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On Hulls of Meromorphy and a Class of Stein Manifolds

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To the memory of my friend G. Lupacciolu

Abstract. If $X$ is a Stein manifold and $K$ a compact subset one may define the meromorphy hull of $K$ in two different ways: with respect to principal hypersurfaces or to arbitrary hypersurfaces. It is shown that the two definitions agree for every compact subset $K$ of $X$ if and only if the following topological condition on $X$ is satisfied: $\text{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$. It is also shown that this condition is equivalent to: for every hypersurface $h \subset X$ and every relatively compact open subset $D \subset X$ there exists $f \in \mathcal{O}(D)$ such that $h \cap D = \{ x \in D | f(x) = 0 \}$.

Finally, several examples are provided, which show that the topological condition $\text{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$ is sharp.

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0. – Introduction

Let $X$ be a Stein manifold. Then it is known (see e.g. [10], p. 181) that the second Cousin problem on $X$ can be solved for an arbitrary divisor if and only if $H^2(X;\mathbb{Z}) = 0$. On the other hand one may consider the strong Poincaré problem on $X$: given a meromorphic function on $F$ on $X$ find holomorphic functions $f, g$ on $X$ such that the germs $f_z, g_z$ are relative prime at any point and $F = f/g$. As in the Cousin second problem, the strong Poincaré problem can be solved for every meromorphic function $F$ on $X$ if and only if $H^2(X;\mathbb{Z}) = 0$ (see [13], p. 250). Therefore there is a strong connection between global properties of meromorphic functions on $X$ and the purely topological invariant $H^2(X;\mathbb{Z})$. A weaker condition than $H^2(X,\mathbb{Z}) = 0$ is $H^2(X,\mathbb{Z})$ is of torsion. One may easily see (Proposition 2) that this is equivalent to: every hypersurface $h \subset X$ (closed analytic subset of codimension 1) can be defined globally by one equation i.e. there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{ f = 0 \}$.

In this paper we study a class of Stein manifolds $X$ which satisfy a weaker condition than $H^2(X,\mathbb{Z})$ is of torsion, namely we consider the topological
condition $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$. This condition is related to the study of hulls of meromorphy of compact subsets $K \subset X$.

When $K \subset X$ is a compact subset one may define in a natural way (see [8], [9]) the following two hulls:

$$h\widehat{K} = \{x \in X \mid \text{every hypersurface passing through } x \text{ intersects } K\}$$

$$h\widehat{K} = \{x \in X \mid \text{every principal hypersurface passing through } x \text{ intersects } K\}$$

Obviously $h\widehat{K} \subset H\widehat{K}$ and it is known [8] that they are compact subsets of $X$. When $X = \mathbb{C}^n$ then $h\widehat{K} = H\widehat{K}$ and it is called the rational convex hull of $K$ (see [16]).

We show (Theorem 1) that on a Stein manifold $X$ the condition

[(a)] 

$$h\widehat{K} = H\widehat{K} \text{ for every compact subset } K \subset X$$

is equivalent to the topological condition

[(b)] 

$$\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0.$$ 

It is also equivalent to each of the following conditions:

[(c)] For every hypersurface $h \subset X$ and every relatively compact open subset $D \subset X$ there exists $f \in \mathcal{O}(D)$ such that $h \cap D = \{x \in D \mid f(x) = 0\}$ (in other words the hypersurfaces on $X$ can be defined, set-theoretically, by one equation on compact subsets).

[(d)] For every $\xi \in H^2(X; \mathbb{Z})$ and every relatively compact open subset $D \subset X$ there exists a positive integer $m$, depending on $\xi$ and $D$, such that $m\xi |_D = 0$ (H^2(X; \mathbb{Z})$ is of torsion on compact subsets).

