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# Stability of the Spectrum for Transfer Operators

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**Abstract.** We prove stability of the isolated eigenvalues of transfer operators satisfying a Lasota-Yorke type inequality under a broad class of random and nonrandom perturbations including Ulam-type discretizations. The results are formulated in an abstract framework.

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## 1. - Introduction

Let $(B, \|\cdot\|)$ be a Banach space which is equipped with a second norm $|\cdot| \leq \|\cdot\|$ with respect to which $B$ is typically non-complete. For any bounded linear operator $Q : B \to B$ let

(1) $$|||Q||| := \sup\{|Qf| : f \in B, \|f\| \leq 1\}.$$

We consider a family $(P_\epsilon)_{\epsilon \geq 0}$ of bounded linear operators on $(B, \|\cdot\|)$ with the following properties: There are $C_1, M > 0$ such that for all $\epsilon \geq 0$

(2) $$|P_\epsilon^n| \leq C_1 M^n \quad \forall n \in \mathbb{N} ;$$

there are $C_2, C_3 > 0$ and $\alpha \in (0, 1)$, $\alpha < M$, such that for all $\epsilon \geq 0$

(3) $$\|P_\epsilon^n f\| \leq C_2 \alpha^n \|f\| + C_3 M^n |f| \quad \forall n \in \mathbb{N} \quad \forall f \in B ;$$

also (but see Remarks 1,2,6 for alternative conditions),

(4) if $z \in \sigma(P_\epsilon)$, $|z| > \alpha$, then $z$ is not in the residual spectrum of $P_\epsilon$;

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and there is a monotone upper-semicontinuous function $\tau : [0, \infty) \to [0, \infty)$ such that $\tau_\epsilon > 0$ if $\epsilon > 0$ and

$$|||P_0 - P_\epsilon||| \le \tau_\epsilon \to 0 \quad \text{as } \epsilon \to 0 \ . \tag{5}$$

For families of operators satisfying (2)-(5) we derive uniform bounds on the resolvents with respect to the norm (1), $|||(z - P_\epsilon)^{-1}|||$, when $z$ is uniformly bounded away from the spectrum $\sigma(P_0)$ of $P_0$, and we show that for such $z$ the difference $|||(z - P_\epsilon)^{-1} - (z - P_0)^{-1}||| = O(\tau_\epsilon^\eta)$ for a suitable $\eta > 0$ (Theorem 1). An immediate corollary to these estimates is the stability of isolated eigenvalues $\lambda$ of $P_0$ with $|\lambda| > \alpha$; stability in the sense that if $\delta > 0$ is such that $B_\delta(\lambda) \cap \sigma(P_0) = \{\lambda\}$ then $\lim_{\epsilon\to 0} |||\Pi_\epsilon - \Pi_0||| = 0$. Here $B_\delta(\lambda)$ denotes the open unit ball of radius $\delta$ around $\lambda$ and $\Pi_\epsilon := \frac{1}{2\pi i}\int_{\partial B_\delta(\lambda)}(z - P_\epsilon)^{-1}\,dz$ is the total spectral projection of $P_\epsilon$ associated with $B_\delta(\lambda) \cap \sigma(P_\epsilon)$ (Corollary 1). More precise statements and further corollaries are deferred to the next section.

REMARK 1.

a) If inequality (3) is satisfied for some $n_0$ such that $C_2^{1/n_0}\alpha =: \alpha_0 < 1$, then it holds for all $n$ with $\alpha$ replaced by $\alpha_0$.

b) It is a simple consequence of (3) that for $C_4 = C_2 + C_3$ holds

$$\|P_\epsilon^n\| \le C_4 M^n \quad \forall \epsilon \ge 0 \qquad \forall n \in \mathbb{N} \ . \tag{6}$$

c) In the mathematical literature there are many examples of single operators $P_0$ satisfying 2) and (3). In nearly all cases the two norms involved have the additional property that

$$\text{the closed unit ball of } (B, \|\cdot\|) \text{ is } |\cdot|\text{-compact,} \tag{7}$$

and in all of these examples this property is the key to proving that the essential spectral radius of $P_\epsilon$ is bounded by $\alpha$ which implies in particular assumption (4), see also Remark 3.

