

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 26,
n° 4 (1998), p. 719-735

http://www.numdam.org/item?id=ASNSP_1998_4_26_4_719_0

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On the Period Map for Abelian Covers of Projective Varieties

RITA PARDINI

Abstract. We show that infinitesimal Torelli for the period of n -forms holds for abelian covers of algebraic varieties of dimension $n \geq 2$, under some explicit ampleness assumptions on the building data of the cover. Moreover, we prove a variational Torelli result for some families of abelian covers.

Mathematics Subject Classification (1991): 14D07.

0. – Introduction

This paper is devoted to the study of the period map for abelian covers of smooth projective varieties of dimension $n \geq 2$. Our viewpoint is very close to that of Green in [9], namely we look for results that hold for abelian covers of an arbitrary variety whenever certain ampleness assumptions on the building data defining the cover are satisfied. We focus on two questions: the infinitesimal and the variational Torelli problems.

Infinitesimal Torelli for the periods of k -forms holds for a smooth projective variety X if the map $H^1(X, T_X) \rightarrow \bigoplus_p \text{Hom}(H^p(X, \Omega_X^{k-p}), H^{p+1}(X, \Omega_X^{k-p-1}))$, expressing the differential of the period map for k -forms, is injective. This is expected to be true as soon as the canonical bundle of X is “sufficiently ample”. There are many results in this direction, concerning special classes of varieties, as hypersurfaces (see, for instance, [9]), complete intersections ([7], [14]) and simple cyclic covers ([12], [15], [18], [20]). Here we continue the work on abelian covers of [17], and prove (see 4.1):

THEOREM 0.1. *Let G be an abelian group and let $f : X \rightarrow Y$ be a G -cover, with X, Y smooth projective varieties of dimension $n \geq 2$. If properties (A) and (B) of 3.1 are satisfied, then infinitesimal Torelli for the periods of n -forms holds for X .*

Properties (A) and (B) amount to the vanishing of certain cohomology groups and are certainly satisfied if the building data of the cover are sufficiently

ample. If Y is a special variety (e.g., $Y = \mathbb{P}^n$), then Theorem 0.1 yields an almost sharp statement (see Theorem 4.2). In general, a result of Ein-Lazarsfeld ([5]) and Griffiths vanishing theorem enable us (see Proposition 3.5) to give explicit conditions under which (A) and (B) are satisfied, and thus to deduce an effective statement from Theorem 0.1 (see Theorem 4.3).

In order to extend to the case of arbitrary varieties the infinitesimal Torelli theorem obtained in [17] for a special class of surfaces, we introduce a generalized notion of prolongation bundle and give a Jacobi ring construction analogous to those of [9] and [13]. This is also a starting point for attacking the variational Torelli problem, which asks whether, given a flat family $\mathcal{X} \rightarrow B$ of smooth polarized varieties, the map associating to a point $b \in B$ the infinitesimal variation of Hodge structure of the fibre X_b is generically injective, up to isomorphism of polarized varieties. A positive answer to this problem has been given for families of projective hypersurfaces ([2],[3]), for hypersurfaces of high degree of arbitrary varieties ([9]), for some complete intersections ([13]) and for simple cyclic covers of high degree ([11]). The most effective tool in handling these problems is the symmetrizer, introduced by Donagi, but unfortunately, an analogous construction does not seem feasible in the case of abelian covers. However, exploiting the variational Torelli result of [9], we are able to obtain, under analogous assumptions, a similar result for a large class of abelian covers. More precisely, we prove (see Theorem 6.1):

THEOREM 0.2 (Notation as in Section 1). *Let Y be a smooth projective variety of dimension $n \geq 2$, with very ample canonical class. Let G be a finite abelian group and let $f : X \rightarrow Y$ be a smooth G -cover with sufficiently ample building data L_χ , D_i , $\chi \in G^*$, $i = 1, \dots, r$. Assume that for every $i = 1, \dots, r$ the identity is the only automorphism of Y that preserves the linear equivalence class of D_i ; moreover, assume that for $i = 1, \dots, r$ there exist a $\chi \in G^*$ (possibly depending on i) such that $\chi(g_i) \neq 1$ and $L_\chi(-D_i)$ is ample. Let $\mathcal{X} \rightarrow \tilde{W}$ be the family of the smooth G -covers of Y obtained by letting the D_i 's vary in their linear equivalence classes: then there is a dense open set $V \subset \tilde{W}$ such that the fibre X_s of \mathcal{X} over $s \in V$ is determined by its IVHS for n -forms plus the natural G -action on it.*

The paper is organized as follows: Section 1 is a brief review of abelian covers, Sections 2 and 3 contain the technical details about prolongation bundles and the Jacobi ring construction, Section 4 contains the statements and proofs of the results on infinitesimal Torelli, and Section 5 contains some technical lemmas that allow us to prove in Section 6 the variational Torelli Theorem 2.

NOTATION AND CONVENTIONS. All varieties are smooth projective varieties of dimension $n \geq 2$ over the field \mathbb{C} of complex numbers. We do not distinguish between vector bundles and locally free sheaves; as a rule, we use the additive notation for divisors and the multiplicative notation for line bundles. Linear equivalence is denoted by \equiv . For a divisor D , $c_1(D)$ denotes the first Chern class of D and $|D|$ the complete linear system of D . If L is a line bundle, we also denote by $|L|$ the complete linear system of L , and we write L^k for $L^{\otimes k}$ and L^{-1} for the dual line bundle. As usual, T_Y denotes the tangent sheaf of Y ,

Ω_Y^k denotes the sheaf of regular k -forms on Y , $\omega_Y = \Omega_Y^n$ denotes the canonical bundle and $\text{Pic}(Y)$ the Picard group. If \mathcal{F} is a locally free sheaf, we denote by \mathcal{F}^* the dual sheaf, by $S^k \mathcal{F}$ the k -th symmetric power of \mathcal{F} and by $\det \mathcal{F}$ the determinant bundle. Consistently, the dual of a vector space U is denoted by U^* ; the group of linear automorphisms of U is denoted by $GL(U)$.

$[x]$ denotes the integral part of the natural number x .

ACKNOWLEDGEMENTS. I wish to thank Mark Green for communicating me the proof of Lemma 5.2.

1. – Abelian covers and projection formulas

In this section we recall some facts about abelian covers that will be needed later. For more details and proofs, see [16].

Let G be a finite abelian group of order m and let $G^* = \text{Hom}(G, \mathbb{C}^*)$ be the group of characters of G . A G -cover of a smooth n -dimensional variety Y is a Galois cover $f : X \rightarrow Y$ with Galois group G , with X normal. Let \mathcal{F} be a G -linearized locally free sheaf of \mathcal{O}_X -modules: under the action of G , the sheaf $f_* \mathcal{F}$ splits as the direct sum of the eigensheaves corresponding to the characters of G . We denote by $(f_* \mathcal{F})^{(\chi)}$ the eigensheaf corresponding to a character $\chi \in G^* \setminus \{1\}$ and by $(f_* \mathcal{F})^{inv}$ the invariant subsheaf. In particular, when $\mathcal{F} = \mathcal{O}_X$, we have $(f_* \mathcal{O}_X)^{inv} = \mathcal{O}_Y$ and $(f_* \mathcal{O}_X)^{(\chi)} = L_\chi^{-1}$, with L_χ a line bundle. Let D_1, \dots, D_r be the irreducible components of the branch locus D of f . For each index i , the subgroup of G consisting of the elements that fix the inverse image of D_i pointwise is a cyclic group H_i , the so-called *inertia subgroup* of D_i . The order m_i of H_i is equal to the order of ramification of f over D_i and the representation of H_i obtained by taking differentials and restricting to the normal space to D_i is a faithful character χ_i . The choice of a primitive m -th root ζ of 1 defines a map from $\{1, \dots, r\}$ to G : the image g_i of i is the generator of H_i that is mapped to ζ^{m/m_i} by χ_i . The line bundles L_χ , $\chi \in G^* \setminus \{1\}$, and the divisors D_i , each “labelled” with an element g_i of G as explained above, are the *building data* of the cover, and determine $f : X \rightarrow Y$ up to isomorphism commuting with the covering maps. The building data satisfy the so-called *fundamental relations*. In order to write these down, we have to set some notation. For $i = 1, \dots, r$ and $\chi \in G^*$, we denote by a_χ^i the smallest non-negative integer such that $\chi(g_i) = \zeta^{ma_\chi^i/m_i}$; for each pair of characters χ, ϕ we set $\epsilon_{\chi, \phi}^i = [(a_\chi^i + a_\phi^i)/m_i]$ (notice that $\epsilon_{\chi, \phi}^i = 0$ or 1) and $D_{\chi, \phi} = \sum_{i=1}^r \epsilon_{\chi, \phi}^i D_i$. In particular, $D_{\chi, \chi^{-1}}$ is the sum of the components D_i of D such that $\chi(g_i) \neq 1$. Then, the fundamental relations of the cover are the following:

$$(1.1) \quad L_\chi + L_\phi \equiv L_{\chi\phi} + D_{\chi, \phi}, \quad \forall \chi, \phi \in G^*$$

When $\phi = \chi^{-1}$, the fundamental relations read:

$$(1.2) \quad L_\chi + L_{\chi^{-1}} \equiv D_{\chi, \chi^{-1}}.$$

The cover $f : X \rightarrow Y$ can be reconstructed from the building data as follows: if one chooses sections s_i of $\mathcal{O}_Y(D_i)$ vanishing on D_i for $i = 1, \dots, r$, then X is defined inside the vector bundle $V = \bigoplus_{\chi \neq 1} L_\chi$ by the equations:

$$(1.3) \quad z_\chi z_\phi = \left(\prod_i s_i^{\epsilon_{\chi, \phi}^i} \right) z_\chi z_\phi, \quad \forall \chi, \phi \in G^* \setminus \{1\}$$

where z_χ denotes the tautological section of the pull-back of L_χ to V . Conversely, for every choice of the sections s_i , (eq. 1.3) define a scheme X , flat over Y , which is smooth iff the zero divisors of the s_i 's are smooth, their union has only normal crossings singularities and, whenever s_{i_1}, \dots, s_{i_t} all vanish at a point y of Y , the group $H_{i_1} \times \dots \times H_{i_t}$ injects into G . So, by letting s_i vary in $H^0(Y, \mathcal{O}_Y(D_i))$, one obtains a flat family \mathcal{X} of smooth G -covers of X , parametrized by an open set $W \subset \bigoplus_i H^0(Y, \mathcal{O}_Y(D_i))$.

Throughout all the paper we will make the following

ASSUMPTION 1.1. *The G -cover $f : X \rightarrow Y$ is smooth of dimension $n \geq 2$; the building data L_χ, D_i and the adjoint bundles $\omega_Y \otimes L_\chi, \omega_Y(D_i)$ are ample for every $\chi \in G^* \setminus \{1\}$ and for every $I = 1, \dots, r$.*

Assumption 1.1 implies that the cover is *totally ramified*, namely that g_1, \dots, g_r generate G . Actually, this is equivalent to the fact that the divisor $D_{\chi, \chi^{-1}}$ is nonempty if the character χ is nontrivial, and also to the fact that none of the line bundles $L_\chi, \chi \in G^* \setminus \{1\}$, is a torsion point in $\text{Pic}(Y)$. Since X is smooth, Assumption 1.1 implies in particular that for each subset $\{i_1, \dots, i_t\} \subset \{1, \dots, r\}$, with $t \leq n$, the cyclic subgroups generated by g_{i_1}, \dots, g_{i_t} give a direct sum inside G .

In principle, all the geometry of X can be recovered from the geometry of Y and from the building data of $f : X \rightarrow Y$. The following proposition is an instance of this philosophy.

PROPOSITION 1.2. *Let $f : X \rightarrow Y$ be a G -cover, with X, Y smooth of dimension n . For $\chi \in G^*$, denote by Δ_χ the sum of the components D_i of D such that $a_i^\chi \neq m_i - 1$. Then, for $1 \leq k \leq n$ there are natural isomorphisms:*

$$\begin{aligned} (f_* \Omega_X^k)^{(x)} &= \Omega_Y^k(\log D_{\chi, \chi^{-1}}) \otimes L_\chi^{-1} \\ (f_* T_X)^{(x)} &= T_Y(-\log \Delta_\chi) \otimes L_\chi^{-1} \end{aligned}$$

and, in particular:

$$(f_* \Omega_X^k)^{inv} = \Omega_Y^k, \quad (f_* T_X)^{inv} = T_Y(-\log D), \quad (f_* \omega_X)^{(x)} = \omega_Y \otimes L_{\chi^{-1}}.$$

PROOF. This is a slight generalization of Proposition 4.1 of [16], and it can be proven along the same lines. The identification $(f_* \omega_X)^{(x)} = \omega_Y \otimes L_{\chi^{-1}}$ follows from the general formula and relations (1.2). \square

We recall the following generalized form of Kodaira vanishing (see [4], page 56):

THEOREM 1.3. *Let Y be a smooth projective n -dimensional variety and let L be an ample line bundle. Then:*

$$H^i(Y, \Omega_Y^k \otimes L^{-1}) = 0, \quad i + k < n.$$

Moreover, if $A + B$ is a reduced effective normal crossing divisor, then:

$$H^i(Y, \Omega_Y^k(\log(A + B)) \otimes L^{-1}(-B)) = 0, \quad i + k < n.$$

From Proposition 1.2, Theorem 1.3 and Assumption 1.1 it follows that the non-invariant part of the cohomology of X is concentrated in dimension n . Thus we will be concerned only with the period map for the periods of n -forms.

2. – Logarithmic forms and sheaf resolutions

In this section we recall the definition and some properties of logarithmic forms and introduce a generalized notion of prolongation bundle. We would like to mention that Konno, when studying in [13] the global Torelli problem for complete intersections, has also introduced a generalization of the definition of prolongation bundle, which is however different from the one used here.

Let D be a normal crossing divisor with smooth components D_1, \dots, D_r on the smooth n -dimensional variety Y . As usual, we denote by $\Omega_Y^k(\log D)$ the sheaf of k -forms with at most logarithmic poles along D_1, \dots, D_r and by $T_Y(-\log D)$ the subsheaf of T_Y consisting of the vector fields tangent to the components of D . Assume that $y \in Y$ lies precisely on the components D_1, \dots, D_t of D , with $t \leq n$. Let x^1, \dots, x^t be local equations for D_1, \dots, D_t and choose x_{t+1}, \dots, x_n such that x_1, \dots, x_n are a set of parameters at y . Then $\frac{dx_1}{x_1}, \dots, \frac{dx_t}{x_t}, dx_{t+1}, \dots, dx_n$ are a set of free generators for $\Omega_Y^1(\log D)$ and $x_1 \frac{\partial}{\partial x_1}, \dots, x_t \frac{\partial}{\partial x_t}, \frac{\partial}{\partial x_{t+1}}, \dots, \frac{\partial}{\partial x_n}$ are free generators for $T_Y(-\log D)$ in a neighbourhood of y . So the sheaves of logarithmic forms are locally free and one has the following canonical identifications: $\Omega_Y^k(\log D) = \wedge^k \Omega_Y^1(\log D)$ and $T_Y(-\log D) = \Omega_Y^1(\log D)^*$, duality being given by contraction of tensors. Moreover, we recall that, if \mathcal{F} is a locally free sheaf of rank m on Y , then the alternation map $\wedge^i \mathcal{F} \otimes \wedge^{m-i} \mathcal{F} \rightarrow \det \mathcal{F}$ is a nondegenerate pairing, which induces a canonical isomorphism $(\wedge^i \mathcal{F})^* \rightarrow \wedge^{m-i} \mathcal{F} \otimes (\det \mathcal{F})^{-1}$. So we have:

$$(2.1) \quad \begin{aligned} T_Y(-\log D) &\cong \Omega_Y^{n-1}(\log D) \otimes (\omega_Y(D))^{-1} \\ \Omega_Y^k(\log D)^* &\cong \Omega_Y^{n-k}(\log D) \otimes (\omega_Y(D))^{-1} \end{aligned}$$