Finally we give examples showing that the statement of Theorem 1 is sharp: namely, there exists Stein manifolds $X$ satisfying the topological condition $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ is not of torsion. In particular, on these manifolds there exists hypersurfaces which cannot be defined globally by one equation but they can be defined on each compact subset of $X$ by one equation.

Acknowledgement. I wish to thank G. Lupacciou who raised me in autumn 1995, when I was visiting the University of Rome “La Sapienza”, the question whether there exist Stein manifolds $X$ and compact subsets $K \subset X$ such that $h\widehat{K} \neq H\widehat{K}$. This was the starting point in writing this paper. I want also to thank my friend and colleague G. Chiriacescu for helpful discussions on the algebraic results needed in the given examples, in particular for the references [1] and [12].
1. – Proof of the results

Let $X$ be a Stein manifold of dimension $n$. A closed analytic subset $h \subset X$ of pure dimension $(n - 1)$ is called a hypersurface. $h$ is called a principal hypersurface iff there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{ f = 0 \}$.

For a compact subset $K \subset X$ we consider the following two hulls:

$$h\hat{K} = \{ x \in X \mid \text{every hypersurface passing through } x \text{ intersects } K \}$$

$$\mathcal{H} \hat{K} = \{ x \in X \mid \text{every principal hypersurface passing through } x \text{ intersects } K \}$$

They are compact subsets of $X$ [8] (p. 50 and p. 52).

Let us recall also the following result [5]

**Lemma 1.** Let $X$ be a Stein manifold, $Y \subset X$ a closed complex submanifold and denote by $N = N_{Y|X}$ the normal bundle of $Y$ in $X$. Then there exists an open neighborhood $U$ of the null section of $N$ biholomorphic to an open neighborhood $U_1$ of $Y$ in $X$ by $\varphi : U \rightarrow U_1$ such that the image of the null section by $\varphi$ is $Y$.

Now we can prove:

**Proposition 1.** Let $X$ be a Stein manifold and assume that $h\hat{K} = \mathcal{H} \hat{K}$ for every compact subset $K \subset X$. Then the Kronecker product $H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ is the null map.

**Proof.** Since $X$ is Stein it follows that $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$. On the other hand every line bundle $L$ on $X$ has a section $s$ whose zero set is smooth (see [9], p. 883) and this section defines a positive divisor which corresponds to $L$. Therefore it is enough to show that the Kronecker product $< c(h), \overline{\alpha} > = 0$, where $h$ is a smooth and connected hypersurface, $c(h) \in H^2(X; \mathbb{Z})$ denotes its Chern class and $\overline{\alpha} \in H_2(X; \mathbb{Z})$. In fact $< c(h), \overline{\alpha} >$ is the intersection number $< h, \overline{\alpha} >$ of $h$ and $\overline{\alpha}$ and can be defined choosing a smooth 2-cycle $\alpha$ (representing $\overline{\alpha}$) intersecting $h$ transversally (see [7], p. 61).

By reductio ad absurdum assume that $z = < h, \overline{\alpha} > \neq 0$. Let $N = N_{h|X}$ be the normal bundle of $h$ in $X$. By Lemma 1 there exists an open neighborhood $U$ of the null section of $N$ biholomorphic to an open neighborhood $U_1$ of $h$ in $X$ by $\varphi : U \rightarrow U_1$, such that the image of the null section by $\varphi$ is $h$. We choose a hermitian metric on $N$ such that $\{ w \in N \mid \| w \| \leq 1 \} \subset U$ and define $V = \varphi(\{ w \in N \mid \| w \| \leq 1 \})$. Then $V$ is a closed neighborhood of $h$ and by Thom’s isomorphism $H_2(X, X \setminus V; \mathbb{Z}) \cong H_0(h; \mathbb{Z}) \cong \mathbb{Z}$. In fact, if $x_0 \in h$ is any point and $B_{x_0} = B(x_0, 1) \subset V$ denotes the corresponding closed ball with center $x_0$ and contained in the fiber then $B_{x_0}$ can be considered as a 2-simplex $s$ of $X$ with boundary contained in $X \setminus \overset{\circ}{V}$, so it defines an element $\overline{s} \in H_2(X, X \setminus \overset{\circ}{V}; \mathbb{Z}) \cong \mathbb{Z}$ and $\overline{s}$ is a generator. In what follows we fix some point $x_0 \in h$. 