In the rest of this introduction we discuss a number of situations where assumptions (2), (3) and (7) are satisfied. This may serve to illustrate the broad applicability of the results proved in this note.

The basic result for this setting is the theorem of Ionescu Tulcea and Marinsecu [15]: If the constant $M$ in (2) and (3) is equal to 1, then $P_0 : B \to B$ has at most finitely many eigenvalues of modulus 1. All these eigenvalues have finite multiplicity, and the rest of the spectrum is contained in a disk around the origin of radius less than 1. In other words: $P_0$ is quasicompact.

The following more concrete setting historically motivated this theorem: $B$ is the space of complex-valued Lipschitz functions on a compact metric space, $\|\cdot\|$ is the Lipschitz norm, and $|\cdot|$ is the supremum norm. $P_0$ is a Markov transition operator of Doeblin-Fortet type, i.e.satisfying (3). For a while this model played a prominent role in mathematical learning theory, see the monographs

[16, Theorem 2.1.40] and [27, Theorem 3.2.1] for a comprehensive treatment and further references. Later, the same analytic setting was used to study Ruelle transfer operators for subshifts of finite type, see e.g. [28] for more information.

Independently of these developments another concrete setting in which the three assumptions (2), (3) and (7) are met emerged in 1973, when Lasota and Yorke [24] studied Perron-Frobenius operators $P_0$ of piecewise $C^2$ and piecewise expanding maps: Now $B$ is the space of functions of bounded variation on an interval, $\|\cdot\|$ is the variation norm, and $|\cdot|$ the usual $L^1$-norm. Inequalities (2) and (3) with $M = 1$ were derived in [24], and the applicability of the theorem of Ionecu Tulcea and Marinescu was noticed later in [18], [19], where [18] in fact deals with an extension of the Lasota-Yorke result to piecewise expanding maps of the unit square. Soon after, Rychlik [30] showed how to bypass the Ionescu Tulcea-Marinescu theorem and proved quasicompactness of $P_0$ more directly, an approach that was exploited in [20] to show that the constant $\alpha$ from (3) is an upper bound for the essential spectral radius of $P_0$. Later, in [12], it was shown how to derive this estimate directly from the Ionescu Tulcea-Marinescu theorem. A good reference for these and related results for Perron-Frobenius operators of one-dimensional maps is the monograph [8]. Variants of this theory for nonexpanding or higher-dimensional maps can be found in [23], [7], [22].

Passing from a single operator $P_0$ to a family $(P_\epsilon)_{\epsilon\geq 0}$ satisfying conditions (2)-(5) can have various interpretations. We mention the following ones: $P_0$ is the Perron-Frobenius operator of a piecewise expanding map $T$ as discussed above.

a) The $P_\epsilon$ are Perron-Frobenius operators of maps $T_\epsilon$ which are "close" to $T$. For maps of the interval $I = [0, 1]$ a suitable notion of closeness is

$$\begin{aligned} d(T, T_\epsilon) := \inf\{\kappa > 0 \mid \exists\, A \subseteq I\, \exists\, \sigma : I \to I \text{ s.th. } m(A) > 1 - \kappa,\ \sigma \text{ is a} \\ \text{diffeomorphism, } T_{\epsilon|A} = T \circ \sigma_{|A}, \text{ and } \forall\, x \in I\text{: } |\sigma(x) - x| < \kappa, \\ |1/\sigma'(x) - 1| < \kappa\}\,. \end{aligned}$$

In this case $\tau_\epsilon = 12\, d(T, T_\epsilon)$, see [21].

b) $P_\epsilon$ is the transition operator of the stochastically perturbed map $T$, where $\epsilon$ is the "size" of the perturbation, see [21], [4], [2], [5]. Typically, $\tau_\epsilon = O(\epsilon)$.

c) $P_\epsilon$ is the transition operator for the Ulam-type discretization of $T$ with grid size $\epsilon$. Again $\tau_\epsilon = O(\epsilon)$, see [25], [21], [3], [9], [5] and for related work also [14], [26], [11], [17].

