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# The Class of Holomorphic Functions Representable by Carleman Formula

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**Abstract.** Carleman formulas, unlike the Cauchy formula, restore a function holomorphic in a domain $\mathcal{D}$ by its values on a part $M$ of the boundary $\partial\mathcal{D}$, provided that $M$ is of positive Lebesgue measure. Naturally arises the following question:

Can we describe the class of holomorphic functions that are represented by Carleman formula?

We consider the simplest Carleman formulas in one and several complex variables on very particular domains. Under these conditions the main result of the present paper is that a necessary and sufficient condition for a holomorphic function $f$ to be represented by Carleman formula over the set $M$ is that $f$ must belong to "the class $\mathcal{H}^1$ near the set $M$".

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## 1. – Introduction

In the theory of boundary values of holomorphic functions of one complex variable a question was raised about the description of the class of holomorphic in a domain $\mathcal{D}$ functions which are represented using their boundary values by the Cauchy integral formula. The answer was very clear and was obtained for the case of the disk by F. and M. Riesz (1916) and for other domains by V. Smirnov (1932). Their result states that this class of functions coincides with the Hardy class $\mathcal{H}^1(\mathcal{D})$.

During the last years there was a number of research papers devoted to the Carleman formulas for holomorphic functions of one or several complex variables (their survey can be found in [1]). These formulas solve the problem of

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the reconstruction of holomophic functions in the interior points of a domain $\mathcal{D}$ from their values on a subset $M \subset \partial\mathcal{D}$ of positive measure, which does not contain the Shilov boundary of the domain. This is exactly the point of essential difference with the Cauchy formulas and its multidimensional analogues. This problem is ill-posed and therefore it is not an accident that the limit appears together with the integration over $M$ in Carleman formulas.

Therefore, naturally arises the following problem:

Can we describe the class of holomorphic functions that are represented by Carleman formula?

Our conjecture is that a necessary and sufficient condition for a holomorphic function $f$ to be represented by Carleman formula over the set $M$ is that $f$ must belong to the "class $\mathcal{H}^1$" near the set $M$.

The solution in this case is more delicate than in the case of Cauchy integral formula described above. The present paper is the first result in this direction. A positive answer is obtained only for the simplest cases of Carleman formulas in one and several complex variables. In one complex variable the conjecture is true when the domain $\mathcal{D}$ in $\mathbf{C}$ is bounded by a Ahlfors-regular curve and a piece of the unit circle. In multidimensional case, the conjecture holds for a strongly convex domain $\mathcal{D}$ in $\mathbf{C}^n$ bounded by a strongly convex hypersurface together with a piece of the boundary of a Reinhardt domain.

Furthermore, we point out that our principal result marks the sharp difference between the Carleman type integrals (i.e. integrals of some $L^1$ function about which it is not known apriori that it comes from a holomorphic function) and the Cauchy type integrals. Cauchy type integrals of $L^1$ functions always exist and are always holomorphic functions in the given domain. On the other hand, Carleman type integrals (for $n = 1$) exist on a sequence of points converging to the boundary of the domain if and only if it is a Carleman integral, i.e. can be continued holomorphically into the domain. In addition, we obtain multidimensional analogues of these statements. Note also that in contrast to the Cauchy formula, even in the case of one complex variable, Carleman formulas depend on the domain $\mathcal{D}$ and on the set $M$. Concluding, we point out that a Carleman type integral is a holomorphic function if and only if it is a Carleman integral.

In Section 2 we solve the problem on domains of particular type with Ahlfors-regular boundary for the simplest Carleman formulas. In Section 3 we are considering some multidimensional analogues of the results from Section 2.

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## 2. – Case of one complex variable

The classical problem of the description of a class of holomorphic functions, representable by their angular boundary values (in the paper we will also use the term boundary values) using the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{f(\zeta)}{\zeta - z} d\zeta \,, \tag{2.1}$$

was solved by F. and M. Riesz (1916) for the disk and by V. Smirnov (1932) [12], [16] for other simply connected domains. Their theorem is the following

THEOREM 2.1. *Let $\mathcal{D}$ be a bounded, simply connected domain with rectifiable boundary. A holomorphic function $f$ with its boundary values belonging to $L^1(\partial\mathcal{D})$ is representable by formulas* (2.1) *if and only if $f \in \mathcal{H}^1(\mathcal{D})$.*

A function $f(z)$ holomorphic in $\mathcal{D}$ belongs to the class $\mathcal{H}^p(\mathcal{D})$, $p > 0$, if there exists a sequence of curves $\gamma_m$ in $\mathcal{D}$ converging to $\partial\mathcal{D}$ such that

$$\int_{\gamma_m} |f(z)|^p |dz| \leq C$$

where $C$ is independent of $m$.