The (*generalized*) *prolongation bundle* P of (D_1, \dots, D_r) is defined as the extension $0 \rightarrow \Omega_Y^1 \rightarrow P \rightarrow \oplus^r \mathcal{O}_Y \rightarrow 0$ associated to the class $(c_1(D_1), \dots, c_1(D_r))$ of $H^1(Y, \oplus^r \Omega_Y^1)$. Let $\{U_\alpha\}$ be a finite affine covering of Y , let x_α^i be local

equations for D_i on U_α , $i = 1 \dots r$, and let $g_{\alpha\beta}^i = x_\alpha^i/x_\beta^i$. Denote by $e_\alpha^1, \dots, e_\alpha^r$ the standard basis of $\bigoplus^r \mathcal{O}_Y|_{U_\alpha}$.

The elements of $P|_{U_\alpha}$ are represented by pairs $(\sigma_\alpha, \sum_i z_\alpha^i e_i)$, where σ_α is a 1-form and the z_α^i 's are regular functions, satisfying the following transition relations on $U_\alpha \cap U_\beta$:

$$\left(\sigma_\alpha, \sum_i z_\alpha^i e_\alpha^i \right) = \left(\sigma_\beta + \sum_i z_\beta^i \frac{dg_{\alpha\beta}^i}{g_{\alpha\beta}^i}, \sum_i z_\beta^i e_\beta^i \right).$$

There is a natural short exact sequence:

$$(2.2) \quad 0 \rightarrow \bigoplus_i \mathcal{O}_Y(-D_i) \rightarrow P \rightarrow \Omega_Y^1(\log D) \rightarrow 0$$

with dual sequence:

$$(2.3) \quad 0 \rightarrow T_Y(-\log D) \rightarrow P^* \rightarrow \bigoplus_i \mathcal{O}_Y(D_i) \rightarrow 0.$$

In local coordinates the map $\bigoplus_i \mathcal{O}_Y(-D_i) \rightarrow P$ is defined by: $x_\alpha^i \mapsto (dx_\alpha^i, x_\alpha^i e_\alpha^i)$ and the map $P \rightarrow \Omega^1(\log D)$ is defined by: $(\sigma_\alpha, \sum_i z_\alpha^i e_\alpha^i) \mapsto \sigma_\alpha - \sum_i z_\alpha^i dx_\alpha^i/x_\alpha^i$.

We close this section by writing down a resolution of the sheaves of logarithmic forms that will be used in Section 3. Given an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of locally free sheaves, for any $k \geq 1$ one has the following long exact sequence (see [8], page 39):

$$(2.4) \quad 0 \rightarrow S^k A \rightarrow B \otimes S^{k-1} A \rightarrow \dots \rightarrow \wedge^{k-1} B \otimes A \rightarrow \wedge^k B \rightarrow \wedge^k C \rightarrow 0.$$

Applying this to (2.2) and setting $V = \bigoplus_i \mathcal{O}_Y(D_i)$ yields:

$$(2.5) \quad \begin{aligned} 0 &\rightarrow S^k V^* \rightarrow S^{k-1} V^* \otimes P \rightarrow \\ \dots &\rightarrow V^* \otimes \wedge^{k-1} P \rightarrow \wedge^k P \rightarrow \Omega_Y^k(\log D) \rightarrow 0 \end{aligned}$$

3. – The algebraic part of the IVHS

The aim of this section is to give, in the case of abelian covers, a construction analogous to the Jacobian ring construction for hypersurfaces of [9].

Let $\mathcal{X} \rightarrow B$ be a flat family of smooth projective varieties of dimension n and let X be the fibre of \mathcal{X} over the point $0 \in B$; the differential of the period map for the periods of k -forms for \mathcal{X} at 0 is the composition of the Kodaira-Spencer map with the following universal map, induced by cup-product:

$$(3.1) \quad H^1(X, T_X) \rightarrow \bigoplus_p \text{Hom}(H^p(X, \Omega_X^{k-p}), H^{p+1}(X, \Omega_X^{k-p-1})),$$

This map is called the *algebraic part of the infinitesimal variation of Hodge structure* of X (IVHS for short). Assume that $f : X \rightarrow Y$ is a smooth G -cover; then the G -action on the tangent sheaf and on the sheaves of differential forms is compatible with cup-product, so the map (3.1) splits as the direct sum of the maps

$$(3.2) \quad \rho_{\chi, \phi}^k : H^1(X, T_X)^{(\chi)} \rightarrow \bigoplus_p \text{Hom}(H^p(X, \Omega_X^{k-p})^{(\phi)}, H^{p+1}(X, \Omega_X^{k-p-1})^{(\chi\phi)})$$

As we have remarked at the end of Section 1, if $f : X \rightarrow Y$ satisfies the Assumption 1.1, then the non invariant part of the Hodge structure is concentrated in the middle dimension n , and so we will only describe the IVHS for $k = n$.

We use the notation of Section 1 and moreover, in order to keep formulas readable, we set:

$$T^{inv} = H^1(Y, T_Y(-\log D)); \quad U^{k, inv} = H^k(Y, \Omega_Y^{n-k})$$

$$U^{k, \chi} = H^k(Y, \Omega_Y^{n-k}(\log D_{\chi, \chi^{-1}}) \otimes L_{\chi}^{-1}), \quad k = 0, \dots, n, \quad \chi \in G^* \setminus \{1\}.$$

Given a character $\chi \in G^*$, let D_{i_1}, \dots, D_{i_s} be the components of $D_{\chi, \chi^{-1}}$; let P^{χ} be the generalized prolongation bundle of $(D_{i_1}, \dots, D_{i_s})$ (see Section 2) and let $V^{\chi} = \bigoplus_j \mathcal{O}_Y(D_{i_j})$.

Consider the map $(P^{\chi})^* \rightarrow V^{\chi}$ defined in sequence (2.3); tensoring this map with $S^{k-1}(V^{\chi})$ and composing with the symmetrization map $S^{k-1}(V^{\chi}) \otimes V^{\chi} \rightarrow S^k(V^{\chi})$, one obtains a map:

$$(3.3) \quad S^{k-1}(V^{\chi}) \otimes (P^{\chi})^* \rightarrow S^k(V^{\chi})$$

Given a line bundle L on Y , we define $R_L^{k, \chi}$ to be the cokernel of the map:

$$H^0(Y, S^{k-1}(V^{\chi}) \otimes (P^{\chi})^* \otimes L) \rightarrow H^0(Y, S^k(V^{\chi}) \otimes L),$$

obtained from (3.3) by tensoring with L and passing to global sections. We set $R_L^{\chi} = \bigoplus_{k \geq 0} R_L^{k, \chi}$; for $L = \mathcal{O}_Y$, $R^{\chi} = R_{\mathcal{O}_Y}^{\chi}$ is a graded ring and, in general R_L^{χ} is a module over R^{χ} , that we call the *Jacobi module* of L . Moreover, if L_1 and L_2 are line bundles on Y , then there is an obvious multiplicative structure:

$$R_{L_1}^{k, \chi} \otimes R_{L_2}^{h, \chi} \rightarrow R_{L_1 \otimes L_2}^{k+h, \chi}.$$

In order to establish the relationship between the Jacobi modules and the IVHS of the cover X , we need some definitions.