For all these families $(P_\epsilon)_{\epsilon\geq 0}$, inequality (3) is satisfied uniformly in $\epsilon$ provided there is an iterate $T^k$ of $T$ with $\inf|(T^k)'| > 2$ and such that there are no discontinuities or turning points $c, c'$ of $T$ with $T^j c = c'$ for some $0 < j \leq k$. But even if this condition is violated certain types of perturbations, notably Ulam discretizations, satisfy (3) uniformly in $\epsilon$, see [5] for a detailed discussion.

REMARK 2. Since in all these situations the $P_\epsilon$ can be interpreted as positive operators on $L^1_m$, their peripheral eigenvalues form a finite cyclic group, see [31], [30], [13].

REMARK 3. For some applications it is interesting to note that the compactness assumption (7) can be replaced by the following weaker set of assumptions:

there is a sequence of linear operators $\pi_k : B \to B$ with $\sup_k \|\pi_k\| < \infty$ and such that

$$\sup\{|f - \pi_k f| : f \in B, \|f\| \le 1\} \le \text{const} \cdot \left(\frac{\alpha}{M}\right)^k$$

and

$$P_\epsilon^k \pi_k \text{ is a compact operator for all } k\,.$$

A simple calculation based only on these assumptions and on (3) shows that there is a constant $C > 0$ such that

$$\|P_\epsilon^k - P_\epsilon^k \pi_k\| \le C\alpha^k \qquad \forall k > 0 \qquad \forall \epsilon \ge 0\,.$$

It follows from [10, Lemma VIII.8.2] that in this case all $P_\epsilon : B \to B$ are quasicompact with essential spectral radius $\le \alpha$ (in particular (4) holds). This is in fact the previously mentioned approach of Rychlik [30]. Related questions for function spaces of higher smoothness are discussed in [1].

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## 2. – The results

For $\delta > 0$ and $r > \alpha$ let

$$V_{\delta,r} := \{z \in \mathbb{C} : |z| \le r \text{ or } \operatorname{dist}(z, \sigma(P_0)) \le \delta\}\,.$$

The main results of this paper are the following bounds on the resolvents $(z - P_\epsilon)^{-1}$.

THEOREM 1. *Suppose that $(P_\epsilon)_{\epsilon \ge 0}$ is a family of linear operators on B satisfying* (2)-(5). *Fix $\delta > 0$ and $r \in (\alpha, M)$ and let $\eta := \frac{\log r/\alpha}{\log M/\alpha}$. Then $\eta > 0$ and there are constants $\epsilon_0 = \epsilon_0(\delta, r) > 0$, $a = a(r) > 0$, $b = b(\delta, r) > 0$, $c = c(\delta, r) > 0$ and $d = d(\delta, r) > 0$ such that for $0 \le \epsilon \le \epsilon_0$ and $z \in \mathbb{C} \setminus V_{\delta,r}$*

$$\|(z - P_\epsilon)^{-1} f\| \le a\,\|f\| + b\,|f| \quad \text{for all } f \in B \tag{8}$$

*and*

$$\left|\left|\left|(z - P_\epsilon)^{-1} - (z - P_0)^{-1}\right|\right|\right| \le \tau_\epsilon^\eta \left(c\,\|(z - P_0)^{-1}\| + d\,\|(z - P_0)^{-1}\|^2\right)\,. \tag{9}$$

Explicit bounds on the constants $\epsilon_0, a, b, c, d$ are given in the proof. They all depend on the operators $P_\epsilon$ via the constants $M, C_1, \ldots, C_4$; $\epsilon_0$ and $b$ depend also on the functions $\epsilon \mapsto \tau_\epsilon$ and $z \mapsto \|(z - P_0)^{-1}\|$.