Carleman formulas, unlike the Cauchy formula, restore a function holomorphic in a domain $\mathcal{D}$ by its values on a part $M$ of the boundary $\partial\mathcal{D}$, provided that $M$ is of positive Lebesgue measure.

Recall that a rectifiable curve $\Gamma$ is called Ahlfors-regular if the following holds

$$l\big(\Gamma \cap K(a, \tau)\big) \leq C\tau \,,$$

where $K(a, \tau)$ is a disk of radius $\tau$ and center at any point $a \in \Gamma$ and $l$ is the length of the curve.

Let us give the simplest Carleman formula (Goluzin-Krylov, 1933) to be found in [13]. Let $\Gamma$ be an Ahlfors-regular simple curve joining two points on the unit circle and lying inside it. We assume that $0 \notin \Gamma$. We define the domain $\mathcal{D}_1$ to be the part of the unit disk $K(0, 1)$ cut off by $\Gamma$ so that $0 \notin \overline{\mathcal{D}}_1$. Then for any function $f$ holomorphic in $\mathcal{D}_1$ and continuous in $\overline{\mathcal{D}}_1$ the following Carleman formula holds

$$f(z) = \lim_{m \mapsto \infty} \frac{1}{2\pi i} \int_{\Gamma} f(\zeta) \left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z} \,. \tag{2.2}$$

The problem of the reconstruction of a function, holomorphic in $\mathcal{D}_1$, by its boundary values on $\Gamma$ is ill-posed (unstable) and therefore it is not surprising that a limit is involved in (2.2). There arises the problem of description of the class of functions, holomorphic in $\mathcal{D}_1$ and representable by (2.2). Let $\rho = \inf_\tau \{\tau : \mathcal{D}_\tau \neq \emptyset\}$, where $\mathcal{D}_\tau = K(0, \tau) \cap \mathcal{D}_1$, $0 < \tau < 1$. Next we state the following

THEOREM 2.2. *Let $f$ be a function holomorphic in $\mathcal{D}_1$ with the property that its boundary values on $\Gamma$ belong to the class $L^1(\Gamma)$.*

1) *If $f \in \mathcal{H}^1(\mathcal{D}_{\tau_k})$ for a sequence $\{\tau_k\}$, $\tau_k > 0$, $\lim_{k \mapsto \infty} \tau_k = 1$, then $f$ is representable by* (2.2).
2) *If $f$ is representable by* (2.2) *then $f \in \mathcal{H}^1(\mathcal{D}_\tau)$ for every $\tau$, $\rho < \tau < 1$.*

PROOF. 1) Let $f \in \mathcal{H}^1(\mathcal{D}_{\tau_k})$, $\tau_k > 0$, $\lim_{k\mapsto\infty} \tau_k = 1$ and $z$ be a fixed point of $\mathcal{D}_1$. Then $z \in \mathcal{D}_{\tau_k}$ for some $k$ and by Cauchy formula we have

$$\frac{f(z)}{z^m} = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_{\tau_k}} \frac{f(\zeta)}{\zeta^m}\frac{d\zeta}{\zeta - z}. \tag{2.3}$$

Furthermore

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial\mathcal{D}_{\tau_k}} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\left(\int_{\Gamma_{\tau_k}} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z} + \int_{\partial\mathcal{D}_{\tau_k}\setminus\Gamma_{\tau_k}} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z}\right), \end{aligned}$$

where $\Gamma_\tau = \Gamma \cap K(0, \tau)$. The second integral tends to zero as $m \mapsto \infty$, hence

$$\begin{aligned} f(z) &= \lim_{m\mapsto\infty} \frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z} \\ &= \lim_{m\mapsto\infty} \frac{1}{2\pi i}\left(\int_{\Gamma} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z} - \int_{\Gamma\setminus\Gamma_{\tau_k}} f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z}\right), \end{aligned}$$

where the second integral tends again to 0. Therefore the Carleman formula (2.2) holds for the function $f$.

2) Assume now that the representation (2.2) holds for $f$ and for $z \in \mathcal{D}_1$. Obviously

$$\frac{\left(\frac{z}{\zeta}\right)^m}{\zeta - z} = \frac{1}{\zeta - z} - \left[\left(\frac{z}{\zeta}\right)^{m-1} + \left(\frac{z}{\zeta}\right)^{m-2} + \cdots + 1\right]\frac{1}{\zeta},$$

hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}d\zeta - \lim_{m\mapsto\infty}\sum_{k=0}^{m-1} a_k z^k, \tag{2.4}$$

where

$$a_k = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta^{k+1}}, \qquad k = 0, 1, 2, \ldots \tag{2.5}$$

Therefore, for $z \in \mathcal{D}_1$ there exists a limit

$$\lim_{m\mapsto\infty}\sum_{k=0}^{m-1} a_k z^k$$

and consequently the series

$$\sum_{k=0}^{\infty} a_k z^k \tag{2.6}$$

has a radius of convergence greater than or equal to 1.

