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Compactness of Conformal Metrics with Positive Gaussian Curvature in \mathbb{R}^2

KUO-SHUNG CHENG – CHANG-SHOU LIN

Abstract. In this paper we consider the compactness of a sequence of solutions u_n of

$$(0.1) \quad \Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2,$$

where $K(x)$ is positive in \mathbb{R}^2 and decays like $|x|^{-b}$ at ∞ for some $b > 0$. Assuming that the limit of the total curvature of u_n satisfies

$$(0.2) \quad 2 - b \neq \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx < 2,$$

we prove that u_n must be bounded in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$. We also construct a specific $K(x) = K(|x|)$ to show that the total curvature of any solution u of equation (0.1) with this $K(|x|)$ must satisfy

$$(0.3) \quad (2 - b) < \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u} dx < 2.$$

This appears to be in contrast with the statement of Theorem A¹ in [A]. In this respect, we show that for any K which decays like $|x|^{-b}$ for $0 < b < 2$, there exists $\alpha_0(K) > \frac{2-b}{2}$ such that the total curvature of any solution u of (0.1) must satisfy

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K e^{2u} dx \geq \alpha_0(K) > \frac{2-b}{2}.$$

1. – Introduction

In this paper, we consider the entire solution of the equation

$$(1.1) \quad \Delta u + K(x)e^{2u} = 0 \quad \text{in } \mathbb{R}^2,$$

where Δ is the Laplacian operator of \mathbb{R}^2 and $K(x)$ is a given function in \mathbb{R}^2 . Equation (1.1) arises in the problem of finding a Riemannian metric which is

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conformal to the flat metric of \mathbb{R}^2 and realizes the given function $K(x)$ as its Gaussian curvature. We refer the reader to [CN1] for a brief description of the background and the history of this problem.

In case K is nonpositive on \mathbb{R}^2 , a fairly complete understanding of the the solution set of (1.1) was achieved in [CN1], [CN2]. To state the results in [CN2], we introduce α_1 as

$$(1.2) \quad \alpha_1 = \sup \left\{ \alpha \in \mathbb{R} \mid \int_{\mathbb{R}^2} |K(x)|(1+|x|^2)^\alpha dx < +\infty \right\}.$$

Then the main result in [CN2] is

THEOREM A. *Suppose that $K \leq 0$ in \mathbb{R}^2 and that*

$$(1.3) \quad |x|^{-m} \leq |K(x)| \leq |x|^m$$

for $|x|$ large and some positive constant m . Then we have:

(I) *If $\alpha_1 \leq 0$, then (1.1) possesses no entire solution in \mathbb{R}^2 .*

(II) *If $\alpha_1 > 0$, then the following conclusions hold:*

(i) *For each $\alpha \in (0, \alpha_1)$, (1.1) possesses a unique solution u_α such that*

$$(1.4) \quad u_\alpha(x) = \alpha \log |x| + O(1) \quad \text{at } \infty.$$

(ii) *The function $U(x)$ given by*

$$U(x) \equiv \sup \{ u(x) \mid u \text{ is an entire solution of (1.1) in } \mathbb{R}^2 \}$$

is well-defined everywhere in \mathbb{R}^2 and is a solution of (1.1) in \mathbb{R}^2 . Moreover, $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$.

(iii) *Let u be an arbitrary solution of (1.1) in \mathbb{R}^2 . Then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \alpha_1)$.*

(iv) *If $0 < \alpha < \beta < \alpha_1$, then $u_\alpha(x) < u_\beta(x) < U(x)$ for all $x \in \mathbb{R}^2$. Furthermore, for any given $\varepsilon > 0$, there exists a constant $R = R(\varepsilon)$ such that for $|x| > R$,*

$$(\alpha_1 - \varepsilon) \log |x| - C \leq U(x) \leq \alpha_1 \log |x| + C.$$

In this paper, K is always assumed locally bounded and positive in \mathbb{R}^2 . A solution u means $u \in W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$ and satisfies (1.1) in the distributional sense. For the case $K(x)$ is positive in \mathbb{R}^2 , it is not expected that results similar to Theorem A should hold. However, for some special $K(x)$ as stated in Theorem 1.1 below, we have the following result in the spirit of Theorem A.