Denote $\sigma_\alpha(P_\epsilon) := \{z \in \mathbb{C} : |z| \leq \alpha\} \cup \sigma(P_\epsilon)$. An immediate consequence of (8) is that

$$S_{\delta,r} := \sup\left\{\|(z - P_\epsilon)^{-1}\| : 0 \leq \epsilon \leq \epsilon_0(\delta, r), z \in \mathbb{C} \setminus V_{\delta,r}\right\} < \infty \tag{10}$$

for all $\delta > 0$ and $r \in (\alpha, M)$. Therefore, all accumulation points (as $\epsilon \to 0$) of spectral values in $\sigma_\alpha(P_\epsilon)$ are contained in $\sigma_\alpha(P_0)$. A more elementary proof of this fact was previously given in [6]. But much more can be deduced from (9). If $\lambda$ is an isolated eigenvalue of $P_0$ with $|\lambda| > \alpha$, then $\delta > 0$ can be chosen so small that $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$ and we can define

$$\Pi_\epsilon^{(\lambda,\delta)} := \frac{1}{2\pi \imath} \int_{\partial B_\delta(\lambda)} (z - P_\epsilon)^{-1}\, dz \ .$$

$\Pi_0^{(\lambda,\delta)}$ does not depend on $\delta$ as long as $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$, and as we just saw, also the projections $\Pi_\epsilon^{(\lambda,\delta)}$ are well defined and independent of $\delta$ for sufficiently small $\epsilon$.

COROLLARY 1. *In the situation of Theorem* 1, *if $\lambda$ is an isolated eigenvalue of $P_0$ with $|\lambda| > r$ and if $\delta > 0$ is such that $B_\delta(\lambda) \cap \sigma_\alpha(P_0) = \{\lambda\}$, we have:*

1) *There is a constant $K_1 = K_1(\delta, r) > 0$ such that $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| \leq K_1 \cdot \tau_\epsilon^\eta$ for all $\epsilon \in [0, \epsilon_0]$.*
2) *There are constants $K_2 = K_2(\delta, r) > 0$ and $\delta_0 = \delta_0(r) > 0$ such that $\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq K_2 \cdot |\Pi_\epsilon^{(\lambda,\delta)} f|$ for all $f \in B$, $\delta \in (0, \delta_0]$ and $\epsilon \in [0, \epsilon_1]$.*
3) *If $\delta \in (0, \delta_0]$, then* $\operatorname{rank}(\Pi_\epsilon^{(\lambda,\delta)}) = \operatorname{rank}(\Pi_0^{(\lambda,\delta)})$ *for $\epsilon$ small enough.*

REMARK 4. If the isolated eigenvalue $\lambda$ of $P_0$ has finite multiplicity, this means in particular that, for $\epsilon$ and $\delta$ small enough, $\sigma_\alpha(P_\epsilon) \cap B_\delta(\lambda)$ consists of eigenvalues $\lambda_j^{(\epsilon)}$, such that $\lim_{\epsilon\to 0} \lambda_j^{(\epsilon)} = \lambda$ for all $j$, and the total multiplicity of the $\lambda_j^{(\epsilon)}$ equals the multiplicity of $\lambda$.

For $r > \alpha$ denote by $\gamma_r$ the circle of radius $r$ around the origin and define

$$\Pi_\epsilon^{(r)} := \frac{1}{2\pi \imath} \int_{\gamma_r} (z - P_\epsilon)^{-1}\, dz \ .$$

COROLLARY 2. *If, in the situation of Theorem* 1, $\sigma(P_0) \cap \{z \in \mathbb{C} : |z| = r\} = \emptyset$, *then*

1) $\lim_{\epsilon\to 0} |||\Pi_\epsilon^{(r)} - \Pi_0^{(r)}||| = 0$
2) *There is $K_3 = K_3(\delta, r) > 0$ such that for all $\epsilon \in [0, \epsilon_0]$*

$$\|P_\epsilon^n \Pi_\epsilon^{(r)}\| \leq K_3 \cdot r^n \quad \forall n \in \mathbb{N} \ .$$

REMARK 5. In many cases the $P_\epsilon$ are $|\cdot|$-contractions, i.e.$M = 1$. If, in such a situation, we considered $r = 1$, i.e. a case which is not covered by Theorem 1, then $\eta$ as given there would evaluate to 1. A look at the proofs reveals, however, that in this case the order of convergence in (9) is only $\tau_\epsilon |\log \tau_\epsilon|$ instead of $\tau_\epsilon$. This is in agreement with results in [21], [17] on the stability of eigenfunctions belonging to eigenvalues of modulus 1.