Consider the exact sequence:

\[ \mathbb{H}_2(X; \mathbb{Z}) \to \mathbb{H}_2(X \setminus \mathbb{V}; \mathbb{Z}) \to \mathbb{H}_2(X; X \setminus \mathbb{V}; \mathbb{Z}) \cong \mathbb{Z} \]

and let us remark that \( i(\alpha) \neq 0 \) (here \( i \) denotes the natural homomorphism at homology induced by the map \( (X, \emptyset) \to (X, X \setminus \mathbb{V}) \) which is the identity on \( X \)).

Otherwise \( \alpha = u + v \) with \( u = \text{boundary in } X \), \( v = \text{cycle in } X \setminus \mathbb{V} \) and it would follow that \( z = \langle h, \alpha \rangle = 0 \) since \( v \) does not meet \( h \). Therefore \( i(\alpha) = \lambda s \) with \( \lambda \in \mathbb{Z} \setminus \{0\} \). We see that

\[ (*) \qquad \alpha - \lambda s = \alpha_1 + b \]

with \( \alpha_1 = \text{chain in } X \setminus \mathbb{V} \) and \( b = \text{boundary in } X \) (and in fact \( \lambda = z \)).

Let us define the compact set \( K = \text{supp } (\alpha_1) \cup \partial B_{x_0} \). By our assumption \( h\tilde{K} = h\tilde{K} \). But \( x_0 \in h \) and \( h \cap K = \emptyset \) (empty set), so there exists a principal hypersurface \( H = \{ f = 0 \}, f \in \mathbb{O}(X) \) with \( x_0 \in H \) and \( H \cap K = \emptyset \). We have \( \langle H, \alpha - \lambda s \rangle = \langle H, \alpha_1 \rangle + \langle H, b \rangle \) where \( \langle, \rangle \) denotes the intersection number.

On the other hand \( \langle H, \alpha \rangle = 0 \) since \( H \) is principal, \( \langle H, b \rangle = 0 \) since \( b \) is a boundary, \( \langle H, \alpha_1 \rangle = 0 \) since \( H \cap \text{supp } (\alpha_1) = \emptyset \). But \( \langle H, s \rangle = 0 \) because \( H \cap \partial B_{x_0} = \emptyset \) and on \( B_{x_0} \) we have a complex structure (see [7] p. 63, [9] Lemme 5.3). We get \( \lambda = 0 \) which is a contradiction. Therefore \( \langle h, \alpha \rangle = 0 \) and the proof of Proposition 1 is complete.

**Theorem 1.** Let \( X \) be a Stein manifold. Then the following conditions are equivalent:

1) \( \langle h, K \rangle = 0 \) for every compact subset \( K \) of \( X \)

2) Hom \( (\mathbb{H}_2(X; \mathbb{Z}); \mathbb{Z}) = 0 \)

3) For every \( \xi \in \mathbb{H}_2(X; \mathbb{Z}) \) and every relatively compact open subset \( D \subset X \) there exists a positive integer \( m = m(D, \xi) \) such that \( m\xi |_D = 0 \).

4) For every hypersurface \( h \subset X \) and every relatively compact open subset \( D \subset X \) there exists a holomorphic function \( f \in \mathbb{O}(D) \) such that one has set-theoretically \( h \cap D = \{ f = 0 \} \).

