohata [10] and Alnihac and Métivier [1]. Let $(\hat{t},\hat{x}) \in (0,T] \times R^n$ and $\Gamma(\hat{t},\hat{x})$ defined in Theorem 1. Denote $B_s(\hat{t},\hat{x}) = \Gamma(\hat{t},\hat{x}) \cap \{t = s\}$. Let a positive integer $N \geq 2$, $t_0 \in [0,T)$ and $0 < r \leq 1$. For $u \in C^0([t_0,T]; L^2_d(R^n))$, we denote

$$|u|^{t,t_0}_{r,N} = \sup_{t_0 \leq s \leq t, 2 \leq |\beta| \leq N} \frac{\|D^\beta_x u\|_{H^m_{\rho(s),d}(B_s(\hat{t},\hat{x}))} r^{|\beta|-2}}{\Gamma_2(|\beta|)}, \tag{3.1}$$

where $\Gamma_2(k) = \lambda_0 k! k^{-2}$ for $k \geq 1$ and $\Gamma_2(0) = \lambda_0$. Then we can choose $\lambda_0 > 0$ such that

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \Gamma_2(|\alpha'| + p)\Gamma_2(|\alpha - \alpha'|) \leq \Gamma_2(|\alpha| + p), \tag{3.2}$$

for $p = 0, 1, 2, \ldots$ and $\alpha \in N^n$. In brief we write $|u|^t_{r,N} = |u|^{t,t_0}_{r,N}$, if there is no confusion.

Lemma 3.1. *For $v_i \in C^0([t_0,T]; L^2_d(R^n)), i = 1, \ldots, n$, denote $v^\beta = v_1^{\beta_1} v_2^{\beta_2} \ldots v_n^{\beta_n}$. Then there is $C_0 > 0$ such that for $2 \leq |\beta| \leq N$, $t \in [t_0, \hat{t}]$,*

$$\text{(i)} \qquad |v^\beta|^t_{r,N} \leq C_0^{|\beta|-1} \left(\sup_{t_0 \leq s \leq t} \|v(s)\|_{H^{m+1}_{\rho(s),d}(B_s(\hat{t},\hat{x}))} + r^2 |v|^t_{r,N} \right)^{|\beta|-1} |v|^t_{r,N},$$

and for $|\beta| > N$, $t \in [t_o, \hat{t}]$

$$\text{(ii)} \qquad \begin{aligned} |v^\beta|^t_{r,N} \leq{}& C_0^{|\beta|-1} \sup_{t_0 \leq s \leq t} \|v(s)\|^{|\beta|-N}_{H^m_{\rho(s),d}(B_s(\hat{t},\hat{x}))} \\ &\times \left(\sup_{t_0 \leq s \leq t} \|v(s)\|_{H^{m+1}_{\rho(s),d}(B_s(\hat{t},\hat{x}))} + r^2 |v|^t_{r,N} \right)^{N-1} |v|^t_{r,N}. \end{aligned}$$

(iii) *Let $d > 1$, $B(R) = \{x \in R^n; |x - \hat{x}| \leq R\}$ and $a(x) \in \mathcal{A}(B(R))$ satisfying $|a|_{B(R),\rho_0,1} < \infty$. Then there is $\tilde{a}(x) \in \gamma^{(d)}(R^n)$ satisfying that $\tilde{a}(x) = a(x)$ in $B(R)$ and for any $\rho > 0$ there are $C > 0$ such that*

$$|\tilde{a}|_{R^n,\rho,d} \leq C |a|_{B(R),\rho_0,1}.$$

(iv) *Let $a(x) \in \mathcal{A}(B_{t_0}(\hat{t},\hat{x}))$ satisfying $|a|_{B_{t_0}(\hat{t},\hat{x}),\rho_0,1} < \infty$. Then there are $C > 0$ such that*

$$|av|^t_{r,N} \leq C |a|_{B_{t_0}(\hat{t},\hat{x}),\rho_0,1} |v|^t_{r,N}.$$

(v) *Let G an open set in R^N and $A(t,x,v)$ is analytic in $(x,v) \in B_{t_0}(\hat{t},\hat{x}) \times G$ and saisfies*

$$|A|^t_{B_{t_0}(\hat{t},\hat{x})\times G,\rho_0,1} = \sup_{t_0 \le s \le t, \alpha,\beta \in N^n, x \in B_{t_0}, v \in G} \frac{|D_x^\alpha D_v^\beta A(t,x,v)|\rho_0^{|\alpha|+|\beta|}}{|\alpha|!|\beta|!} < \infty .$$