THEOREM 1.1. *Let $K(x) \equiv 1$ for $|x| \leq 1$ and $K(x) \equiv |x|^{-b}$ for $|x| \geq 1$ for some constant $b > 0$. Then the following statements hold:*

- (i) *For every α satisfying $-2 < \alpha < \min\{0, b - 2\}$, (1.1) possesses a unique C^2 radial solution $u_\alpha(r)$ satisfying (1.4).*
- (ii) *Let u be an arbitrary solution of (1.1) satisfying (1.4) for some α , then α satisfies $-2 < \alpha < \min\{0, b - 2\}$ and $u(x) \equiv u_\alpha(x)$ where $u_\alpha(x)$ is the solution in (i) above.*

REMARK 1.2. On the contrast to the case $K \leq 0$, the family of solution $u_\alpha(x)$ in Theorem 1.1 does not have the monotone property in α as the case in Theorem A. In fact, by the concrete construction of solutions in the proof of Theorem 1.1, it can be seen that $u_\alpha(r)$ and $u_\beta(r)$ exactly intersects once for $\alpha \neq \beta$. We hope that it will be useful in a future study.

Although Theorem 1.1 are only concerned with some specific $K(x)$, it still provides an interesting example to the situation when $K(x)$ is positive in \mathbb{R}^2 . In [A], Aviles proved the following theorem, (See Theorem A¹ in [A]).

THEOREM B. *Assume $K(x) > 0$ in \mathbb{R}^2 and $\lim_{|x| \rightarrow +\infty} K(x)|x|^b = 1$ for some positive constant $b > 0$. Then, for any α satisfying*

$$(1.5) \quad -2 < \alpha < \min\left(0, \frac{b-2}{2}\right),$$

there exists a solution u of (1.1) satisfying

$$u(x) = \alpha \log |x| + O(1) \quad \text{at } \infty.$$

Let $K(x)$ be the specific function given in Theorem 1.1 with $0 < b < 2$. Then Theorem 1.1 contradicts to the result of Theorem B. In fact, Theorem 1.1 is not an isolated case to show that Theorem B does not hold. For a general $K(x)$, set

$$(1.6) \quad \alpha_0 = \sup\{\alpha \mid \text{there is an entire solution } u \text{ of (1.1) such that } u(x) = \alpha \log |x| + O(1) \text{ at } \infty\}.$$

Our main result is

THEOREM 1.2. *Suppose that $K(x)$ is positive and locally bounded in \mathbb{R}^2 and satisfies*

$$(1.7) \quad B|x|^{-b} \leq K(x) \leq A|x|^{-b}$$

for $|x| \geq 1$ and for positive constants A, B and $0 < b < 2$. Then $\alpha_0 < -\frac{2-b}{2}$, where α_0 is given in (1.6).

Obviously, Theorem 1.2 implies that Theorem B does not hold in general. We note that the real number α_1 in (1.2) is $-\frac{2-b}{2}$ if $K(x)$ satisfies (1.7). Theorem 1.2 provides a major contrast to Theorem A for the case $K(x) \leq 0$. We would like to remark that solutions possessing the asymptotic behavior (1.4) have a geometric meaning. Following conventional notations, a solution $u(x)$ of (1.1) is said to have a finite total curvature if $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} dx$ is called the total curvature of u . Assume $K(x)$ satisfies (1.7). A consequence of our previous results in [CLn] is that a solution u has a finite total curvature if and only if u possesses the asymptotic behavior (1.4), or more precisely, $\lim_{|x| \rightarrow +\infty} u(x)/\log|x|$ exists, and the identity

$$\frac{-1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} dx = \lim_{|x| \rightarrow +\infty} \frac{u(x)}{\log|x|}$$

are always true. Please see Lemma 2.1 in Section 2. Thus, it is interesting to know what is the possible range of α or equivalently, the possible range of the total curvature of solutions. In [M], McOwen proved that if $0 < K(x) \leq C|x|^{-b}$ at ∞ , then for every $\alpha \in (-2, (b-2)^-)$ where $(b-2)^- = \min(0, b-2)$, there exists a solution of (1.1) satisfying (1.4). Together with Theorem 1.1, we see that the result of McOwen is the best possible for a general K which decays like $|x|^{-b}$ at ∞ .