Let us consider the Cauchy type integral

$$F_{\pm}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \,,$$

where $F_+$ is holomorphic in $\mathcal{D}_1$ and $F_-$ is holomorphic in $K(0,1) \setminus \overline{\mathcal{D}}_1$. In a neighborhood of the origin

$$F_-(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta \left(1 - \dfrac{z}{\zeta}\right)} = \sum_{k=0}^{\infty} a_k z^k \,,$$

where $a_k$ are defined by (2.5), so $F_-$ has analytic continuation into $K(0,1)$. The function $F_+$ being the Cauchy type integral belongs to the space $\mathcal{H}^p(\mathcal{D}_1)$, for all $0 < p < 1$ (V. Smirnov, 1928 in [16], Th. 3.5 in [10]). Smirnov's result was proven for the unit disk, however it does hold for simply connected domain with Ahlfors-regular boundary [7], [8], [9].

Since $F_+ \in \mathcal{H}^p(\mathcal{D}_1)$, for $0 < p < 1$, we have that the subharmonic function $|f(z)|^p$ has a harmonic majorant in $\mathcal{D}_1$ (p. 168 in [10]). Therefore $|f(z)|^p$ has a harmonic majorant in $\mathcal{D}_\tau \subset \mathcal{D}_1$ for all $\rho < \tau < 1$. Hence by (2.4) we deduce that

$$F_+ - F_- = f \in \mathcal{H}^p(\mathcal{D}_\tau)$$

for any $\tau$, $\rho < \tau < 1$. Since $f \in \mathcal{H}^p(\mathcal{D}_\tau)$, it has angular boundary values almost everywhere on $\Gamma$. Thus, for almost all $\tau$, $\rho < \tau < 1$, we have that $f \in L^\infty(\gamma_\tau)$, where $\gamma_\tau = \{z : \ |z| = \tau, \ z \in \mathcal{D}_1\}$. Since $f \in L^1(\Gamma)$, we deduce that $f \in L^1(\partial \mathcal{D}_\tau)$ for almost all $\tau$. We recall now a Smirnov's theorem (1928) ([16]; chap. 9, Section 4, Th. 4 in [12]), which states that if a function belongs to the class $\mathcal{H}^p$ and its values on the boundary of the domain are in the space $L^q$, $q > p$ then the same function belongs to the class $\mathcal{H}^q$. From this theorem and the fact that $f \in \mathcal{H}^p(\mathcal{D}_\tau)$ we conclude that $f \in \mathcal{H}^1(\mathcal{D}_\tau)$ for almost all $\tau$, $\rho < \tau < 1$. Applying once more the argument about the harmonic majorant, mentioned above, we obtain that $f \in \mathcal{H}^1(\mathcal{D}_\tau)$ for every $\tau$, $\rho < \tau < 1$. □

COROLLARY 2.1. *If $f \in L^1(\Gamma)$ and $a_k$ are defined by* (2.5) *then*

1) *If $f$ has analytic continuation into $\mathcal{D}_1$ as $\mathcal{H}^1(\mathcal{D}_{\tau_k})$-function, where $\{\tau_k\}$ is a sequence of positive numbers increasing to* 1, *and so that $\rho < \tau_k < 1$ then*

$$\limsup_{k \mapsto \infty} |a_k|^{\frac{1}{k}} \leq 1 \,. \tag{2.7}$$

2) *If* (2.7) *holds then* $f$ *has analytic continuation into* $\mathcal{D}_1$ *as* $\mathcal{H}^1(\mathcal{D}_\tau)$ *function for all* $\tau$, $\rho < \tau < 1$.
3) *If* $f|_\Gamma$ *is not almost everywhere zero then* (2.7) *in* 1) *and* 2) *is equivalent to*

$$\limsup_{k\mapsto\infty} |a_k|^{\frac{1}{k}} = 1\,. \tag{2.8}$$

Proof. The parts 1) and 2) are essentially proven above.

Part 3) (for $f \in C(\Gamma)$ one can consult [2]) follows from the fact that the Cauchy type integral

$$F(z) = \frac{1}{2\pi}\int \frac{f(\zeta)d\zeta}{\zeta - z}$$

defines a holomorphic function in the domain $\overline{\mathbf{C}} \setminus \Gamma$ such that $F(\infty) = 0$. In an open neighborhood of the 0 the function $F(z)$ can be developed into the series (2.6). This series, if $\limsup_{k\mapsto\infty} |a_k|^{\frac{1}{k}} < 1$, converges in a disc of radius $R > 1$. Therefore $\Gamma \subset U \subset D(0, R)$, for some open neighborhood $U$ of $\Gamma$. Hence the singularities disappear and $F(z) = 0$. It means that in this case the functions $F_+$, $F_-$ are analytic continuations of each other and $F_+ \equiv F_- \equiv 0$. Hence $F_+ - F_-|_\Gamma = f = 0$ almost everywhere. □

Remark 2.1. The conditions (2.7), (2.8) first appeared in the work of the first author (1990, 1992) as a condition on analytic continuation of a function $f \in C(\Gamma)$. The corresponding references can be found in [1], [2].