Then if $\|v(t)\|_{H^{m+1}_{\rho(t),d}(B_{t_0}(\hat{t},\hat{x}))} < \rho_0/n$, the composite function $A \circ v(t,x) = A(t,x,v(t,x))$ satisfies that there is $C > 0$ such that for $t \in [t_0,\hat{t}]$

$$|A \circ v|^t_{r,N} < C|A|^t_{B_{t_0}(\hat{t},\hat{x})\times G,\rho_0,1} \times \left(1 + \sum_{j=0}^{N-1}\left(\sup_{t_0 \le s \le t}\|v(s)\|_{H^{m+1}_{\rho(s),d}(B_s(\hat{t},\hat{x}))} + r^2|v|^t_{r,N}\right)^j |v|^t_{r,N}\right) .$$

The proof of this lemma will be given in the appendix. We remark that it follows from (v) of Lemma 3.1 that if $A(t,x,v)$ is analytic in $(x,v) \in B_{t_0}(\hat{t},\hat{x}) \times G$, we have for $2 \le |\alpha| \le N, t \in [t_0,\hat{t})$

$$\begin{aligned} &\|D_x^\alpha(A \circ v)(t)\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \le Cr^{-|\alpha|+2}\Gamma_2(|\alpha|) \\ &\times \left\{1 + \sum_{j=0}^{N-1}\left(\sup_{t_0 \le s \le t}\|v(s)\|_{H^{m+1}_{\rho(s),d}(B_s(\hat{t},\hat{x}))} + r^2|v|^t_{r,N}\right)^j |v|^t_{r,N}\right\} , \end{aligned} \tag{3.3}$$

where C is independent of N.

THE PROOF OF THEOREM 3. Let u be satisfied with (1)-(2). Since $u \in C^0([0,T]; L^2_d(R^n))$, for any $\epsilon > 0$ there is $\tau > 0$ such that

$$\|u(t,\cdot) - u(i\tau,\cdot)\|_{H^{m+1}_{\rho(t),d}(R^n)} < \epsilon , \tag{3.4}$$

for $t \in [i\tau,(i+1)\tau], i = 0,1,\dots,[T/\tau]-1$ and $t \in [[T/\tau]\tau,T]$, where $[\cdot]$ stands for the Gauss notation. We shall prove that $u(t,x)$ is analytic in $x \in B_t(\hat{t},\hat{x})$ for $t \in [i\tau,(i+1)\tau]$ by induction of i. First we can prove our assertion for $i = 0$ applying the Cauchy-Kowalevski's theorem , letting $\tau > 0$ small if necessary. Assume that $u(i\tau,x)$ is analytic in $x \in B_{i\tau}(\hat{t},\hat{x})$. Then we may assume that

$$\|D_x^\alpha u(i\tau,\cdot)\|_{H^m_{\rho(i\tau),d}(B_{i\tau}(\hat{t},\hat{x}))} \le Cr_0^{-|\alpha|}|\alpha|! . \tag{3.5}$$

Put $v(t,x) = u(t,x) - u(i\tau,x)$. Since u is a solution of (1), v satisfies

$$\begin{aligned} &\frac{\partial v}{\partial t} - \sum_{j=1}^n \tilde{A}_j(t,x,v)\frac{\partial v}{\partial x_j} - \tilde{A}_0(t,x,v)v \\ &= \tilde{f}(t,x,v), (t,x) \in (i\tau,(i+1)\tau) \times R^n , \\ &v(i\tau,x) = 0, x \in R^n , \end{aligned} \tag{3.6}$$

where $\tilde{A}_j(t,x,v) = A_j(t,x,v+u(i\tau,x))$ and

$$\tilde{f}(t,x,v) = f(t,x) - \sum_{j=1}^{n} A_j(t,x,v(t,x)+u(i\tau,x))\frac{\partial u(i\tau,x)}{\partial x_j} - A_0(t,x,v+u(i\tau,x))u(i\tau,x)\,. \tag{3.7}$$