**Proof.**

1) \( \implies \) 2)

It is known ([6], p. 132) that the natural morphism (induced by the Kronecker product) \( \mathbb{H}_2(X; \mathbb{Z}) \to \text{Hom } (\mathbb{H}_2(X; \mathbb{Z}); \mathbb{Z}) \) is surjective. Therefore every \( u \in \text{Hom } (\mathbb{H}_2(X; \mathbb{Z}); \mathbb{Z}) \) is of the form \( u(\alpha) = \langle \xi, \alpha \rangle \) for some \( \xi \in \mathbb{H}_2(X; \mathbb{Z}) \).

By Proposition 1 it follows that \( \text{Hom } (\mathbb{H}_2(X; \mathbb{Z}); \mathbb{Z}) = 0 \).

2) \( \implies \) 3)

Let \( \xi \in \mathbb{H}_2(X; \mathbb{Z}), D \subset X \) a relatively compact open subset, and we have to find a positive integer \( m \) such that \( m\xi |_D = 0 \). We may assume that the boundary of \( D \) is smooth, therefore the homology groups of \( D \) are finitely generated. We define \( \xi_1 = \xi |_D \in \mathbb{H}_2(D; \mathbb{Z}) \). By our assumption \( \langle \xi_1, \alpha \rangle = 0 \) for every \( \alpha \in \mathbb{H}_2(D; \mathbb{Z}) \). Since the homology groups of \( D \) are finitely generated it follows from ([6], p. 136) that \( \xi_1 \) is a torsion element of \( \mathbb{H}_2(D; \mathbb{Z}) \).
Let \( h \subset X \) be a hypersurface and \( D \subset X \) a relatively compact open subset. We may assume that \( D \) is Stein, therefore \( H^2(D; \mathbb{Z}) \cong H^1(D, \mathcal{O}^*) \cdot h \) defines a line bundle \( L \in H^1(X, \mathcal{O}^*) \) which has a canonical section \( s \in \Gamma(X, L) \) with \( h = \{ s = 0 \} \). Define \( \xi = c(L) \in H^2(X; \mathbb{Z}) \) the Chern class of \( L \). By our assumption there exists a positive integer \( m \) with \( m\xi = 0 \) on \( D \). Therefore \( L^m \mid_D \) is trivial. Also \( s^m \in \Gamma(X, L^m) \) and \( f = s^m \mid_D \) is a holomorphic function on \( D \) with \( h \cap D = \{ f = 0 \} \).

Let \( K \subset X \) be a compact subset and \( x_0 \in X \) such that there exists a hypersurface \( h \subset X \) with \( x_0 \in h \) and \( h \cap K = \emptyset \). We have to find a principal hypersurface \( H \subset X \) with \( x_0 \in H \) and \( H \cap K = \emptyset \). Let \( D \subset X \) be a Runge domain with \( K \cup \{ x_0 \} \subset D \). By our assumption there is a holomorphic function \( H_1 \in \mathcal{O}(D) \) with \( \{ H_1 = 0 \} = h \cap D \) (set-theoretically).

Let \( \epsilon_0 = \inf \{ |H_1(x)| : x \in K \} > 0 \). Since \( D \subset X \) is Runge we can approximate \( H_1 \) on \( K \cup \{ x_0 \} \subset D \) by \( \tilde{H}_1 \in \mathcal{O}(X) \) such that \( |\tilde{H}_1(x) - H_1(x)| \leq \epsilon_0/4 \) if \( x \in K \) and \( |\tilde{H}_1(x_0)| = |\tilde{H}_1(x_0) - H_1(x_0)| \leq \epsilon_0/4 \). Define \( H(x) = H_1(x) - \tilde{H}_1(x_0) \). Then obviously \( H(x_0) = 0 \) and \( H(x) \neq 0 \) if \( x \in K \).

Thus our theorem is completely proved.

**Corollary 1.** Let \( X \) be a Stein manifold such that \( H_1(X; \mathbb{Z}), H_2(X; \mathbb{Z}) \) are finitely generated (e.g. \( X \) is affine algebraic).