DEFINITION 3.1. For a G -cover $f : X \rightarrow Y$ satisfying Assumption 1.1, let Γ_f be the semigroup of $\text{Pic}(Y)$ generated by the building data. We say that: X has property (A) iff $H^k(Y, \Omega_Y^j \otimes L) = 0$ and $H^k(Y, \Omega_Y^j \otimes \omega_Y \otimes L) = 0$ for $k > 0$, $j \geq 0$, $L \in \Gamma_f \setminus \{0\}$;

X has property (B) iff for L_1, L_2 in $\Gamma_f \setminus \{0\}$ the multiplication map $H^0(Y, \omega_Y \otimes L_1) \otimes H^0(Y, \omega_Y \otimes L_2) \rightarrow H^0(Y, \omega_Y^2 \otimes L_1 \otimes L_2)$ is surjective.

REMARK 3.2. If M is an ample line bundle such that $H^k(Y, \Omega_Y^j \otimes M) = 0$ for $k > 0$ and $j \geq 0$, then the cohomology groups $H^k(Y, \wedge^j P \otimes M^{-1})$ vanish for $j \geq 0$ and $k < n$.

PROOF. By Serre duality, it is equivalent to show that $H^r(Y, \wedge^j P^* \otimes M \otimes \omega_Y) = 0$ for $r > 0$. In turn, this can be proven by induction on j , by looking at the hypercohomology of the complex obtained by applying (2.4) to the sequence $0 \rightarrow \oplus_i \mathcal{O}_Y \rightarrow P^* \rightarrow T_Y \rightarrow 0$. \square

We also introduce the following

NOTATION 3.3. Let L and M be line bundles on the smooth variety Y ; if $L \otimes M^{-1}$ is ample, then we write $L > M$ and, if $L \otimes M^{-1}$ is nef, then we write $L \geq M$. We use the same notation for divisors.

REMARK 3.4. Properties (A) and (B) are easily checked for coverings of certain varieties Y ; for instance, if $Y = \mathbb{P}^n$, then by Bott vanishing theorem it is enough to require that $\omega_Y \otimes L_\chi > 0$ and $\omega_Y(D_i) > 0$ for every $\chi \neq 1$ and for $i = 1, \dots, r$.

The next proposition yields an effective criterion for (A) and (B) in case Y is an arbitrary variety.

PROPOSITION 3.5. Let $f : X \rightarrow Y$ be a G -cover satisfying Assumption 1.1 and let E be a very ample divisor on Y . Define $c(n) = \binom{n-1}{(n-1)/2}$ if n is odd and $c(n) = \binom{n-1}{n/2}$ if n is even, and set $E_n = (\omega_Y(2nE))^{c(n)}$.

- i) if $D_i, L_\chi, \omega_Y(D_i), \omega_Y \otimes L_\chi > E_n$ for $\chi \neq 1$ and for $i = 1, \dots, r$, then (A) is satisfied.
- ii) if L_χ and $D_i \geq (n+1)E$ for $\chi \neq 1$ and for $i = 1, \dots, r$, then (B) is satisfied.

PROOF. The complete linear system $|E|$ embeds Y in a projective space \mathbb{P} ; since $\Omega_{\mathbb{P}}^j(j+1)$ is generated by global sections, the sheaf $W^j = \Omega_Y^j((j+1)E)$, being a quotient of the former bundle, is also generated by global sections. By Griffiths vanishing Theorem ([19], Theorem 5.52), if N is an ample line bundle, then the cohomology group $H^k(Y, W^j \otimes \omega_Y \otimes \det(W^j) \otimes N)$ vanishes for $k > 0$. We recall from adjunction theory that $\omega_Y((n+1)E)$ is base point free and therefore nef; using this fact, it is easy to check that $E_n \geq \det(W^j)$ for $j \geq 0$. Statement i) now follows immediately from Griffiths vanishing.

In order to prove ii), set $V_1 = H^0(Y, \omega_Y \otimes L_1)$. The assumptions imply that V_1 is base-point free, so evaluation of sections gives the following short exact sequence of locally free sheaves: $0 \rightarrow K_1 \rightarrow \mathcal{O}_Y \otimes V_1 \rightarrow \omega_Y \otimes L_1 \rightarrow 0$. Twisting with $\omega_Y \otimes L_2$ and passing to cohomology, one sees that the statement follows if $H^1(Y, K_1 \otimes \omega_Y \otimes L_2) = 0$. In turn, this is precisely case $k = q = 1$ of Theorem 2.1 of [5]. \square

LEMMA 3.6. *Let $f : X \rightarrow Y$ be a G -cover satisfying property (A) and let L be a line bundle on Y ; if L or $L \otimes \omega_Y^{-1}$ belong to $\Gamma_f \setminus \{0\}$, then for every $\chi \in G^* \setminus \{1\}$ there is a natural isomorphism:*

$$H^k \left(Y, \Omega_Y^{n-k} \left(\log D_{\chi, \chi^{-1}} \right) \otimes L^{-1} \right) \cong \left(R_{\omega_Y \otimes L}^{n-k, \chi} \right)^*$$

In particular, there are natural isomorphisms:

$$T^{inv} \cong \left(R_{\omega_Y^2(D)}^{n-1, 1} \right)^*; \quad U^{k, \chi} \cong \left(R_{\omega_Y \otimes L}^{n-k, \chi} \right)^*, \quad \chi \neq 1$$

$$H^1 \left(Y, \Omega_Y^{n-1} \left(\log D_{\chi, \chi^{-1}} \right) \otimes (L_\chi \otimes L_{\phi^{-1}} \otimes \omega_Y)^{-1} \right) \cong \left(R_{\omega_Y^2 \otimes L_\chi \otimes L_{\phi^{-1}}}^{n-1, \chi} \right)^*, \quad \chi \neq 1.$$

PROOF. For $k = n$, the statement is just Serre duality. For $k < n$, we compute $H^k \left(Y, \Omega_Y^{n-k} \left(\log D_{\chi, \chi^{-1}} \right) \otimes L^{-1} \right)$ by tensoring the resolution (2.5) of the sheaf $\Omega_Y^{n-k} \left(\log D_{\chi, \chi^{-1}} \right)$ with L^{-1} and breaking up the resolution thus obtained into short exact sequences. Remark 3.2 and Theorem 1.3 imply that the cohomology groups $H^{n-k+j} \left(Y, S^j(V_\chi)^* \otimes L^{-1} \otimes \wedge^{n-k-j} P \right)$ vanish for $0 \leq j < n - k$; thus the group $H^k \left(Y, \Omega_Y^{n-k} \left(\log D_{\chi, \chi^{-1}} \right) \otimes L^{-1} \right)$ and the kernel of the map $H^n \left(Y, S^{n-k}(V_\chi)^* \otimes L^{-1} \right) \rightarrow H^n \left(Y, S^{n-k-1}(V_\chi)^* \otimes P^\chi \otimes L^{-1} \right)$ are naturally isomorphic. By Serre duality, the latter group is dual to $R_{\omega_Y \otimes L}^{n-k, \chi}$. \square

The next result is the analogue in our setting of Macaulay's duality theorem.