Then it follows from (3.5) that $\tilde{A}_j$ and $\tilde{f}$ satisfy the condition of (v) in Lemma 3.1. Therefore $\tilde{A}_j$ and $\tilde{f}$ satisfy (3.3) with $t_0 = i\tau$. From now on we take $t_0 = i\tau$ and denote $|v|^t_{r,N} = |v|^{t,i\tau}_{r,N}$. Differentiating (3.6) with respect to x we get

$$L[D^\alpha v] = f_\alpha,\, D^\alpha v(i\tau) = 0\,, \tag{3.8}$$

where

$$f_\alpha = D^\alpha \tilde{f} + \sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'}\left(\sum_{j=1}^{n} D^{\alpha-\alpha'}\tilde{A}_j D^{\alpha'}D_j v + D^{\alpha-\alpha'}\tilde{A}_0 D^{\alpha'}v\right)\,. \tag{3.9}$$

Therefore by use of (1.13) of Theorem 1.4 we obtain

$$\|D^\alpha v(t)\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \le C\int_{i\tau}^{t} \|\tilde{f}_\alpha(s)\|_{H^m_{\rho(s),d}(B_s(\hat{t},\hat{x}))}ds\,. \tag{3.10}$$

On the other hand from (3.2), (3.3) and (3.4) we have if in brief $\|\cdot\| = \|\cdot\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))}$,

$$\begin{aligned}
\|f_\alpha(t)\| \le{}& \|D^\alpha\tilde{f}\| + \sum_{|\alpha'|=0,|\alpha|-1}\binom{\alpha}{\alpha'}\sum_{j=1}^{n}\|D^{\alpha-\alpha'}\tilde{A}_j\|\|D^{\alpha'}D_j v\| \\
&+ \sum_{1\le|\alpha'|\le|\alpha|-2}\binom{\alpha}{\alpha'}\sum_{j=1}^{n}\|D^{\alpha-\alpha'}\tilde{A}_j\|\|D^{\alpha'}D_j v\| \\
&+ \sum_{|\alpha'|=0,1,|\alpha|-1}\binom{\alpha}{\alpha'}\|D^{\alpha-\alpha'}\tilde{A}_0\|\|D^{\alpha'}v\| \\
&+ \sum_{2\le|\alpha'|\le|\alpha|-2}\binom{\alpha}{\alpha'}\|D^{\alpha-\alpha'}\tilde{A}_0\|\|D^{\alpha'}v\| \\
\le{}& |f|^t_{r,N} r^{-|\alpha|+2}\Gamma_2(|\alpha|) \\
&+ C\left(1+\sum_{j=0}^{N-1}(\epsilon + r^2|v|^t_{r,N})^j|v|^t_{r,N}\right) r^{-|\alpha|+2}\Gamma_2(|\alpha|)
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&+C \sum_{1\le|\alpha'|\le|\alpha|-2} \binom{\alpha}{\alpha'} r^{-(|\alpha-\alpha'|-2)}\Gamma_2(|\alpha-\alpha'|)\\
&\times\left\{1+\sum_{j=0}^{N-1}(\epsilon+r^2|v|_{r,N}^t)^j|v|_{r,N}^t\right\}|v|_{r,N}^t r^{-|\alpha'|+1}\Gamma_2(|\alpha'|+1)\\
\le&\left\{|f|_{r,N}^t+C\left(1+\sum_{j=0}^{N-1}(\epsilon+r^2|v|_{r,N}^t)^j|v|_{r,N}^t\right)\right\}r^{-|\alpha|+2}\Gamma_2(|\alpha|)\\
&+C\left\{1+\sum_{j=0}^{N-1}(\epsilon+r^2|v|_{r,N}^t)^j|v|_{r,N}^t\right\}r^{-|\alpha|+3}|v|_{r,N}^t\Gamma_2(|\alpha|+1)\,.
\end{aligned}
$$