Then the following conditions are equivalent:

1) every compact subset \( K \subset X \).

2) \( H^2(X; \mathbb{Z}) \) is of torsion.

3) For every hypersurface \( h \subset X \) there exists \( f \in \mathcal{O}(X) \) such that one has set-theoretically \( h = \{ f = 0 \} \).

**Proof.** Since \( H_1(X; \mathbb{Z}), H_2(X; \mathbb{Z}) \) are finitely generated it follows from ([6], p. 136) that we have a (non-canonical) isomorphism:

\[
(*) \quad H^2(X; \mathbb{Z}) \cong \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) \oplus T_1
\]

where \( T_1 \) denotes the torsion part of \( H_1(X; \mathbb{Z}) \). Now the corollary follows immediately from (*) and Theorem 1.

**Proposition 2.** Let \( X \) be a connected Stein manifold. Then the following two conditions are equivalent:

1) \( H^2(X; \mathbb{Z}) \) is of torsion.

2) For every hypersurface \( h \subset X \) there exists \( f \in \mathcal{O}(X) \) such that one has set-theoretically \( h = \{ f = 0 \} \).

We first show that 1) \( \implies \) 2).

Let \( h \subset X \) be a hypersurface and let \( L \in H^1(X, \mathcal{O}^*) \) be the corresponding line bundle, therefore there is a canonical section \( s \in \Gamma(X, L) \) with \( h = \{ s = 0 \} \). Since \( X \) is Stein \( H^2(X; \mathbb{Z}) \cong H^1(X, \mathcal{O}^*) \), hence there is a positive integer \( m \)
such that \( L^m \) is trivial. \( s^m \) is a section in \( \Gamma(X, L^m) \) and if we set \( f = s^m \) then \( f \) is a holomorphic function on \( X \) such that \( h = \{ f = 0 \} \).

We prove now that \( 2) \implies 1 \).

We recall the following result (see [3]): If \( L \) is a line bundle over a connected Stein manifold \( X \) then there is a section \( s \in \Gamma(X, L) \) such that \( \{ s = 0 \} \) is irreducible (in fact the set of sections \( s \in \Gamma(X, L) \) with \( \{ s = 0 \} \) irreducible is dense in \( \Gamma(X, L) \)).

Let now \( \xi \in H^2(X; \mathbb{Z}) \cong H^1(X, \mathcal{O}^*) \) and let \( L \in H^1(X, \mathcal{O}^*) \) be the corresponding line bundle. We choose \( s \in \Gamma(X, L) \) such that \( h = \{ s = 0 \} \) is irreducible. If we consider \( (h) \) as a divisor there is a positive integer \( n \) such that \( L = n(h) \) \((n \) is the order of \( s \) along \( h \), which is well defined because \( h \) is irreducible\). On the hand there exists \( f \in \mathcal{O}(X) \) with \( h = \{ f = 0 \} \) (set-theoretically). If \( m \) is the order of \( f \) along \( h \) then \( m(h) = 0 \). Therefore \( L^m \) is the trivial line bundle and consequently \( m\xi = 0 \). So we have showed that \( H^2(X; \mathbb{Z}) \) is of torsion, and the proof of Proposition 2 is complete.

REMARK 1. There is a surjective homomorphism group (see [6], p. 132)

\[
H^2(X; \mathbb{Z}) \to \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z})
\]

from which it follows that:

\[
H^2(X; \mathbb{Z}) \text{ is of torsion} \implies \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0.
\]

We shall give examples of Stein manifolds \( X \) such that \( \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0 \) but \( H^2(X; \mathbb{Z}) \) contains nontorsion elements. Of course, for such Stein manifolds \( X \), every \( \xi \in H^2(X; \mathbb{Z}) \) is of torsion on compact subsets, i.e. for every \( D \subset X \) there is a positive integer \( m = m(D, \xi) \) with \( m\xi = 0 \) on \( D \). But it is possible that \( m \to \infty \) as it will be shown by our next examples.