## 3. – Proofs

In the proofs we use the following abbreviating notation:

$$Q_\epsilon := (z - P_\epsilon)$$

where we suppress the dependence of $Q_\epsilon$ on $z$. Recall that since $z \notin V_{\delta,r}$ always $|z| \geq r > \alpha$ and $Q_0^{-1} = (z - P_0)^{-1}$ exists as a bounded linear operator on $B$.

LEMMA 1. *For each $r \in (\alpha, M)$ holds*

$$\|f\| \leq C_5 \|Q_\epsilon f\| + C_6 |f| \quad \textit{for all } f \in B \textit{ and } \epsilon \geq 0$$

*where*

$$C_5 := \frac{2C_4}{M-r}\left(\frac{M}{r}\right)^{n_1} \quad \textit{and} \quad C_6 := 2C_3\left(\frac{M}{r}\right)^{n_1} \quad \textit{with} \quad n_1 := \left\lceil \frac{\log 2C_2}{\log r/\alpha} \right\rceil$$

*and $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.*

PROOF. For $f \in B$ holds

$$\begin{aligned} |z|^n \|f\| = \|z^n f\| &\leq \|(z^n - P_\epsilon^n) f\| + \|P_\epsilon^n f\| \\ &\leq \sum_{i=0}^{n-1} |z^i| \, \|P_\epsilon^{n-1-i}\| \, \|(z - P_\epsilon) f\| + C_2 \alpha^n \|f\| + C_3 M^n |f| \end{aligned}$$

in view of (3). Observing (6), this yields

$$\|f\| \leq \frac{C_4}{M-r}\left(\frac{M}{r}\right)^n \|Q_\epsilon f\| + C_2 \left(\frac{\alpha}{r}\right)^n \|f\| + C_3 \left(\frac{M}{r}\right)^n |f|.$$

Let $n = n_1$. Then $C_2 \left(\frac{\alpha}{r}\right)^{n_1} \leq \frac{1}{2}$ so that

$$\|f\| \leq \frac{2C_4}{M-r}\left(\frac{M}{r}\right)^{n_1} \|Q_\epsilon f\| + 2C_3 \left(\frac{M}{r}\right)^{n_1} |f|. \qquad \square$$

PROOF OF THEOREM 1. Consider any $h \in B$ and let $g := Q_0 h$. As

$$Q_0^{-1} = (z - P_0)^{-1} = z^{-n} Q_0^{-1} P_0^n + \sum_{j=0}^{n-1} z^{-j-1} P_0^j$$

we have

$$\begin{aligned} |h| = |Q_0^{-1} g| &\leq |z|^{-n} \| Q_0^{-1} P_0^n g \| + \sum_{j=0}^{n-1} |z|^{-j-1} |P_0^j g| \\ &\leq \| Q_0^{-1} \| \, C_2 \left(\frac{\alpha}{r}\right)^n \|g\| + \|Q_0^{-1}\| \, C_3 \left(\frac{M}{r}\right)^n |g| \\ &\quad + \frac{C_1}{M - r} \left(\frac{M}{r}\right)^n |g| . \end{aligned} \tag{11}$$

As $g = (Q_0 - Q_\epsilon) Q_0^{-1} g + Q_\epsilon Q_0^{-1} g = (P_\epsilon - P_0) h + Q_\epsilon h$, we have

$$|g| \leq |(P_\epsilon - P_0) h| + |Q_\epsilon h| \leq \tau_\epsilon \|h\| + |Q_\epsilon h| \ .$$

Combining these two estimates yields

$$|h| \leq \|Q_0^{-1}\| \, C_2 \left(\frac{\alpha}{r}\right)^n \|g\| + \left( \|Q_0^{-1}\| \, C_3 + \frac{C_1}{M - r} \right) \left(\frac{M}{r}\right)^n (\tau_\epsilon \|h\| + |Q_\epsilon h|) \ .$$

Recall that $g = Q_0 h = (z - P_0) h$ so that $\|g\| \leq (C_4 M + |z|) \|h\|$. Therefore,

$$\begin{aligned} |h| \leq \Bigg( & \|Q_0^{-1}\| C_2 (C_4 M + |z|) \left(\frac{\alpha}{r}\right)^n \\ & + \left( \|Q_0^{-1}\| C_3 + \frac{C_1}{M - r} \right) \tau_\epsilon \left(\frac{M}{r}\right)^n \Bigg) \|h\| \\ & + \left( \|Q_0^{-1}\| C_3 + \frac{C_1}{M - r} \right) \left(\frac{M}{r}\right)^n |Q_\epsilon h| . \end{aligned} \tag{12}$$