Here we choose $r=r(t)=r_0e^{-\gamma(t-i\tau)}$, where $0<r_0\le 1$ and $\gamma>0$. Denote

$$
y(t)=\sup_{i\tau\le s\le t} r(s)|v|_{r(\cdot),N}^s\,.
$$

Noting that $r(t)\le 1$ we have from (3.11)

$$
\begin{aligned}
\|f_\alpha\|\le&\left\{|f|_{r,N}^t+C\left(1+\sum_{j=0}^{N-1}(\epsilon+y(t))^j|v|_{r,N}^t\right)\right\}r^{-|\alpha|+2}\Gamma_2(|\alpha|)\\
&+C\left\{1+\sum_{j=0}^{N-1}(\epsilon+y(t))^j r|v|_{r,N}^t\right\}y(t)r^{-|\alpha|+2}\Gamma_2(|\alpha|+1)\,.
\end{aligned}
$$

Hence for $|\alpha|>2$ we have from (3.10) and (3.11)

$$
\begin{aligned}
\|D^\alpha v(t)\|_{H^m_{\rho(t),d}(B_t(\hat t,\hat x))}\le&\int_{i\tau}^t\left\{\left(|f|_{r,N}^s+C\left(1+\sum_{j=0}^{N-1}(\epsilon+y(s))^j|v|_{r,N}^s\right)\right)r(s)^{-|\alpha|+2}\Gamma_2(|\alpha|)\right.\\
&\left.+C\left(1+\sum_{j=0}^{N-1}(\epsilon+y(s))^j|v|_{r(\cdot),N}^s\right)y(s)r(s)^{-|\alpha|+2}\Gamma_2(|\alpha|+1)\right\}ds\\
\le&\, C\left\{|f|_{r,N}^t r(t)^{-|\alpha|+2}\Gamma_2(|\alpha|)+\sum_{j=0}^{N}(\epsilon+y(t))^j|v|_{r(\cdot),N}^t\right.\\
&\left.\times\int_{i\tau}^t r(s)^{-|\alpha|+2}ds\,\Gamma_2(|\alpha|+1)\right\},
\end{aligned}
$$

and noting that $1\le r(s)r(t)^{-1}e^{\gamma\tau}$ for $s\le t$, we get for $|\alpha|=2$

$$
\begin{aligned}
\|D^\alpha v(t)\|_{H^m_{\rho(t),d}(B_t(\hat t,\hat x))}\le&\, C\int_{i\tau}^t\left\{|f|_{r,N}^s ds\,\Gamma_2(|\alpha|)\right.\\
&\left.+\sum_{j=0}^{N}(\epsilon+y(s))^j|v|_{r,N}^s r(s)\right\}ds\,\Gamma_2(|\alpha|)r(t)^{-1}e^{\gamma\tau}\,.
\end{aligned}
$$

Since $|f|^t_{r,N} \le C, r(t) \le r_0$ and $\int_{i\tau}^t r(s)^{-|\alpha|+2} ds \le r(t)^{-|\alpha|+2}(|\alpha|-2)^{-1}\gamma^{-1}$ for $|\alpha| > 2$, we get from the above estimates,

$$(3.12)\quad y(t) \le C \left\{ r_0 + \gamma^{-1} \sum_{j=0}^{N} (\epsilon + y(t))^j y(t) + e^{\gamma\tau} \int_{i\tau}^{t} \sum_{j=1}^{N} (\epsilon + y(s))^j y(s) ds \right\},$$

for $t \in [i\tau, (i+1)\tau]$. We shall prove that $y(t) < \epsilon$ for $t \in [i\tau, (i+1)\tau]$, if we choose $r_0 > 0$ small enough and $\gamma > 0$ sufficiently large. Assume that there is $t_1 \in [i\tau, (i+1)\tau]$ such that $y(t_1) = \epsilon$ and $y(t) < \epsilon$ for $t \in (i\tau, t_1)$. Since $y(i\tau) = 0$, we have $t_1 > i\tau$. It follows from (3.12) that

$$(3.13)\qquad y(t) \le C \left(r_0 + \gamma^{-1} \frac{y(t)}{1-2\epsilon} + e^{\gamma\tau} \int_{i\tau}^{t} \frac{y(s)}{1-2\epsilon} ds \right),$$

for $t \in [i\tau, t_1)$. Here we take $\gamma > 0$ satisfying $\frac{2C\gamma^{-1}}{(1-2\epsilon)} = 1/2$. Hence we get from (3.13) and $r(t) \le r_0$,

$$y(t) \le 2C \left(r_0 + e^{\gamma\tau} \int_{i\tau}^{t} \frac{y(s)}{1-2\epsilon} ds \right).$$