EXAMPLE 1. In [11] it is given an example of a Stein domain \( X \subset \mathbb{C}^2 \) with \( H_1(X; \mathbb{Z}) = \mathbb{Q} \) (rational numbers) and \( H_2(X; \mathbb{Z}) = 0 \).

Let us study \( H^2X; \mathbb{Z} \). There is an exact sequence ([4], p. 153):

\[
0 \to \text{Ext}(H_1(X; \mathbb{Z}); \mathbb{Z}) \to H^2(X; \mathbb{Z}) \to \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) \to 0
\]

Therefore we get \( H^2(X; \mathbb{Z}) = \text{Ext}(\mathbb{Q}; \mathbb{Z}) \). Clearly \( \text{Ext}(\mathbb{Q}; \mathbb{Z}) \) is a \( \mathbb{Q} \) vector space, so every \( \xi \in \text{Ext}(\mathbb{Q}; \mathbb{Z}) \setminus \{0\} \) is a nontorsion element. We shall prove that \( \dim_{\mathbb{Q}} \text{Ext}(\mathbb{Q}; \mathbb{Z}) = \infty \).

If \( p \) is a prime we denote by \( p = \text{the additive group of those rational numbers whose denominators are powers of } p \) and by \( \mathbb{Z}(p^\infty) \) the quotient \( \mathbb{Z}/P \). There is a group isomorphism (see [12], p. 6): \( \mathbb{Q}/\mathbb{Z} \cong \bigoplus \mathbb{Z}(p^\infty) \). It follows:

\[
\text{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\mathbb{Q}; \bigoplus \mathbb{Z}(p^\infty)) = \bigoplus \text{Hom}(\mathbb{Q}; \mathbb{Z}(p^\infty)).
\]
Now for every prime $p$ since we have a surjective homomorphism $\mathbb{Q} \to \mathbb{Z}(p\infty)$ obtained from the composition of two surjective homomorphisms $\mathbb{Q} \to \mathbb{Z}(p\infty)$. If follows that $\dim_\mathbb{Q} \text{Hom}(Q; \mathbb{Q}/\mathbb{Z}) = \infty$.

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ applying $\text{Hom}_\mathbb{Z}(Q; \cdot)$ we get the exact sequence of $\mathbb{Q}$ vector spaces:

$$0 \to \text{Hom}(Q; \mathbb{Z}) \to \text{Hom}(Q; \mathbb{Q}) \to \text{Hom}(Q; \mathbb{Q}/\mathbb{Z}) \to \text{Ext}(Q; \mathbb{Z}) \to \text{Ext}(Q; \mathbb{Q}) = 0$$

Since $\text{Hom}(Q; \mathbb{Z}) = 0$, $\text{Hom}(Q; \mathbb{Q})$ has dimension 1 as a $\mathbb{Q}$ vector space and $\dim_\mathbb{Q} \text{Hom}(Q; \mathbb{Q}/\mathbb{Z}) = \infty$ it follows that $\dim_\mathbb{Q} \text{Ext}(Q; \mathbb{Z}) = \infty$.

**Example 2.** For each integer $m \geq 2$ consider the map $\varphi : D \to \mathbb{R}^4$ (is the closed unit disc, i.e. $D = \{ z \in \mathbb{C} | |z| \leq 1 \}$ given by $\varphi(z) = (z^m, (1 - |z|)z)$ where $\mathbb{R}^4$ is identified with $\mathbb{C}^2$ in the usual way. Then $\varphi|_D$ is injective and $\varphi|_\partial D$ has degree $m$.

Since $\partial D = S^1$, if we set $K_m = \varphi(\overline{D})$, then $K_m$ is obtained from $S^1$ by adding a two cell by a map of degree $m$ (see [6], p. 83). It follows from ([6], p. 89) that $H_1(K_m; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ and $H_2(K_m; \mathbb{Z}) = 0$.