PROPOSITION 3.7. *Assume that the cover $f : X \rightarrow Y$ satisfies property (A). For $\chi \in G^* \setminus \{1\}$, set $\omega_\chi = \omega_Y^2(D_{\chi, \chi^{-1}})$; then:*

i) *there is a natural isomorphism $R_{\omega_\chi}^{n, \chi} \cong \mathbb{C}$*

ii) *let L be a line bundle on Y such that L and $L^{-1}(D_{\chi, \chi^{-1}})$ (or $L \otimes \omega_Y^{-1}$ and $(\omega_Y \otimes L)^{-1}(D_{\chi, \chi^{-1}})$) belong to $\Gamma_f \setminus \{0\}$; then the multiplication map $R_{\omega_Y \otimes L}^{k, \chi} \otimes R_{\omega_Y \otimes L^{-1}(D_{\chi, \chi^{-1}})}^{n-k, \chi} \rightarrow R_{\omega_\chi}^{n, \chi}$ is a perfect pairing, corresponding to Serre duality via the isomorphism of Lemma 3.6. In particular, one has natural isomorphisms:*

$$U^{k, \chi} \cong R_{\omega_Y \otimes L}^{k, \chi}.$$

PROOF. In order to prove i), consider the complex (2.5) for $k = n$: twisting it by $\omega_Y(D_{\chi, \chi^{-1}})^{-1}$ and arguing as in the proof of Lemma 3.6, one shows the existence of a natural isomorphism between $R_{\omega_\chi}^{n, \chi}$ and $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$. In order to prove statement ii), one remarks that the group $H^k \left(Y, \Omega_Y^{n-k} \left(\log D_{\chi, \chi^{-1}} \right) \otimes L^{-1} \right)$ is Serre dual to $H^{n-k} \left(Y, \Omega_Y^k \left(\log D_{\chi, \chi^{-1}} \right) \otimes L(-D_{\chi, \chi^{-1}}) \right)$ by (2.1). By Lemma 3.6 the latter group equals $\left(R_{\omega_Y \otimes L^{-1}(D_{\chi, \chi^{-1}})}^{k, \chi} \right)^*$. Both these isomorphisms and the multiplication map are natural, and therefore compatible with Serre duality. The last claim follows in view of (1.2). \square

4. – Infinitesimal Torelli

In this section we exploit the algebraic description of the IVHS of a G -cover to prove an infinitesimal Torelli theorem. We will use freely the notation introduced in Section 3.

We recall that *infinitesimal Torelli* for the periods of k -forms holds for a variety X if the map (3.1) is injective. By the remarks at the beginning of Section 3, a G -cover $f : X \rightarrow Y$ satisfies infinitesimal Torelli property if for each character $\chi \in G^*$ the intersection, as ϕ varies in G^* , of the kernels of the maps $\rho_{\chi,\phi}^k$ of (3.2) is equal to zero. The next theorem shows that this is actually the case for $k = n$, under some ampleness assumptions on the building data of $f : X \rightarrow Y$.

THEOREM 4.1. *Let X, Y be smooth complete algebraic varieties of dimension $n \geq 2$ and let $f : X \rightarrow Y$ be a G -cover with building data $L_\chi, D_i, \chi \in G^* \setminus \{1\}, i = 1, \dots, r$. If properties (A) and (B) are satisfied, then the following map is injective:*

$$H^1(X, T_X) \rightarrow \text{Hom}(H^0(X, \omega_X), H^1(X, \Omega_X^{n-1})),$$

and, as a consequence, *infinitesimal Torelli for the periods of n -forms holds for X .*

Before giving the proof, we deduce two effective results from Theorem 4.1

THEOREM 4.2. *Let $f : X \rightarrow \mathbb{P}^n, n \geq 2$, be a G -cover with building data $L_\chi, D_i, \chi \in G^* \setminus \{1\}, i = 1, \dots, r$. Assume that $L_\chi \otimes \omega_{\mathbb{P}^n} > 0$ and $D_i \otimes \omega_{\mathbb{P}^n} > 0$ for $\chi \in G^* \setminus \{1\}, i = 1, \dots, r$; then *infinitesimal Torelli for the periods of n -forms holds for X .**

PROOF. By Remark 3.4, properties (A) and (B) are satisfied in this case. \square

THEOREM 4.3. *Let X, Y be smooth complete algebraic varieties of dimension $n \geq 2$ and let $f : X \rightarrow Y$ be a G -cover with building data $L_\chi, D_i, \chi \in G^* \setminus \{1\}, i = 1, \dots, r$. Let E be a very ample divisor on Y and let E_n be defined as in Proposition 3.5; if $D_i, L_\chi, \omega_Y(D_i), \omega_Y \otimes L_\chi > E_n$ and $L_\chi, D_i \geq (n + 1)E$ for $\chi \in G^* \setminus \{1\}, i = 1, \dots, r$, then *infinitesimal Torelli for the periods of n -forms holds for X .**

PROOF. Follows from Theorem 4.1 together with Proposition 3.5. \square

PROOF OF THEOREM 4.1. Since all cohomology groups appearing in this proof are computed on Y , we will omit Y from the notation.

By Proposition 1.2 and by the discussion at the beginning of the section, we have to show that for every $\chi \in G^*$ the intersection of the kernels of the maps

$$\rho_{\chi,\phi} : H^1(T_Y(-\log \Delta_\chi) \otimes L_\chi^{-1}) \rightarrow \text{Hom}(U^{0,\phi}, U^{1,\chi\phi}),$$

as ϕ varies in G^* , is equal to zero. For $\chi, \phi \in G^*$, set $R_{\chi,\phi} = D_{\chi\phi,(\chi\phi)^{-1}} - D_{\chi\phi,\phi^{-1}}$. Notice that $R_{\chi,\phi}$ is effective. By (2.1) and (1.1) there is a natural identification:

$$\Omega_Y^{n-1}(\log D_{\chi\phi,(\chi\phi)^{-1}}) \otimes (\omega_Y \otimes L_{\chi\phi} \otimes L_{\phi^{-1}})^{-1} = T_Y(-\log D_{\chi\phi,(\chi\phi)^{-1}}) \otimes L_\chi^{-1}(R_{\chi,\phi}).$$

So the map $\rho_{\chi,\phi}$ can be viewed as the composition of the map

$$i_{\chi,\phi} : H^1(T_Y(-\log \Delta_\chi) \otimes L_\chi^{-1}) \rightarrow H^1(T_Y(-\log D_{\chi\phi,(\chi\phi)^{-1}}) \otimes L_\chi^{-1}(R_{\chi,\phi})),$$

induced by inclusion of sheaves, and of the map

$$r_{\chi\phi,\phi} : H^1(\Omega_Y^{n-1}(\log D_{\chi\phi,(\chi\phi)^{-1}}) \otimes (\omega_Y \otimes L_{\chi\phi} \otimes L_{\phi^{-1}})^{-1}) \rightarrow \text{Hom}(U^{0,\phi}, U^{1,\chi\phi})$$

induced by cup-product. Arguing as in the proof of Theorem 3.1 of [17], one can show that, for fixed $\chi \in G^*$, the intersection of the kernels of $i_{\chi,\phi}$, as ϕ varies in $G^* \setminus \{1, \chi^{-1}\}$, is zero. (Notice that Lemma 3.1 of [17], although stated for surfaces, actually holds for varieties of any dimension, and that the ampleness assumptions on the building data allow one to apply it, in view of Theorem 1.3). So the statement will follow if we prove that the map $r_{\chi,\phi} : H^1(\Omega_Y^{n-1}(\log D_{\chi,\chi^{-1}}) \otimes (\omega_Y \otimes L_\chi \otimes L_{\phi^{-1}})^{-1}) \rightarrow \text{Hom}(U^{0,\phi}, U^{1,\chi})$ is injective for every pair χ, ϕ of nontrivial characters. By Lemma 3.6, the map $r_{\chi,\phi}$ may be rewritten as: $r_{\chi,\phi} : (R_{\omega_Y^2 \otimes L_\chi \otimes L_{\phi^{-1}}}^{n-1,\chi})^* \rightarrow (R_{\omega_Y \otimes L_{\phi^{-1}}}^{0,\chi})^* \otimes (R_{\omega_Y \otimes L_\chi}^{n-1,\chi})^*$. We prove

that $r_{\chi,\phi}$ is injective by showing that the dual map $r_{\chi,\phi}^* : R_{\omega_Y \otimes L_{\phi^{-1}}}^{0,\chi} \otimes R_{\omega_Y \otimes L_\chi}^{n-1,\chi} \rightarrow R_{\omega_Y^2 \otimes L_\chi \otimes L_{\phi^{-1}}}^{n-1,\chi}$, induced by multiplication, is surjective. In order to do this, it is sufficient to observe that the multiplication map

$$H^0(Y, \omega_Y \otimes L_{\phi^{-1}}) \otimes H^0(Y, S^{n-1}(V^\chi) \otimes \omega_Y \otimes L_\chi) \rightarrow H^0(Y, S^{n-1}(V^\chi) \otimes \omega_Y^2 \otimes L_{\phi^{-1}} \otimes L_\chi)$$

is surjective by property (B). □

REMARK 4.4. In Section 6 of [6], it is proven that for any abelian group G there exist families of smooth G -covers with ample canonical class that are *generically complete*. In those cases, Theorem 4.1 means that the period map is étale on a whole component of the moduli space.