One can consider now the Carleman type integral

$$\lim_{m\mapsto\infty} \frac{1}{2\pi i}\int_\Gamma f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z}, \tag{2.9}$$

where $f \in L^1(\Gamma)$. Next, we state the following

Corollary 2.2. *Let the limit in* (2.9) *exists for a sequence* $\{z_j\}$ *of points in* $\mathcal{D}_1$, *such that* $\lim_{j\mapsto\infty} |z_j| = 1$, *then* $f$ *has analytic continuation into* $\mathcal{D}_1$ *as* $\mathcal{H}^1(\mathcal{D}_\tau)$-*function for all* $\tau$, $\rho < \tau < 1$.

Proof. If the limit (2.9) exists for a sequence $\{z_j\}$, then, as is pointed out in the proof of Theorem 2.2, the series (2.6) converges at these points $\{z_j\}$. Hence its radius of converges is greater than or equal to 1. Therefore (2.7) holds. □

Theorem 2.3. *Let* $f$ *be a function holomorphic in* $\mathcal{D}_1$, *with its boundary values belonging to the space* $L^p(\Gamma)$ *for some* $1 < p < \infty$. *If* $f$ *is representable by* (2.2) *then* $f \in \mathcal{H}^p(\mathcal{D}_\tau)$ *for all* $\tau$, $\rho < \tau < 1$.

Proof. The proof essentially repeats the arguments in the part 2) of the previous theorem. But in the present case we obtain the fact that $F_+ \in \mathcal{H}^p(\mathcal{D}_\tau)$ for all $\tau$, $\rho < \tau < 1$. Therefore one is able to conclude that $f = F_+ - F_- \in \mathcal{H}^p(\mathcal{D}_\tau)$. This last step is based upon the fact that for the domains with Ahlfors-regular boundary the Cauchy type integral of a function from the space $L^p$, $1 < p < \infty$, is a function that belongs to the class $\mathcal{H}^p$, [7], [8], [9]. □

As an application of this theorem we have two corollaries, similar to Corollaries 2.1 and 2.2.

COROLLARY 2.3. *If* $f \in L^p(\Gamma)$ *for some* $1 < p < \infty$ *and the relation* (2.7) *holds, where* $a_k$ *are given by* (2.5)*, then the function* $f$ *has analytic continuation into the domain* $\mathcal{D}_1$ *as an* $\mathcal{H}^p(\mathcal{D}_\tau)$*-function for every* $\tau \in (\rho, 1)$.

COROLLARY 2.4. *Let* $f \in L^p(\Gamma)$ (*with* $1 < p < \infty$) *and a sequence* $\{z_j\} \subset \mathcal{D}_1$ *be such that* $\lim_{j\mapsto\infty} |z_j| = 1$ *and the limit* (2.9) *exists for all* $j$. *Then the function* $f$ *has analytic continuation into the domain* $\mathcal{D}_1$ *as an* $\mathcal{H}^p(\mathcal{D}_\tau)$*-function for every* $\tau \in (\rho, 1)$.

In order to illustrate the main points of the above theorems we conclude the present section with the following

EXAMPLE 2.1. We want to construct a function which satisfies the condition 1) of Theorem 2.2, and therefore is representable by Carleman formula (2.2), but is not representable by the Cauchy formula. The reason is that the function under construction will not belong to the Hardy class $\mathcal{H}^p$, for any $p > 0$.

Consider the curve

$$\Gamma = \left\{z : z = x + iy,\ x = \frac{1}{2},\ -\frac{1}{2} \le y < \frac{\sqrt{3}}{2}\right\}$$
$$\cup \left\{z :\ \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}, \frac{1}{2} \le x < 1,\ y < 0\right\},$$

that is the curve $\Gamma$ is the union of a vertical segment and of an arc of a circle tangent to the unit circle from the inside. Consider the function

$$f(z) = e^{\frac{1+z}{1-z}}.$$

Then $f \notin \mathcal{H}^\delta(K(0,1))$, for all $\delta > 0$ [16], since the modulus $|f(z)|$ grows faster than it should in order to be in the class $\mathcal{H}^\delta$, whenever $z \mapsto 1^-$. For the same reason this function does not belong to any class $\mathcal{H}^\delta(\mathcal{D}_1)$ and therefore is not representable by Cauchy formula. On the other hand $f(z) \in L^\infty(\Gamma)$, since it is holomorphic on the vertical part of $\Gamma$ and on the arc of $\{z :\ |z - \frac{1}{2}| = \frac{1}{2}\}$ we have that $|f(z)| = e$. Furthermore, in any domain $\mathcal{D}_\tau$, $\rho < \tau < 1$, this function belongs to the class $\mathcal{H}^1(\mathcal{D}_\tau)$ since it is even holomorphic in $\overline{\mathcal{D}}_\tau$. Hence $f(z)$ is representable by Carleman formula (2.2).