THEOREM 1.3. *Suppose $K(x)$ is a positive continuous function in \mathbb{R}^2 and satisfies $\lim_{|x| \rightarrow +\infty} K(x)|x|^b = 1$ for some $0 \leq b < 2$. Assume u_n is a sequence of solutions of (1.1) such that*

$$(1.8) \quad 2 - b \neq \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx < 2$$

Then u_n is bounded in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$. Furthermore if u_n converges to u in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$, then

$$(1.9) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx = \int_{\mathbb{R}^2} K(x)e^{2u(x)} dx.$$

COROLLARY 1.4. *Suppose K satisfies the assumption of Theorem 1.3 and u_n is a sequence of solutions of (1.1). If $|u_n(0)| \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx \leq 2 - \varepsilon_0$ for some $\varepsilon_0 > 0$, then we always have*

$$(1.10) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx = 2\pi(2 - b).$$

COROLLARY 1.5. *Suppose K satisfies the assumption of Theorem 1.3 and $\alpha_0(K)$ is defined in (1.6). If $\alpha_0(K) > -(2 - b)$, then $\alpha_0(K)$ is achieved, i.e. there exists a solution u of (1.1) with*

$$(1.11) \quad -\alpha_0(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} dx.$$

REMARK 1.6. When $K(x)$ decays like $|x|^{-b}$ for $b \geq 2$, and u_n is a sequence of solutions of (1.4) satisfying

$$0 < \varepsilon_0 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx \leq 2 - \varepsilon_0$$

for some $\varepsilon_0 > 0$, then u_n is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. The proof is easy, and will be omitted.

The paper is organized as follows. In Section 2, we will give a proof of Theorem 1.1. Both Theorem 1.2 and Theorem 1.3 will be proved in Section 3.

2. – Proof of Theorem 1.1

Let K be positive in \mathbb{R}^2 and satisfy

$$(2.1) \quad |x|^{-m} \leq K(x) \leq |x|^m$$

for $|x|$ large, where m is a positive constant. A solution u of (1.1) is said to have a finite total curvature if $Ke^{2u} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi} \int Ke^{2u} dx$ is called the *total curvature* of u . Theorem 1.1 in [CLn] says that if u is a solution of (1.1) with a finite total curvature, then $\lim_{|x| \rightarrow +\infty} \frac{u(x)}{\log|x|}$ exists and

$$(2.2) \quad \lim_{|x| \rightarrow +\infty} \frac{u(x)}{\log|x|} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u} dx.$$

Conversely, it is easy to see that if $\lim_{|x| \rightarrow +\infty} \frac{u(x)}{\log|x|}$ exists, then $Ke^{2u} \in L^1(\mathbb{R}^2)$ and (2.2) holds. Hence, we have

LEMMA 2.1. *Suppose K satisfies (2.1). Then $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$ if and only if $\lim_{|x| \rightarrow +\infty} \frac{u(x)}{\log|x|}$ exists. Moreover, (2.2) always holds.*

REMARK 2.2. In fact, Theorem 1.1 in [CLn] also shows that for a solution u of (1.1) having a finite total curvature α , there exists a constant C such that

$$(2.3) \quad \alpha \log|x| - C \leq u(x)$$

holds. Hence, if $C_2|x|^{-b} \leq K(x) \leq C_1|x|^{-b}$ for large $|x|$, then $\alpha < -\frac{(2-b)^+}{2}$ where $(2-b)^+ = \max\{2-b, 0\}$.