Solving this inequality, we have $y(t) \le 2Cr_0(1 + (1-2\epsilon)^{-1}\tau e^{\frac{2C\gamma\tau}{1-2\epsilon}})$ for $t \in [i\tau, t_1)$. This contradicts $y(t_1) = \epsilon$, if we choose $r_0 > 0$ such that

$$2Cr_0\big(1 + (1-2\epsilon)^{-1}\tau e^{\frac{2C\gamma\tau}{1-2\epsilon}}\big) = \epsilon/2.$$

Thus we can get $y(t) \le \epsilon$ for $t \in [i\tau, (i+1)\tau]$ and consequently we have proved Theorem 3.

Appendix

Here we shall prove Lemma 3.1. First we note that it follows from (2.4)

$$(A.1)\qquad \|u(t)v(t)\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \le C_m \|u\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))} \|v\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))},$$

for $u, v \in C^0([0,T]; L^2_d(R^n))$. For simplicity we denote $\|u\|_m = \|u\|_{H^m_{\rho(t),d}(B_t(\hat{t},\hat{x}))}$ and $|u| = |u|^t_{r,N}$.

PROOF OF (i). We prove this by induction of $|\beta|$. Let $|\beta| = 2$. We write $v^\beta = v_1 v_2$. Then by use of (A.1) and (3.2) we have

$$\begin{aligned}
\|D^\alpha v^\beta\|_m &\le C_m \sum_{|\alpha'|=0,1,|\alpha|-1,|\alpha|} \binom{\alpha}{\alpha'} \|D^{\alpha-\alpha'} v_1\|_m \|D^{\alpha'} v_2\|_m \\
&\quad + C_m \sum_{2\le|\alpha'|\le|\alpha|-2} \binom{\alpha}{\alpha'} \|D^{\alpha-\alpha'} v_1\|_m \|D^{\alpha'} v_2\|_m \\
&\le C_m \Big\{ C_n |v| \Gamma_2(|\alpha|) r^{-|\alpha|+2} \|v\|_{m+1} \\
&\quad + \sum_{2\le|\alpha'|\le|\alpha|-2} \binom{\alpha}{\alpha'} r^{-|\alpha|+4} \Gamma_2(|\alpha-\alpha'|)\Gamma_2(|\alpha'|)|v|^2 \Big\} \\
&\le C_m |v| (C_n \|v\|_{m+1} + r^2|v|) r^{-|\alpha|+2} \Gamma(|\alpha|)\,,
\end{aligned}$$

which proves (i) for $|\beta| = 2$, if we take $C_0 \ge C_m(C_n + 1)$. Assume (i) is valid for $|\beta| - 1 \ge 2$. We write $v^\beta = v^{\beta-e} v$, where $|e| = 1$ and denote $v = v^e$ for simplicity. Then

$$\begin{aligned}
\|D^\alpha v^\beta\|_m &\le C_m \sum_{|\alpha'|=0,1,|\alpha|-1,|\alpha|} \binom{\alpha}{\alpha'} \|D^{\alpha-\alpha'} v^{\beta-e}\|_m \|D^{\alpha'} v\|_m \\
&\quad + C_m \sum_{2\le|\alpha'|\le|\alpha|-2} \binom{\alpha}{\alpha'} \|D^{\alpha-\alpha'} v^{\beta-e}\|_m \|D^{\alpha'} v\|_m \\
&\le C_m \Big\{ C_0^{|\beta|-2} C_n \left(\sup_{0\le s\le t} \|v(s)\|_{m+1} + r^2|v| \right)^{|\beta|-2} \|v\|_{m+1} \\
&\quad + (C_m \|v\|_{m+1})^{|\beta|-1} \Big\} |v| \Gamma_2(|\alpha|) r^{-|\alpha|+2} \\
&\quad + \sum_{2\le|\alpha'|\le|\alpha|-2} \binom{\alpha}{\alpha'} r^{-|\alpha|+4} \Gamma_2(|\alpha-\alpha'|)\Gamma_2(|\alpha'|)|v|^2 C_0^{|\beta|-2} \\
&\qquad \times \left(\sup_{0\le s\le t} \|v(s)\|_{m+1} + r^2|v| \right)^{|\beta|-2} \\
&\le C_m C_0^{|\beta|-2} |v| \left(\sup_{0\le s\le t} \|v(s)\|_{m+1} + r^2|v| \right)^{|\beta|-2} (C_n \|v\|_{m+1} + r^2|v|) \\
&\qquad \times r^{-|\alpha|+2} \Gamma(|\alpha|)\,,
\end{aligned}$$