Consider in $\mathbb{R}^4$ an infinite real line $d$, and on $d$ we add the compacts $K_m$ such that $K_m \cap d = y_d$, $K_m \cap K_n = \emptyset$ if $m \neq n$ and $(K_m)$ is locally finite. Thus we get a locally finite cellular complex $M \subset \mathbb{R}^4$. One may easily see that $H_1(M; \mathbb{Z}) = \oplus_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$ and $H_2(M; \mathbb{Z}) = 0$.

From the exact sequence:

$$0 \to \text{Ext}(H_1(M; \mathbb{Z}); \mathbb{Z}) \to H^2(M; \mathbb{Z}) \to \text{Hom}(H_2(M; \mathbb{Z}); \mathbb{Z}) \to 0$$

we get $H^2(M; \mathbb{Z}) = \text{Ext}(\oplus_{m \geq 2} \mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$. From ([1], § 5, Prop. 7, p. 89) $\text{Ext}(\oplus G_m; \mathbb{Z}) \cong \prod \text{Ext}(G_m; \mathbb{Z})$ and by ([4], p. 148) $\text{Ext}(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$.

We deduce that $H^2(M; \mathbb{Z}) = \prod_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$. We take now and open neighborhood $U$ of $M$ in $\mathbb{R}^4$ such that $M$ is a deformation retract of $U$. Considering the inclusion $R^4 \subset \mathbb{C}^4$ given by $y_1 = \ldots = y_4 = 0$ where $z_k = x_k + iy_k$ are the complex coordinates on $\mathbb{C}^4$, there exists by [14] a Stein domain $X \subset \mathbb{C}^4$ such that $X \cap \mathbb{R}^4 = U$ and $U$ is a deformation retract of $X$. We have $H^2(X; \mathbb{Z}) = \prod_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$ and $H_2(X; \mathbb{Z}) = 0$. The element $(1, 1, \ldots, 1, \ldots)$ (taking 1 on all factors of the infinite product) is a nontorsion element of $H^2(X; \mathbb{Z})$.

**Example 3.** In examples 1) and 2) we have $H_2(X; \mathbb{Z}) = 0$. But it is possible to find $X$ with $H_2(X; \mathbb{Z}) \neq 0$, $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X; \mathbb{Z})$ has nontorsion elements. To see this we replace in example 1) $X$ by $X \times \{ z \in \mathbb{C} | 0 < |z| < 1 \}$. Then by Künneth formula $H_2(X_1; \mathbb{Z}) = \mathbb{Q}$ and $H_1(X_1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Q}$. If follows that $\text{Hom}(H_2(X_1; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X_1; \mathbb{Z}) = \text{Ext}(H_1(X_1; \mathbb{Z}); \mathbb{Z}) = \text{Ext}(\mathbb{Z} \oplus \mathbb{Q}; \mathbb{Z}) = \text{Ext}(\mathbb{Q}; \mathbb{Z}) \neq 0$.

Similarly we may replace $X$ in example 2) by its product with $\{ z \in \mathbb{C} | 0 < |z| < 1 \}$.

**Example 4.** By [15] it is possible to construct, for every countable torsion free abelian group $G$, a compact connected subset $K \subset \mathbb{R}^3$ (in fact a curve)
such that $H_1(\mathbb{R}^3 \setminus K; \mathbb{Z}) \cong G$ and $H_2(\mathbb{R}^3 \setminus K; \mathbb{Z}) = 0$. Taking $G$ such that $\text{Ext}(G; \mathbb{Z})$ contains nontorsion elements and $X \subset \mathbb{C}^3$ a Stein open subset such that $X \cap R^3 = \mathbb{R}^3 \setminus K$ and $\mathbb{R}^3 \setminus K$ is a deformation retract of $X$, one gets as above examples of Stein manifolds $X$ with $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ has nontorsion elements.

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