## 3. – The case of several complex variables

Let $\Omega = \{z \in \mathbf{C}^n :\ \varrho(z) < 0\}$ be a convex, bounded Reinhardt domain with a boundary of class $\mathcal{C}^2$. Consider a strictly convex hypersurface $\Gamma$ intersecting

$\Omega$ and cutting from it the domain $\mathcal{D} = \{z \in \mathbf{C}^n : \varrho_0(z) < 0\}$ with a smooth boundary $\partial\mathcal{D}$ and such that on the $\Gamma \cap \partial\Omega$ the functions $\varrho$ and $\varrho_0$ coincide up to the second order derivatives. We assume also that $0 \notin \overline{\mathcal{D}}$. The Cauchy-Fantappiè kernel is given by

$$\omega(\zeta - z, w) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{k=1}^{n} (-1)^{k-1} w_k dw[k] \wedge d\zeta}{\langle w, \zeta - z\rangle^n},$$

where $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, $dw[k] = dw_1 \wedge \cdots \wedge dw_{k-1} \wedge dw_{k+1} \wedge \cdots \wedge dw_n$, $\langle w, z\rangle = \sum_{i=1}^{n} w_i z_i$. Then for every function $f$ holomorphic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$ the following Carleman formula is valid [1], [2],

$$\begin{aligned} f(z) = \lim_{m \mapsto \infty} \Bigg[ & \int_\Gamma f(\zeta)\omega(\zeta - z, \varrho_0') \\ & - \sum_{k=0}^{m} \frac{(k+n-1)!}{(n-1)!k!} \int_\Gamma f(\zeta) < \frac{\varrho'(\zeta)}{\langle \varrho'(\zeta), \zeta >}, z\rangle^k \omega(\zeta, \varrho') \Bigg], \end{aligned} \tag{3.1}$$

where $\varrho' = grad_\zeta \varrho = (\varrho'_{\zeta_1}, \dots, \varrho'_{\zeta_n})$, $\varrho_0' = grad_\zeta \varrho_0$.

For small enough positive number $\tau$ let also $\Omega_\tau = \{z \in \mathbf{C}^n : \varrho(z) < -\tau\}$ be a convex Reinhardt domain. Then we obtain the domains $\mathcal{D}_\tau = \mathcal{D} \cap \Omega_\tau$. Define $\rho = \inf_\tau\{\tau : \mathcal{D}_\tau \neq \emptyset\}$.

The Hardy class $\mathcal{H}^p(\mathcal{D})$ consists of such functions $f$ holomorphic in $\mathcal{D}$ for which

$$\limsup_{\epsilon \mapsto 0} \int_{\partial\mathcal{D}} |f(\zeta - \epsilon\nu_\zeta)|^p d\sigma_\zeta < \infty,$$

where $\nu_\zeta$ is the unit vector of the exterior normal to $\partial\mathcal{D}$ at the point $\zeta$ and $d\sigma$ is the area element of the Lebesgue measure, $0 < p < \infty$.

Before proceeding any further we give the following

DEFINITION 3.1. Denote by $A(E)$ the complete Reinhardt envelope of the set $E$, that is, the smallest complete Reinhardt domain containing the set $E$.

Now we are ready to formulate the next result.

THEOREM 3.1. *Let $f$ be a function holomorphic in the domain $\mathcal{D}$ with the property that its boundary values exist on $\Gamma$ almost everywhere and belong to the class $L^1(\Gamma)$.*

1) *If $f \in \mathcal{H}^1(\mathcal{D}_{\tau_k})$ for some sequence $\{\tau_k\}$, $\rho < \tau_k < 1$, $\lim_{k \mapsto \infty} \tau_k = 1$, then $f$ is representable by the formula* (3.1).
2) *Consider the domain $\Omega = A(\mathcal{D})$. If $f$ is representable by the formula* (3.1) *then $f$ belongs to $\mathcal{H}^1(\mathcal{D}_\tau)$ for all $\tau$, $\rho < \tau < 1$.*