which implies (i) for β, if we take $C_0 \ge C_m(C_m + C_n + 1)$.

PROOF OF (ii). For $M = |\beta| > N$ we write $v^\beta = w_1 w_2 \dots w_M$. Then

$$\begin{aligned} D^\alpha v^\beta &= D^\alpha (w_1 w_2 \dots w_M) \\ &= \sum_{\tilde{\alpha}^M = \alpha} \binom{\alpha}{\alpha^1} \binom{\alpha - \tilde{\alpha}^1}{\alpha^2} \cdots \binom{\alpha - \tilde{\alpha}^{M-2}}{\alpha^{M-1}} D^{\alpha^1} w_1 D^{\alpha^2} w_2 \dots D^{\alpha^M} w_M , \end{aligned}$$

where $\tilde{\alpha}^k = \sum_{j=1}^k \alpha^j$. Since $M > N \geq |\alpha| = L$, we can arange

$$\begin{aligned} D^\alpha v^\beta = \sum_{\tilde{\alpha}^L = \alpha} & \binom{\alpha}{\alpha^{i_1}} \binom{\alpha - \tilde{\alpha}^{i_1}}{\alpha^{i_2}} \cdots \binom{\alpha - \tilde{\alpha}^{i_{L-2}}}{\alpha^{i_{L-1}}} \\ & \times D^{\alpha^{i_1}} w_{i_1} D^{\alpha^{i_2}} w_{i_2} \dots D^{\alpha^{i_L}} w_{i_L} w_{i_{L+1}} \dots w_M , \end{aligned}$$

where the summation of $(i_1, i_2, \dots , i_L)$ runs over $(1, 2, \dots , M)$. Let $|\alpha_{i_j}| \geq 2$ for $j = 1, 2, \dots , \ell$ and $|\alpha_{i_j}| = 1$ for $j = \ell + 1, \dots , L$. Taking account of $\|D^\alpha w\|_{m+1} \leq |v| r^{-|\alpha|+2} \Gamma_2(|\alpha|)$ for $|\alpha| \geq 2$ and $\|D^\alpha w\|_m \leq \|v\|_{m+1}$ for $|\alpha| \leq 1$, we can estimate,

$$\begin{aligned} \|D^\alpha v^\beta\|_m \leq C_m^{M-1} & \sum_{\tilde{\alpha}^L = \alpha} \binom{\alpha}{\alpha^{i_1}} \binom{\alpha - \tilde{\alpha}^{i_1}}{\alpha^{i_2}} \cdots \binom{\alpha - \tilde{\alpha}^{i_{L-2}}}{\alpha^{i_{L-1}}} \\ & \times \|D^{\alpha^{i_1}} w_{i_1}\|_m \|D^{\alpha^{i_2}} w_{i_2}\|_m \dots \|D^{\alpha^{i_L}} w_{i_L}\|_m \|w_{i_{L+1}}\|_m \dots \|w_M\|_m, \\ \leq C_m^{M-1} & \sum_{\tilde{\alpha}^L = \alpha} \binom{\alpha}{\alpha^{i_1}} \binom{\alpha - \tilde{\alpha}^{i_1}}{\alpha^{i_2}} \cdots \binom{\alpha - \tilde{\alpha}^{i_{L-2}}}{\alpha^{i_{L-1}}} \\ & \times \Gamma_2(|\alpha^{i_1}|) \Gamma_2(|\alpha^{i_2}|) \dots \Gamma_2(|\alpha^{i_\ell}|) |v|^\ell r^{-|\tilde{\alpha}_{i_\ell}|+2\ell} \|v\|_{m+1}^{L-\ell} \|v\|_m^{M-L} . \end{aligned}$$