5. – Sufficiently ample line bundles

We take up the following definition from [9]

DEFINITION 5.1. A property is said to hold for a *sufficiently ample* line bundle L on the smooth projective variety Y if there exists an ample line bundle L_0 such that the property holds whenever the bundle $L \otimes L_0^{-1}$ is ample. We will denote this by writing that the property holds for $L \gg 0$.

In this section, we collect some facts about sufficiently ample line bundles that will be used to prove our variational Torelli result. In particular, we prove a variant of Proposition 5.1 of [11] to the effect that, given sufficiently ample line bundles L_1 and L_2 on Y , it is possible to recover Y from the kernel of the multiplication map $H^0(Y, L_1) \otimes H^0(Y, L_2) \rightarrow H^0(Y, L_1 \otimes L_2)$.

The next lemma is “folklore”. The proof given here has been communicated to the author by Mark Green.

LEMMA 5.2. *Let \mathcal{F} be a coherent sheaf on Y and let L be a line bundle on Y . Then, if $L \gg 0$,*

$$H^i(Y, \mathcal{F} \otimes L) = 0, \quad i > 0.$$

PROOF. We proceed by descending induction on i . If $i > n$, then the statement is evident. Otherwise, fix an ample divisor E and an integer m such that $\mathcal{F}(mE)$ is generated by global sections. This gives rise to an exact sequence: $0 \rightarrow \mathcal{F}_1 \rightarrow \oplus \mathcal{O}_Y(-mE) \rightarrow \mathcal{F} \rightarrow 0$. Tensoring with L and considering the corresponding long cohomology sequence, one sees that it is enough that $H^i(Y, L(-mE)) = H^{i+1}(Y, \mathcal{F}_1 \otimes L) = 0$, for $L \gg 0$. The vanishing of the former group follows from Kodaira vanishing and the vanishing of the latter follows from the inductive hypothesis. \square

LEMMA 5.3. *Let $|E|$ be a very ample linear system on the smooth projective variety Y of dimension n . Denote by \mathcal{M}_y the ideal sheaf of a point $y \in Y$: if L is a sufficiently ample line bundle on Y , then the multiplication map:*

$$H^0(Y, L) \otimes H^0(Y, \mathcal{M}_y(E)) \rightarrow H^0(Y, \mathcal{M}_y(L + E))$$

is surjective for every $y \in Y$.

PROOF. For $y \in Y$, set $V_y = H^0(Y, \mathcal{M}_y(E))$ and consider the natural exact sequence $0 \rightarrow N_y \rightarrow V_y \otimes \mathcal{O}_Y \rightarrow \mathcal{M}_y(E) \rightarrow 0$; tensoring with L and considering the corresponding cohomology sequence, one sees that if $H^1(Y, N_y \otimes L) = 0$ then the map $H^0(Y, L) \otimes V_y \rightarrow H^0(Y, \mathcal{M}_y(L + E))$ is surjective. By Lemma 5.2, there exists an ample line bundle L_y such that $H^1(Y, N_y \otimes L) = 0$ if $L > L_y$. In order to deduce from this the existence of a line bundle L_0 such that $H^1(Y, N_y \otimes L) = 0$ for every $y \in Y$ if $L > L_0$, we proceed as follows. Consider the product $Y \times Y$, with projections p_i , $i = 1, 2$, denote by \mathcal{I}_Δ the ideal sheaf of the diagonal in $Y \times Y$ and set $\mathcal{V} = p_{2*}(p_1^* \mathcal{O}_Y(E) \otimes \mathcal{I}_\Delta)$. \mathcal{V} is a fibre bundle on Y such that the fibre of \mathcal{V} at y can be naturally identified with V_y . We define the sheaf \mathcal{N} on $Y \times Y$ to be the kernel of the map $p_2^* \mathcal{V} \rightarrow p_1^* \mathcal{O}_Y(E) \otimes \mathcal{I}_\Delta$; the restriction of \mathcal{N} to $p_2^{-1}(y)$ is precisely N_y . For any fixed line bundle L on Y , $h^1(Y, N_y \otimes L) = h^1(p_2^{-1}(y), \mathcal{N} \otimes p_1^* L|_{p_2^{-1}(y)})$ is an upper-semicontinuous function of y . Thus, we may find a finite open covering U_1, \dots, U_k of Y and ample line bundles L_1, \dots, L_k such that $h^1(Y, N_y \otimes L) = 0$ for $y \in U_i$ if $L \otimes L_i^{-1}$ is ample. To finish the proof it is enough to set $L_0 = L_1 \otimes \dots \otimes L_k$. \square

LEMMA 5.4. *Let Y be a smooth projective variety of dimension $n \geq 2$, $Y \neq \mathbb{P}^n$. If E is a very ample divisor on Y , then:*

- i) *if $L > \omega_Y((n-1)E)$, then L is base point free.*
- ii) *if $L > \omega_Y(nE)$, then L is very ample.*

PROOF. In order to prove the claim, it is enough to show that if C is a smooth curve on Y which is the intersection of $n-1$ divisors of $|E|$, then $L|_C$ is base point free (very ample) and $|L|$ restricts to the complete linear $L|_C$. In view of the assumption, the former statement follows from adjunction on Y and Riemann-Roch on C , and the latter follows from the vanishing of $H^1(Y, \mathcal{I}_C \otimes L)$,

where \mathcal{I}_C is the ideal sheaf of C . In turn, this vanishing can be shown by means of the Koszul complex resolution:

$$0 \rightarrow L((1-n)E) \rightarrow L \otimes \wedge^{n-2} \left(\bigoplus_1^{n-1} \mathcal{O}_Y(-E) \right) \dots \rightarrow \bigoplus_1^{n-1} L(-E) \rightarrow \mathcal{I}_C \otimes L \rightarrow 0. \quad \square$$

The following proposition is a variant for sufficiently ample line bundles of Proposition 5.1 of [11]. Our statement is weaker, but we do not need to assume that one of the line bundles involved is much more ample than the other one.

PROPOSITION 5.5. *Let L_1 and L_2 be very ample line bundles on a smooth projective variety Y . Let $\phi_1 : Y \rightarrow \mathbb{P}_1$ and $\phi_2 : Y \rightarrow \mathbb{P}_2$ be the corresponding embeddings into projective space, and let $f : Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_2$ be the composition of the diagonal embedding of Y in $Y \times Y$ with the product map $\phi_1 \times \phi_2$. If $L_1, L_2 \gg 0$, then $f(Y)$ is the zero set of the elements of $H^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_2}(1, 1))$ vanishing on it.*