PROOF. 1) If $z$ is a fixed point of the domain $\mathcal{D}$ then $z \in \mathcal{D}_{\tau_k}$ for sufficiently large $k$. By Cauchy-Fantappie formula [1], [2], [4]

$$\begin{aligned} f(z) = \int_{\Gamma_{\tau_k} \cap \partial\Omega_{\tau_k}} f(\zeta) R(z,\zeta) + \int_{\Gamma_{\tau_k}} f(\zeta)\omega(\zeta - z, \varrho_0') \\ + \int_{\partial\mathcal{D}_{\tau_k} \setminus \Gamma_{\tau_k}} f(\zeta)\omega(\zeta - z, \varrho'), \end{aligned} \tag{3.2}$$

where $\Gamma_\tau = \Gamma \cap \mathcal{D}_\tau$ and the kernel $R(z,\zeta)$ is a differential form depending on $\varrho$, $\varrho_0$ and their first derivatives. The form $R(z,\zeta)$ is equal to zero on the faces of integration if the functions $\varrho, \varrho_0$ and their first derivatives coincide there.

A convex Reinhardt domain is also linearly convex, that is, the analytic tangent plane $\{z : < \varrho', \zeta - z >= 0\}$, where $\zeta \in \partial\Omega_\tau \cap \mathcal{D}$, does not intersect $\Omega_\tau$. In other words for $\zeta \in \partial\Omega_\tau \cap \mathcal{D}$, $z \in \Omega_\tau$ the relation $< \varrho', \zeta - z > \neq 0$ holds or equivalently

$$\frac{\langle \varrho', z\rangle}{\langle \varrho', \zeta\rangle} \neq 1. \tag{3.3}$$

Moreover, if $z \in \Omega_\tau$, then $ze^{it} \in \Omega_\tau$, for $0 \le t \le 2\pi$. It follows from (3.3) that

$$\left|\frac{\langle \varrho', z\rangle}{\langle \varrho', \zeta\rangle}\right| \neq 1.$$

This and the fact that $0 \in \Omega_\tau$ imply that

$$\left|\left\langle \frac{\varrho'(\zeta)}{\langle \varrho', \zeta\rangle}, z\right\rangle\right| < 1, \; z \in \Omega_\tau, \; \zeta \in \partial\Omega_\tau \cap \mathcal{D}.$$

Consequently the kernel of the third integral in (3.2) has a series expansion for $z \in \mathcal{D}_{\tau_k}$, $\zeta \in \partial\Omega_{\tau_k} \cap \mathcal{D} = \partial\mathcal{D}_{\tau_k} \setminus \Gamma_{\tau_k}$ as follows

$$\omega(\zeta - z, \varrho') = \sum_{l=0}^{\infty} \frac{(l+n-1)!}{l!(n-1)!} \left\langle \frac{\varrho'(\zeta)}{\langle \varrho'(\zeta), \zeta\rangle}, z\right\rangle^l \omega(\zeta, \varrho'),$$

which converges uniformly with respect to $\zeta \in \partial\mathcal{D}_{\tau_k} \setminus \Gamma_{\tau_k}$ for fixed $z \in \mathcal{D}_{\tau_k}$. The form

$$\phi_l = \left\langle \frac{\varrho'(\zeta)}{\langle \varrho'(\zeta), \zeta\rangle}, z\right\rangle^l \omega(\zeta, \varrho')$$

is of class $\mathcal{Z}^1_{n,n-1}(\overline{\mathcal{D}})$ (the class of closed exterior differential forms with $C^1$ coefficients and of type $(n, n-1)$) for every $l$. Hence this form is orthogonal to holomorphic functions when integrating over $\partial\mathcal{D}_{\tau_k}$, that is

$$\begin{aligned} 0 = \frac{1}{(n-1)!}\Bigg[\int_{\Gamma_{\tau_k}} f(\zeta) \sum_{l=0}^{m} \frac{(l+n-1)!}{l!}\phi_l \\ + \int_{\partial\mathcal{D}_{\tau_k} \setminus \Gamma_{\tau_k}} f(\zeta) \sum_{l=0}^{m} \frac{(l+n-1)!}{l!}\phi_l\Bigg]. \end{aligned} \tag{3.4}$$

Substracting the equality (3.4) from the equality (3.2) and passing to the limit as $m \mapsto \infty$, we observe that the second integral in the obtained equality tends to zero. This implies that

$$f(z) = \lim_{m\mapsto\infty} \int_{\Gamma_{\tau_k}} f(\zeta) \left[\omega(\zeta - z, \varrho_0') - \frac{1}{(n-1)!}\sum_{l=0}^{m} \frac{(l+n-1)!}{l!}\phi_l\right] + \int_{\Gamma_{\tau_k}\cap\partial\Omega_{\tau_k}} f(\zeta) R(\zeta, z).$$

Furthermore, we have

$$f(z) = \lim_{k\mapsto\infty}\lim_{m\mapsto\infty} \left[\int_{\Gamma_{\tau_k}} f(\zeta) \left[\omega(\zeta - z, \varrho_0') - \frac{1}{(n-1)!}\sum_{l=0}^{m} \frac{(l+n-1)!}{l!}\phi_l\right] + \int_{\Gamma_{\tau_k}\cap\partial\Omega_{\tau_k}} f(\zeta) R(\zeta, z)\right]. \tag{3.5}$$

Uniform convergence in (3.5) implies that we can interchange the order of limits and hence obtain the desired formula (3.1), since on the face $\Gamma \cap \Omega$ the kernel $R(z, \zeta)$ tends to 0.