Noting that $\Gamma_2(|\alpha|) = \lambda_0 \leq 1$ for $|\alpha| \leq 1$ and using (3.2),

$$\begin{aligned} & \sum_{\tilde{\alpha}^L = \alpha} \binom{\alpha}{\alpha^{i_1}} \binom{\alpha - \tilde{\alpha}^{i_1}}{\alpha^{i_2}} \cdots \binom{\alpha - \tilde{\alpha}^{i_{L-2}}}{\alpha^{i_{L-1}}} \Gamma_2(|\alpha^{i_1}|) \Gamma_2(|\alpha^{i_2}|) \dots \Gamma_2(|\alpha^{i_\ell}|) \\ & \leq \sum_{\tilde{\alpha}^M = \alpha} \binom{\alpha}{\alpha^1} \binom{\alpha - \tilde{\alpha}^1}{\alpha^2} \cdots \binom{\alpha - \tilde{\alpha}^{M-2}}{\alpha^{M-1}} \Gamma_2(|\alpha^1|) \Gamma_2(|\alpha^2|) \dots \Gamma_2(|\alpha^{M-1}|) \lambda_0^{\ell - M} \\ & \leq \lambda_0^{-M+1} \Gamma_2(|\alpha|) . \end{aligned}$$

Thus we obtain from the above inequality,

$$\|D^\alpha v^\beta\| \leq C_m^{M-1} C_n^{M-1} \|v\|_{m+1}^{|\beta|-N} (\|v\|_{m+1} + r^2 |v|)^{N-1} |v| r^{-|\alpha|+2} \Gamma_2(|\alpha|) ,$$

which proves (ii).

PROOF OF (iii). Since $a(x)$ is analytic in $x \in B(R)$ and $B(R)$ is a closed set, there is $\epsilon > 0$ and $\hat{a}(x)$ such that $\hat{a}(x)$ is analytic in $B(R+\epsilon)$ and equal to $a(x)$ in $B(R)$. Let $\chi(t) \in \gamma^{(d)}(R)$ satisfying that $\chi(t) = 1$ for $|t| \leq R$ and $\chi(t) = 0$ for $|t| \geq R + \epsilon/2$. Define $\tilde{a}(x) = \hat{a}(x)\chi(x)$. It is easily seen that $\tilde{a}(x)$ satisfies (iii).

PROOF OF (iv). For $a \in \mathcal{A}(B(R))$ we take $\tilde{a}$ defined in (iii). Then we can estimate applying Proposition 1.1,

$$\|\tilde{a}u\|_{H^m_{\rho,d}(R^n)} \leq C|\tilde{a}|_{R^n,\rho_1,d}\|u\|_{H^m_{\rho,d}(R^n)},$$

for $u \in L^2_d(R^n)$. Hence we get (iv) from the definition (3.1) and (iii).

PROOF OF (v). We have by Taylor's expansion,

$$A(t) \circ v(x) = \sum_{\beta} \left(\frac{\partial}{\partial v}\right)^{\beta} A(t,x,0) v^{\beta} \beta!^{-1}.$$

Hence we get from (iv)

$$|A(t) \circ v|^t_{\rho,d} \leq \sum \left| \left(\frac{\partial}{\partial v}\right)^{\beta} A(t,x,0)\right|_{B_{t_0},\rho_0,1} |v^{\beta}|^t_{\rho,N}. \tag{A.2}$$

Taking account that

$$\left| \left(\frac{\partial}{\partial v}\right)^{\beta} A(t,\cdot,0)\right|_{B_{t_0},\rho_0,1} \leq |A|^t_{B_{t_0}\times G,\rho_0,1}\rho_0^{-|\beta|}|\beta|!,$$

we obtain (v) from (A.2), (i) and (ii).

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Institute of Mathematics
University of Tsukuba
305 Tsukuba Ibaraki, Japan