PROOF. By [11], Proposition 5.1, we may find very ample line bundles $M_i, i = 1, 2$ such that, if $\psi_i : Y \rightarrow \mathbb{Q}_i$ are the corresponding embeddings in projective space and $g : Y \rightarrow \mathbb{Q}_1 \times \mathbb{Q}_2$ is the composition of the diagonal embedding of Y in $Y \times Y$ with $\psi_1 \times \psi_2$, then $g(Y)$ is the scheme-theoretic intersection of the elements of $H^0(\mathbb{Q}_1 \times \mathbb{Q}_2, \mathcal{O}_{\mathbb{Q}_1 \times \mathbb{Q}_2}(1, 1))$ vanishing on it. By Lemma 5.4, we may assume that $L_i \otimes M_i^{-1}$ is very ample, $i = 1, 2$. To a divisor D_i in $|L_i \otimes M_i^{-1}|$ there corresponds a projection $p_{D_i} : \mathbb{P}_i \cdots \rightarrow \mathbb{Q}_i$ such that $\psi_i = p_{D_i} \circ \phi_i$. The claim will follow if we show that for $(x_1, x_2) \notin f(Y)$ one can find $D_i \in |L_i \otimes M_i^{-1}|$ such that p_{D_i} is defined at $x_i, i = 1, 2$, and $(p_{D_1}(x_1), p_{D_2}(x_2)) \notin g(Y)$. In fact, this implies that there exists $s \in H^0(\mathbb{Q}_1 \times \mathbb{Q}_2, \mathcal{O}_{\mathbb{Q}_1 \times \mathbb{Q}_2}(1, 1))$ that vanishes on $g(Y)$ and does not vanish at $(p_{D_1}(x_1), p_{D_2}(x_2))$, so that the pull-back of s via $p_{D_1} \times p_{D_2}$ is a section of $H^0(\mathbb{P}_1 \times \mathbb{P}_2, \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_2}(1, 1))$ that vanishes on $f(Y)$ and does not vanish at (x_1, x_2) .

By Lemma 5.3, if $L_i \gg 0$, then the multiplication map $H^0(Y, L_i \otimes M_i^{-1}) \otimes H^0(Y, M_i \otimes \mathcal{M}_y) \rightarrow H^0(Y, L_i \otimes \mathcal{M}_y)$ is surjective $\forall y \in Y$ and for $i = 1, 2$. Notice that, in particular, this implies that the map $H^0(Y, L_i \otimes M_i^{-1}) \otimes H^0(Y, M_i) \rightarrow H^0(Y, M_i \otimes L_i)$ is surjective for $i = 1, 2$, so that, for a generic choice of D_i , the projection p_{D_i} is defined at x_i . If either $(x_1, x_2) \in \phi_1(Y) \times \phi_2(Y)$ or there exists a divisor D_i , for $i = 1$ or $i = 2$, such that $p_{D_i}(x_i) \notin \psi_i(Y)$, then we are set. So assume that, say, $x_1 \notin \phi_1(Y)$ and that $p_{D_i}(x_i) \in \psi_i(Y)$, for a generic choice of D_i , for $i = 1, 2$. Fix a divisor D_2 such that p_{D_2} is defined at x_2 and write $p_{D_2}(x_2) = \psi_2(y_2)$, with $y_2 \in Y$. Now it is enough to show that there exists D_1 such that p_{D_1} is defined at x_1 and $p_{D_1}(x_1) \neq \psi_1(y_2)$. Since the multiplication map $H^0(Y, L_1 \otimes M_1^{-1}) \otimes H^0(Y, M_1 \otimes \mathcal{M}_{y_2}) \rightarrow H^0(Y, L_1 \otimes \mathcal{M}_{y_2})$ is surjective, there exist $\sigma \in H^0(Y, L_1 \otimes M_1^{-1})$ and $\tau \in H^0(Y, M_1 \otimes \mathcal{M}_{y_2})$ such that $\sigma \tau$ corresponds to a hyperplane of \mathbb{P}_1 passing through $\phi_1(y_2)$ but not through x_1 . If D_1 is the divisor of σ , then the projection p_{D_1} is defined at x_1 and $p_{D_1}(x_1) \neq \psi_1(y_2)$. \square

6. – A variational Torelli theorem

In this section we prove a variational Torelli theorem for the family $\mathcal{X} \rightarrow W$ of abelian covers with fixed basis and fixed L_χ 's.

Let $\mathcal{Z} \rightarrow B$ be a flat family of smooth projective polarized varieties on which G acts fibrewise, and let Z be the fibre of \mathcal{Z} over the point $0 \in B$. It is possible to show that the monodromy action on the cohomology of X preserves the group action; therefore one may define a G -period map, by dividing the period domain D by the subgroup of linear transformations that, beside preserving the integral lattice and the polarization, are compatible with the G -action. In particular, let $f : X \rightarrow Y$ be a G -cover, with X and Y smooth projective varieties of dimension $n \geq 2$ and with building data L_χ, D_i . In Section 1, we have introduced the family $\mathcal{X} \rightarrow W$ of G -covers of Y , obtained by letting the sections $s_i \in H^0(Y, \mathcal{O}_Y(D_i))$ vary in (eq. 1.3). There is an obvious G -action on \mathcal{X} , and the choice of an ample divisor E on Y gives a G -invariant polarization of \mathcal{X} . Since two elements (s_1, \dots, s_r) and (s'_1, \dots, s'_r) of W represent the same G -cover iff there exist $\lambda_i \in \mathbb{C}^*$ such that $s'_i = \lambda_i s_i$, the G -period map can be regarded as being defined on the image \tilde{W} of W in $|D_1| \times \dots \times |D_r|$. We denote by s the image in \tilde{W} of the point (s_1, \dots, s_r) , by X_s the corresponding G -cover of Y , by T_s the tangent space to \tilde{W} at s and, consistently with the notation of Section 3, by $U_s^{k,\chi}$ the subspace of $H^k(X_s, \Omega_{X_s}^{n-k})$ on which G acts via the character χ . Remark that the space T_s can be naturally identified with $\bigoplus_i H^0(Y, \mathcal{O}_Y(D_i))/(s_i)$. Denote by Γ the subgroup of $GL(T_s) \times GL(H^0(X_s, \omega_{X_s})) \times GL(H^1(X_s, \Omega_{X_s}^{n-1}))$ that preserves the G -action on the cohomology of X_s .

THEOREM 6.1. *Assume that the dimension n of Y is ≥ 2 , that the canonical class ω_Y of Y is very ample and that $L_\chi, D_i \gg 0, \chi \neq 1, i = 1, \dots, r$. Assume that for $i = 1, \dots, r$ the identity is the only automorphism of Y that preserves the linear equivalence class of D_i ; moreover assume that $\forall i = 1, \dots, r$ there exists $\chi \in G^*$ (possibly depending on i) such that $\chi(g_i) \neq 1$ and $L_\chi \gg D_i$. Then a generic point $s \in \tilde{W}$ is determined by the Γ -class of the linear map:*

$$T_s \rightarrow \bigoplus_\chi \text{Hom}(U_s^{0,\chi}, U_s^{1,\chi}),$$

which represents the first piece of the algebraic part of the IVHS for n -forms.

REMARK 6.2. The assumption, made in Theorem 6.1 and Corollary 6.3, that for every $i = 1, \dots, r$ there exists $\chi \in G^*$ such that $\chi(g_i) \neq 1$ and $L_\chi \gg D_i$ is not satisfied by simple cyclic covers, namely totally ramified covers branched on an irreducible divisor. Still there are many cases in which our results apply: for instance, construction 6.2 of [6] provides examples with $G = \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_{r-1}, r - 1 \geq n$, branched on r algebraically equivalent divisors D_1, \dots, D_r . The ramification order over D_i is equal to m_i for $i < r$, and it is equal to the least common multiple of the m_i 's for $i = r$. (The assumption, made in [6], that $m_i | m_{i+1}$ is actually unnecessary in order to make

the construction.) Apart from the case $r = m_1 = m_2 = 2$, if the branch divisors are ample, then for every i there exists χ such that $L_\chi(-D_i)$ is ample.

Before giving the proof of Theorem 6.1, we state:

COROLLARY 6.3. *Under the same assumptions as in Theorem 6.1 the G -period map for n -forms has degree 1 on \tilde{W} .*

PROOF. This is just a variation of the main result of [1], taking care of the case in which there is a group action, and it can be proved exactly in the same way. □

PROOF OF THEOREM 6.1. Whenever confusion is not likely to arise, we omit to write the space where cohomology groups are computed.