For the proof of assertion (8) we may assume that $|z| \leq 2M$, because for $|z| > 2M$ a von Neumann series representation gives immediately that $\|(z - P_\epsilon)^{-1} f\| \leq C_4 M^{-1} \|f\|$. Let

$$\begin{aligned} H = H(\delta, r) &:= \sup\{ \|(z - P_0)^{-1}\| : z \in \mathbb{C} \setminus V_{\delta, r} \} , \\ n = n_2(\delta, r) &:= \left\lceil \frac{\log 4 C_6 H C_2 (C_4 + 2) M}{\log r / \alpha} \right\rceil \end{aligned} \tag{13}$$

and

$$\epsilon_1 = \epsilon_1(\delta, r) := \sup \left\{ \epsilon > 0 : \tau_\epsilon \left( H C_3 + \frac{C_1}{M - r} \right) \left(\frac{M}{r}\right)^{n_2} \leq \frac{1}{4 C_6} \right\} . \tag{14}$$

Since $\|h\| \le C_5 \|Q_\epsilon h\| + C_6 |h|$ by Lemma 1, this and (12) yields for $0 \le \epsilon \le \epsilon_1$

$$|h| \le \frac{1}{2}|h| + \frac{C_5}{2C_6}\|Q_\epsilon h\| + \frac{1}{4\tau_{\epsilon_1} C_6} |Q_\epsilon h|$$

so that

$$|h| \le \frac{C_5}{C_6}\|Q_\epsilon h\| + \frac{1}{2\tau_{\epsilon_1} C_6} |Q_\epsilon h| \ .$$

Applying Lemma 1 once more, we arrive at

$$\|h\| \le 2C_5 \|Q_\epsilon h\| + \frac{1}{2\tau_{\epsilon_1}} |Q_\epsilon h| \le \left(2C_5 + \frac{1}{2\tau_{\epsilon_1}}\right) \|Q_\epsilon h\| \tag{15}$$

for $0 \le \epsilon \le \epsilon_1$. By (15) it follows that $Q_\epsilon$ is invertible, its range is closed, and the inverse is bounded, hence $Q_\epsilon^{-1}$ exists as a bounded operator on $B$ since $z$ cannot belong to the residual spectrum by assumption (4). Applying this inequality to $h := Q_\epsilon^{-1} f$, this is (8) with $a := \max\{2C_5, C_4 M^{-1}\}$ and $b := \frac{1}{2\tau_{\epsilon_1}}$.

We turn to the proof of (9) and keep the abbreviation $h = Q_\epsilon^{-1} f$. Consider estimate (12) for $n = \lceil \frac{\log \tau_\epsilon}{\log \alpha / M} \rceil = \lceil \eta \frac{\log \tau_\epsilon}{\log \alpha / r} \rceil$. Then

$$\begin{aligned} |h| \le \tau_\epsilon^\eta &\left( \|Q_0^{-1}\| \,(C_2(C_4 M + |z|) + C_3) + \frac{C_1}{M - r} \right) \|h\| \\ &+ \tau_\epsilon^{\eta - 1} \left( \|Q_0^{-1}\| \, C_3 + \frac{C_1}{M - r} \right) |f| \ . \end{aligned}$$

As $|z| \, \|Q_0^{-1}\| \le 2C_4$ if $|z| \ge 2M$, this yields

$$\begin{aligned} |h| \le \tau_\epsilon^\eta &\left( \|Q_0^{-1}\| \underbrace{(C_2(C_4 + 2)M + C_3)}_{=:A} + \underbrace{\left(2C_2C_4 + \frac{C_1}{M - r}\right)}_{=:B} \right) \|h\| \\ &+ \tau_\epsilon^{\eta - 1} \left( \|Q_0^{-1}\| \, C_3 + \frac{C_1}{M - r} \right) |f| \ . \end{aligned}$$