2) At this stage of the proof we recall the standard multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$. For the Reinhardt domain $\Omega$ we can write the formula (3.1) in a different form. We use instead of the sequence of partial sums of series of homogeneous polynomials the sequence of partial sums of the corresponding power series and get the following

$$f(z) = \lim_{m\mapsto\infty} \int_{\Gamma} f(\zeta) \left[\omega(\zeta - z, \varrho_0') - \sum_{\alpha=0}^{m} \frac{(|\alpha|+n-1)!}{(n-1)!\alpha!} \mathcal{G}_\alpha(z, \zeta)\omega(\zeta, \varrho')\right], \tag{3.6}$$

where $\mathcal{G}_\alpha(z, \zeta) = (\frac{\varrho'(\zeta)}{\langle \varrho'(\zeta), \zeta\rangle})^\alpha z^\alpha$. Hence the representation of the function $f(z)$ by the Carleman formula (3.1) implies the convergence in the domain $\mathcal{D}$ of the power series

$$\sum_{|\alpha|\geq 0} c_\alpha z^\alpha, \tag{3.7}$$

where

$$c_\alpha = \frac{(|\alpha|+n-1)!}{(n-1)!\alpha!} \int_{\Gamma} f(\zeta) \left(\frac{\varrho'(\zeta)}{\langle \varrho'(\zeta), \zeta\rangle}\right)^\alpha \omega(\zeta, \varrho'). \tag{3.8}$$

Since the complete Reinhardt envelope of $\mathcal{D}$ is $\Omega$ the series (3.7) converges in the domain $\Omega$ also. We denote this sum by $F_-(z)$ and by $F_+(z)$ the integral

$$\int_{\Gamma} f(\zeta)\omega(\zeta - z, \varrho_0'). \tag{3.9}$$

Then $f(z) = F_+(z) - F_-(z)$ for all $z \in \mathcal{D}$. Since $\Gamma$ is strictly convex hypersurface of order $\mathcal{C}^2$ then $F_+$ as an integral of Cauchy-Fantappiè type belongs to the Hardy class $\mathcal{H}^p(\mathcal{D})$ for every $0 < p < 1$. This result for the unit sphere can be found in [17], but it also holds in our case [18]. On the other hand for functions from the space $\mathcal{H}^p(\mathcal{D})$ we have the following estimate

$$|f(z)| \leq \frac{C}{d(z, \partial\mathcal{D})^{\frac{n}{p}}} \tag{3.10}$$

The estimate (3.10) is proven for the unit ball in [17], but, as it is pointed out there, it can be extended to strictly pseudoconvex domains together with the majority of the other results (see, for example, [14]). The constant $C$ in (3.10) depends on the function $f$. From (3.10) we infer that the integrals

$$\int_{\partial\mathcal{D}_\tau \setminus \Gamma_\tau} |f|^p d\sigma \leq C_1 , \tag{3.11}$$

where $d\sigma$ is an area element corresponding to the hypersurface (we consider the case when $n > 1$, and the hypersurface of integration in (3.11) has dimension $2n - 1$). The same result follows easily in a different manner from the multidimensional version of the Carleson theorem [15]. In addition, since $f \in L^p(\Gamma)$ and $f \in \mathcal{H}^p(\mathcal{D})$, for a sequence of hypersurfaces $\Gamma_\tau^\epsilon$ from $\mathcal{D}_\tau$, where

$$\Gamma_\tau^\epsilon = \{z : z \in \mathcal{D}_\tau,\ d(z, \Gamma_\tau) = \epsilon\}$$

the following will hold

$$\lim_{\epsilon \mapsto 0} \int_{\Gamma_\tau^\epsilon} |f|^p d\sigma = \int_{\Gamma_\tau} |f|^p d\sigma .$$

Therefore $f \in \mathcal{H}^p(\mathcal{D}_\tau)$ for any $\tau,\ \rho < \tau < 1$. On the other hand (3.11), the fact that $f \in L^1(\Gamma_\tau)$ and the multidimensional analogue of Smirnov's theorem (look at the next Lemma 3.1) imply that $f \in \mathcal{H}^1(\mathcal{D}_\tau)$ for any $\tau,\ \rho < \tau < 1$. □

In order to complete the proof of the last theorem we state and prove a multidimensional variation of Smirnov's theorem. In the next lemma $\mathcal{D}$ is a Liapunov domain in $\mathbf{C}^n$, that is at any point $\zeta \in \partial\mathcal{D}$ there exists the exterior normal and the unit normal is continuous vector-function satisfying the Hölder condition.