By Theorem 0.3 of [9], if $D_i \gg 0$ for $i = 1, \dots, r$, then the period map P for $n - 1$ forms has degree 1 on $|D_i|$. Denote by U_i the open subset of $|D_i|$ consisting of the points z such that $P^{-1}(P(z)) = \{z\}$, and fix $s \in \tilde{W} \cap (U_1 \times \dots \times U_r)$. From now on we will drop the subscript s and write X for X_s , T for T_s , and so on. For each index $i \in \{1, \dots, r\}$ set $S_i = \{\chi \in G^* | \chi(g_i) = 1\}$; as a first step, we show that the subspace $H^0(\mathcal{O}_Y(D_i))/(s_i)$ of T is the intersection of the kernels of the maps $\rho_\chi : T \rightarrow \text{Hom}(U^{0,\chi}, U^{1,\chi})$, as χ varies in $S_i \setminus \{1\}$. Since ω_Y is ample, $H^0(T_Y) = 0$ by Theorem 1.3, and so T equals $R_{\mathcal{O}_Y}^{1,1}$. The map ρ_χ factors through the surjection $R_{\mathcal{O}_Y}^{1,1} \rightarrow R_{\mathcal{O}_Y}^{1,\chi}$, whose kernel is $\bigoplus_{(i|\chi(g_i)=1)} H^0(Y, \mathcal{O}_Y(D_i))/(s_i)$. In turn, by sequence 2.3, $R_{\mathcal{O}_Y}^{1,\chi}$ injects in $H^1(Y, T_Y(-\log D_{\chi,\chi-1}))$. We have shown in the proof of Theorem 4.1 that the map $H^1(T_Y(-\log D_{\chi,\chi-1})) \rightarrow \text{Hom}(U^{0,\chi}, U^{1,\chi})$ is injective. So $\ker \rho_\chi = \bigoplus_{(i|\chi(g_i)=1)} H^0(Y, \mathcal{O}_Y(D_i))/(s_i)$. As we have remarked in Section 1, if $i \neq j$, then the subgroups of G generated by g_i and g_j intersect only in $\{0\}$, and so i is the only index such that $\chi(g_i) = 1$ for all $\chi \in S_i$. We conclude that $\bigcap_{\chi \in S_i \setminus \{1\}} \ker \rho_\chi = H^0(Y, \mathcal{O}_Y(D_i))/(s_i)$.

Now fix $i \in \{1, \dots, r\}$ and let $\chi \in G^* \setminus S_i$ be such that $L_\chi > D_i$: the restriction to $H^0(\mathcal{O}_Y(D_i))/(s_i)$ of the map $T \otimes U^{0,\chi^{-1}} \rightarrow U^{1,\chi^{-1}}$ is the multiplication map $H^0(\mathcal{O}_Y(D_i))/(s_i) \otimes H^0(\omega_Y \otimes L_{\chi^{-1}}) \rightarrow H^0(Y, \omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i}$, followed by the inclusion $H^0(\omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i} \rightarrow U^{1,\chi^{-1}}$.

CLAIM. The kernel of the latter map is equal to zero.

If we assume that the claim holds, then we have recovered the multiplication map $H^0(\mathcal{O}_Y(D_i))/(s_i) \otimes H^0(\omega_Y \otimes L_{\chi^{-1}}) \rightarrow H^0(Y, \omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i}$. The right kernel of this map is $(s_i)H^0(\omega_Y \otimes L_{\chi^{-1}}(-D_i))$. So we can reconstruct the map $H^0(\mathcal{O}_Y(D_i))|_{D_i} \otimes H^0(\omega_Y \otimes L_{\chi^{-1}})|_{D_i} \rightarrow H^0(\omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i}$. Let $\phi_1 : Y \rightarrow \mathbb{P}_1$ be the embedding defined by the linear system $|D_i|$, let $\phi_2 : Y \rightarrow \mathbb{P}_2$ be the embedding defined by the linear system $|\omega_Y \otimes L_{\chi^{-1}}|$ and let $f : Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_2$ be the composition of the diagonal embedding $Y \rightarrow Y \times Y$ with the product map $\phi_1 \times \phi_2$: by Proposition 5.5, $f(Y)$ is the zero set of the elements of the kernel of $H^0(\mathcal{O}_Y(D_i)) \otimes H^0(\omega_Y \otimes L_{\chi^{-1}}) \rightarrow H^0(\omega_Y \otimes L_{\chi^{-1}}(D_i))$. This implies that $f(D_i)$

the zero set in $\mathbb{P}(\mathbb{H}^0(\mathcal{O}_Y(D_i))|_{D_i}) \times \mathbb{P}(\mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}})|_{D_i})$ of the elements of the kernel of $\mathbb{H}^0(\mathcal{O}_Y(D_i))|_{D_i} \otimes \mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}})|_{D_i} \rightarrow \mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i}$. Thus it is possible to recover D_i as an abstract variety, for every $i = 1, \dots, r$. Since $s \in U_1 \times \dots \times U_r$, this is enough to determine the D_i 's as divisors on Y and, in turn, the point s .

In order to complete the proof, we have to prove the claim. Let D_{i_1}, \dots, D_{i_s} be the components of $D_{\chi, \chi^{-1}}$. (Recall that there exists j_0 such that $i = i_{j_0}$.) Let $V = \bigoplus_j \mathcal{O}_Y(D_{i_j})$, let $V_i = \bigoplus_{j \neq j_0} \mathcal{O}_Y(D_{i_j})$, let P be the generalized prolongation bundle associated to $(D_{i_1}, \dots, D_{i_s})$ and let P_i be the generalized prolongation bundle associated to $(D_{i_1}, \dots, \hat{D}_i, \dots, D_{i_s})$. There is a natural short exact sequence $0 \rightarrow P_i \rightarrow P \rightarrow \mathcal{O}_Y \rightarrow 0$, with dual sequence $0 \rightarrow \mathcal{O}_Y \rightarrow P^* \rightarrow P_i^* \rightarrow 0$. From this and sequence 2.3, tensoring with $\omega_Y \otimes L_{\chi^{-1}}$ and taking global sections, one deduces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 \rightarrow \mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}}) & \rightarrow & \mathbb{H}^0(P^* \otimes \omega_Y \otimes L_{\chi^{-1}}) & \rightarrow & \mathbb{H}^0(P_i^* \otimes \omega_Y \otimes L_{\chi^{-1}}) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow \mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}}(D_i)) & \rightarrow & \mathbb{H}^0(V \otimes \omega_Y \otimes L_{\chi^{-1}}) & \rightarrow & \mathbb{H}^0(V_i \otimes \omega_Y \otimes L_{\chi^{-1}}) & \rightarrow & 0
 \end{array}$$

In view of sequence 2.3, by applying snake's lemma to this diagram one obtains the following exact sequence: $\mathbb{H}^0(T_Y(-\log(D_{\chi, \chi^{-1}} - D_i)) \otimes \omega_Y \otimes L_{\chi^{-1}}) \rightarrow \mathbb{H}^0(\omega_Y \otimes L_{\chi^{-1}}(D_i))|_{D_i} \rightarrow U^{1, \chi^{-1}}$. So it is enough to show that $\mathbb{H}^0(T_Y(-\log(D_{\chi, \chi^{-1}} - D_i)) \otimes L_{\chi^{-1}}) = 0$. Using the isomorphism (2.1) and the relations (1.2), one has $T_Y(-\log(D_{\chi, \chi^{-1}} - D_i)) \otimes \omega_Y \otimes L_{\chi^{-1}} \cong \Omega_Y^{n-1}(\log(D_{\chi, \chi^{-1}} - D_i) \otimes L_{\chi^{-1}}^{-1}(D_i))$. In view of the assumptions, the required vanishing now follows from Theorem 1.3. □

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