Since $\|h\| \le C_5 \|f\| + C_6 |h|$ by Lemma 1 again, this yields

$$\begin{aligned} |h| \le \tau_\epsilon^\eta &\left( \|Q_0^{-1}\| \, A + B \right) C_5 \|f\| + \tau_\epsilon^\eta \left( \|Q_0^{-1}\| \, A + B \right) C_6 |h| \\ &+ \tau_\epsilon^{\eta - 1} \left( \|Q_0^{-1}\| \, C_3 + \frac{C_1}{M - r} \right) |f| \ . \end{aligned}$$

Recall $H = H(\delta, r)$ from (13) and let

$$\epsilon_0 = \epsilon_0(\delta, r) := \sup\left\{ \epsilon \in (0, \epsilon_1] : \tau_\epsilon^\eta (HA + B) C_6 \le \frac{1}{2} \right\} \ .$$

Then

$$|Q_\epsilon^{-1} f| = |h| \leq 2\,\tau_\epsilon^\eta \left( \left( \|Q_0^{-1}\| A + B \right) C_5 \, \|f\| + \tau_\epsilon^{-1} \left( \|Q_0^{-1}\| C_3 + \frac{C_1}{M-r} \right) |f| \right)$$

for $0 \leq \epsilon \leq \epsilon_0$.

We apply this estimate to $(P_\epsilon - P_0) Q_0^{-1} f$ instead of $f$:

$$\begin{aligned} |(Q_\epsilon^{-1} - Q_0^{-1}) f| &= |Q_\epsilon^{-1} (P_\epsilon - P_0) Q_0^{-1} f| \\ &\leq 2\tau_\epsilon^\eta \left( \left( \|Q_0^{-1}\| A + B \right) C_5 \, \|(P_\epsilon - P_0) Q_0^{-1} f\| \right. \\ &\quad \left. + \tau_\epsilon^{-1} \left( \|Q_0^{-1}\| C_3 + \frac{C_1}{M-r} \right) |(P_\epsilon - P_0) Q_0^{-1} f| \right) \\ &\leq 2\|Q_0^{-1}\|^2 \, \tau_\epsilon^\eta \, (2AC_4C_5M + C_3) \, \|f\| \\ &\quad + 2\|Q_0^{-1}\| \, \tau_\epsilon^\eta \left( 2BC_4C_5M + \frac{C_1}{M-r} \right) \|f\| \end{aligned}$$

which proves (9) with $c := 2(2BC_4C_5M + \frac{C_1}{M-r})$ and $d := 2(2AC_4C_5M + C_3)$ for $0 \leq \epsilon \leq \epsilon_0$. □

REMARK 6. In the preceding proof assumption (4) was used only to conclude that if $z \in \mathbb{C} \setminus V_{\delta,r}$ then the range of the operator $Q_\epsilon = z - P_\epsilon$ must be all $B$. In some situations the following argument might be used instead of assumption (4): Let $A$ denote a connected component of $\mathbb{C} \setminus V_{\delta,r}$, so $A$ is open. We claim that

$$\text{either } A \subseteq \sigma(P_\epsilon) \text{ or } A \cap \sigma(P_\epsilon) = \emptyset\,. \tag{16}$$

Indeed, estimate (15) guarantees that $\operatorname{dist}(z, \sigma(P_\epsilon)) \geq \frac{1}{C}$ for $z \in \mathbb{C} \setminus (\sigma(P_\epsilon) \cup V_{\delta,r})$ where $C = 2C_5 + (2\tau_{\epsilon_1})^{-1}$, and it is an easy exercise to derive (16) from this. In view of the alternative (16) the unbounded component of $\mathbb{C} \setminus V_{\delta,r}$ is certainly disjoint from $\sigma(P_\epsilon)$.