LEMMA 3.1. *If $f \in \mathcal{H}^p(\mathcal{D})$ and its boundary values belong to the space $L^q(\partial\mathcal{D})$, where $q > p$ then $f \in \mathcal{H}^q(\mathcal{D})$.*

PROOF. It was pointed out in [3], Prop.0.8 in [1] that if we consider the family of complex lines $\{\beta\}$ passing through a common fixed point or a family of parallel complex lines then $f \in \mathcal{H}^p(\mathcal{D})$ if and only if the following two conditions are satisfied

1) $f \in \mathcal{H}^p(\beta \cap \mathcal{D})$ for almost all lines $\beta$ of the family.
2) The boundary values of the function $f$ belong to the space $L^p(\partial\mathcal{D})$.

If $f \in L^q(\partial\mathcal{D})$, then by Fubini's theorem we have that $f \in L^q(\beta \cap \partial\mathcal{D})$ for almost all $\beta$. Thus, by Smirnov's theorem, we deduce that $f \in \mathcal{H}^q(\beta \cap \mathcal{D})$. From the above equivalence condition it follows that $f \in \mathcal{H}^q(\mathcal{D})$. □

REMARK 3.1. It is possible that the last lemma is already a known result. However, we were not able to locate the suitable reference.

Using the standard multi-index notation we state the following

COROLLARY 3.1. *If a function $f$ is holomorphic in $\mathcal{D}$ and has boundary values on $\Gamma$ which belong to the class $L^1(\Gamma)$, then*

1) *If $f \in \mathcal{H}^1(\mathcal{D}_{\tau_k})$ for some sequence $\{\tau_k\}_k$, $\rho < \tau_k < 1$, $\lim_{k\mapsto\infty} \tau_k = 1$, then*

$$\limsup_{|\alpha|\mapsto\infty} \left(|c_\alpha| d_\alpha(\Omega)\right)^{\frac{1}{|\alpha|}} \leq 1\,, \tag{3.12}$$

*where $\Omega = A(\mathcal{D})$, $d_\alpha(\Omega) = max_{\overline{\Omega}}|z|^\alpha$ and $c_\alpha$ are defined by* (3.8).

2) *If $f$ is a CR function on $\Gamma$ and* (3.12) *holds then $f$ is extendible into $\mathcal{D}$ as a $\mathcal{H}^1(\mathcal{D}_\tau)$-function for all $\tau$, $\rho < \tau < 1$.*

PROOF. 1) The proof goes as in the part 1) of Theorem 3.1. We get that the series (3.7) converges in $\mathcal{D}$, but then the same series convergs in $\Omega$, since the set $\Omega$ is the complete Reinhardt envelope of $\mathcal{D}$. Applying a theorem from [5] we get (3.12).

2) This part was essentially proven in [6], but there only the part about the analytic continuation was marked out. □

Consider now the Carleman type integral

$$\lim_{m\mapsto\infty} \int_\Gamma f(\zeta) \left[\omega(\zeta - z, \varrho_0') - \sum_{|\alpha|=0}^{m} \frac{(|\alpha| + n - 1)!}{(n-1)!\alpha!} \mathcal{G}_\alpha(z,\zeta)\omega(\zeta, \varrho')\right]\,, \tag{3.13}$$

with $f \in L^1(\Gamma)$, $\mathcal{G}_\alpha(z,\zeta) = (\frac{\varrho'(\zeta)}{\langle\varrho'(\zeta),\zeta\rangle})^\alpha z^\alpha$.

REMARK 3.2. The condition (3.12) first appeared in [6] as a condition on analytic continuation of a function $f \in C(\Gamma)$, however the multidimensional version of 3) from Corollary 2.1 is not true.

COROLLARY 3.2. *Let $f$ be a $CR$-function on $\Gamma$. Suppose that there exists a sequence of $\{z^j\} \subset \mathcal{D}$ such that $A(\{z^j\}) = \Omega$ and the limit in* (3.13) *exists for any $z = z^j$, $j = 1, 2\ldots$ Then $f$ can be extended into the domain $\mathcal{D}$ as a $\mathcal{H}^1(\mathcal{D}_\tau)$-function for all $\tau \in (\rho, 1)$.*

PROOF. The assumption implies that the series (3.7), where the coefficients $c_\alpha$ are given by (3.8) converges in all of the domain $\Omega$, and therefore (3.12) holds,[5]. The conclusion of the corollary now follows from the part 2) of the Cor. 3.1. □