PROOF OF THEOREM 1.1. Let

$$(2.4) \quad u_\alpha(r) = \frac{1}{2} \log(4B_1) - \log[1 + B_1 r^2], \quad r \in [0, 1]$$

and

$$(2.5) \quad u_\alpha(r) = \frac{1}{2} \log(4A_2^2 B_2) + \left(A_2 - 1 + \frac{b}{2} \right) \log r - \log[1 + B_2 r^{2A_2}],$$

$$r \in [1, \infty),$$

where $B_1 > 0$ is a constant and $\alpha = -A_2 - 1 + \frac{b}{2}$. Then it is not very difficult to verify that u_α is a C^2 -solution of (1.1) provided that

$$(2.6) \quad A_2 = \left\{ \frac{4B_1 + \left[B_1 \left(1 + \frac{b}{2} \right) - \left(1 - \frac{b}{2} \right) \right]^2}{(1 + B_1)^2} \right\}^{\frac{1}{2}},$$

$$(2.7) \quad B_2 = \frac{A_2(1 + B_1) + \left[B_1 \left(1 + \frac{b}{2} \right) - \left(1 - \frac{b}{2} \right) \right]}{A_2(1 + B_1) - \left[B_1 \left(1 + \frac{b}{2} \right) - \left(1 - \frac{b}{2} \right) \right]}.$$

Since $u_\alpha(0) = \frac{1}{2} \log(4B_1)$, we see that $B_1 > 0$ exhausts all radial solutions. It is easy to see that u_α satisfies (1.4) with $\alpha = -A_2 - 1 + \frac{b}{2}$. Now A_2 is a monotonic function of B_1 satisfying

$$\lim_{B_1 \rightarrow 0^+} A_2(B_1) = \left| \frac{b}{2} - 1 \right| \quad \text{and} \quad \lim_{B_1 \rightarrow \infty} A_2(B_1) = \frac{b}{2} + 1.$$

Hence α satisfies $-2 < \alpha < \min\{0, b - 2\}$. This proves (i).

Now suppose that u be an arbitrary solution of (1.1) with finite total curvature. Since $K(x) = K(|x|)$ is nonincreasing in r and $K(r) \geq e^{-r^\beta}$ for any $0 < \beta < 1$, then from Theorem 1.7 in [CLn], we conclude that u must be a radial function. Hence $u \equiv u_\alpha$ for some α in the range $-2 < \alpha < \min\{0, b - 2\}$, where u_α is defined in (2.4) and (2.5). This proves (ii). \square

3. – Proofs of compactness theorems

In this section, we begin with a proof of Theorem 1.2. First, we need the following result which was proved in [BM].

THEOREM 3.1 (Theorem 3 in [BM]). *Assume u_n is a sequence of solutions of*

$$(3.1) \quad \Delta u_n + K_n e^{2u_n} = 0 \quad \text{in } \Omega$$

satisfying

$$(3.2) \quad 0 \leq K_n \leq C_1 \quad \text{in } \Omega,$$

and

$$(3.3) \quad \|e^{2u_n}\|_{L^1(\Omega)} \leq C_2$$

for two constants C_1 and C_2 . Then either u_n is bounded in $L_{\text{loc}}^\infty(\Omega)$ or there exists a subsequence of u_n (still denoted by u_n) such that either $u_n \rightarrow -\infty$ uniformly on any compact sets of Ω or the blow-up set S is a set of finite number of points, $u_n \rightarrow -\infty$ uniformly on any compact set of $\Omega \setminus S$, and $K_n e^{2u_n}$ converges to $\sum_i \alpha_i \delta_{p_i}$ with $\alpha_i \geq 2\pi$ and $S = \cup_i \{p_i\}$.

REMARK 3.2. When either K_n is uniformly convergent or converges to a positive constant then Theorem 3.1 can be improved to have $\alpha_i \geq 4\pi$.