If $\mathbb{C} \setminus \sigma_\alpha(P_0)$ is connected and if $z \in \mathbb{C} \setminus \sigma_\alpha(P_0)$, then $z$ belongs to the unbounded connected component of $\mathbb{C} \setminus V_{\delta,r}$ provided $r - \alpha$ and $\delta$ are sufficiently small, and it follows that $z \in \mathbb{C} \setminus \sigma(P_\epsilon)$ if $\epsilon < \epsilon_0(\delta, r)$. In other words: Given $z \in \mathbb{C} \setminus \sigma_\alpha(P_0)$, there are $\delta, r > 0$ (possibly depending on $z$) such that $(z - P_\epsilon)^{-1}$ exists as a bounded linear operator for $\epsilon \in [0, \epsilon_0(\delta, r)]$. In particular, (4) can be replaced by "$\mathbb{C} \setminus V_{\delta,r}$ is connected for each $r - \alpha$ and $\delta$ sufficiently small".

PROOF OF COROLLARY 1.

1) $$\begin{aligned} |||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| &\leq \frac{1}{2\pi} \int_{\partial B_\delta(\lambda)} |||(z - P_\epsilon)^{-1} - (z - P_0)^{-1}||| \, dz \\ &\leq \delta \, \tau_\epsilon^\eta (c\, H(\delta, r) + d\, H(\delta, r)^2) \text{ with } H(\delta, r) \text{ from (13).} \end{aligned}$$

Note that this implies $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| \to 0$ as $\epsilon \to 0$.

2) Using (8) follows

$$\begin{aligned}
\|\Pi_\epsilon^{(\lambda,\delta)} f\| &\leq \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} \|(z-P_\epsilon)^{-1} f\|\, dz \\
&\leq \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} (a(r)\,\|f\| + b(\delta, r)\,|f|)\, dz \\
&\leq \delta\, a(r)\,\|f\| + \delta\, b(\delta, r)\,|f| \ .
\end{aligned}$$

Fixing $r$ and choosing $\delta > 0$ such that $\delta\, a(r) \leq \frac{1}{2}$ this yields for sufficiently small $\epsilon$

$$\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq \frac{1}{2}\|f\| + \text{const}\cdot|f| \ .$$

Applied to $\Pi_\epsilon^{(\lambda,\delta)} f$ instead of $f$ we can conclude that $\|\Pi_\epsilon^{(\lambda,\delta)} f\| \leq K_2\cdot|\Pi_\epsilon^{(\lambda,\delta)} f|$ for a suitable constant $K_2 > 0$.

3) Let us consider a $n$-dimensional subspace $\mathbb{V}_n$ of $\Pi_\epsilon^{(\lambda,\delta)}(B)$. In view of part (1) we can choose $\epsilon$ and $\delta$ small enough such that $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| < \frac{1}{2K_2}$. Then, for $f \in \mathbb{V}_n$,

$$|f - \Pi_0^{(\lambda,\,\delta)} f| = |\Pi_\varepsilon^{(\lambda,\,\delta)} f - \Pi_0^{(\lambda,\,\delta)} f| \leq \frac{1}{2K_2}\|f\| \leq \frac{1}{2}|f|\,,$$

where we have used the result of part (2) above.

This means that the unit ball of the subspace $\mathbb{V}_n$ is contained in a $\frac{1}{2}$-neighborhood of the subspace $\Pi_0^{(\lambda,\,\delta)}(B)$. In this situation $n \leq \text{rank}(\Pi_0^{(\lambda,\delta)})$ by Tichomirov's theorem (e.g.[29, Theorem 1.5]), and, by the arbitrariness of $n$, $\text{rank}(\Pi_\epsilon^{(\lambda,\delta)}) \leq \text{rank}(\Pi_0^{(\lambda,\delta)})$. The reverse inequality follows by interchanging the roles of $\Pi_\epsilon^{(\lambda,\delta)}$ and $\Pi_0^{(\lambda,\delta)}$. □

PROOF OF COROLLARY 2.

1) This is proved just like the first assertion of Corollary 1.

2) As observed in (10), $\|(z-P_\epsilon)^{-1}\| \leq S_{\delta,r}$ for $z \in \gamma_r$ and $\epsilon \in [0, \epsilon_0]$. Therefore

$$\|P_\epsilon^n \Pi_\epsilon^{(r)}\| \leq \frac{1}{2\pi}\int_{\gamma_r} |z|^n\, \|(z-P_\epsilon)^{-1} f\|\, dz \leq S_{\delta,r}\, r^{n+1} \ .$$

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