PROOF OF THEOREM 1.2. Suppose $\alpha_0 = -(\frac{2-b}{2})$. Since $\alpha_0 = -\frac{2-b}{2}$ can not be achieved by some solution of (1.1) by Remark 2.2, there exists a sequence of solutions of u_n such that the total curvature

$$(3.4) \quad \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx = \frac{2-b}{2} < 1.$$

Since K has a lower positive bound in any compact set of \mathbb{R}^2 , by Theorem 3.1, we have either u_n is uniformly bounded in any compact set or u_n is uniformly convergent to $-\infty$ in any compact set of \mathbb{R}^2 .

STEP 1. We claim that $u_n \rightarrow -\infty$ uniformly in any compact set of \mathbb{R}^2 . Suppose u_n is uniformly bounded in any compact set of \mathbb{R}^2 . By the elliptic estimates, we may assume $u_n \rightarrow u$ in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$. In particular, u satisfies (1.1) and the total curvature

$$\frac{1}{2\pi} \int K(x) e^{2u(x)} dx \leq \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int K(x) e^{2u_n(x)} dx = \frac{2-b}{2},$$

which yields a contradiction by Remark 2.2. Hence, by Theorem 3.1, we have $u_n \rightarrow -\infty$ uniformly in any compact set of \mathbb{R}^2 .

STEP 2. We claim there exists a constant $C > 0$ such that

$$(3.5) \quad K(x) e^{2u_n(x)} \leq C|x|^{-2} \quad \text{for } x \in \mathbb{R}^2.$$

To prove the claim, we assume there exists $x_n \in \mathbb{R}^2$ such that $u_n(x_n) + \frac{(2-b)}{2} \log|x_n| \rightarrow +\infty$. By Step 1, we have $|x_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. Set

$$v_n(y) = u_n(x_n + |x_n|y) + \frac{2-b}{2} \log|x_n|.$$

Then v_n satisfies

$$(3.6) \quad \Delta v_n + K_n(y)e^{2v_n(y)} = 0 \quad \text{in } |y| < \frac{1}{2},$$

where $K_n(y) = |x_n|^b K_n(x_n + |x_n|y)$. By the assumption on K , $0 < \bar{C}_1 \leq K_n(y) \leq \bar{C}_2$ for $|y| < \frac{1}{2}$, and

$$\int_{|y| < \frac{1}{2}} K_n(y)e^{2v_n(y)} dy \leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx < 2\pi.$$

By Theorem 3.1, we conclude that $v_n(0) \leq C$ for some constant C , which yields a contradiction to the assumption.

STEP 3. There exists a positive constant C such that $|\nabla u_n(x)| \leq C|x|^{-1}$ and $|u_n(x) - u_n(y)| \leq C$ for $|x| = |y|$.

In [CLn], we have proved that u_n has the following representation

$$(3.7) \quad u_n(x) = u_n(0) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|y|}{|x-y|} K(y)e^{2u_n(y)} dy.$$

Thus, we have

$$\begin{aligned} |\nabla u_n(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |x-y|^{-1} K(y)e^{2u_n(y)} dy \\ &= \frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \\ &\quad + \frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \end{aligned}$$

By Step 2, the first integral can be estimated by

$$\frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \leq C_1|x|^{-2} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} dy = C_2|x|^{-1}.$$

For the second integral, we have

$$\frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \leq \frac{1}{\pi|x|} \int_{\mathbb{R}^2} K(y)e^{2u_n(y)} dy.$$

Combined these two estimates together, we have

$$|\nabla u_n(x)| \leq C_3|x|^{-1}.$$

Set $w_n(x) = e^{2u_n(x)}$. Then $w_n(x)$ satisfies

$$(3.8) \quad \Delta w_n(x) + 4(Ke^{2u_n} + |\nabla u_n|^2)w_n = 0.$$

Since $K(x)e^{2u_n(x)} + |\nabla u_n|^2 \leq C_4|x|^{-2}$ for some constant C_4 , by Harnack inequality, for any $a \geq 1$, there exists a positive constant $C_5 = C_5(a)$ such that

$$(3.9) \quad \sup_{a^{-1}r \leq |x| \leq ar} w_n(x) \leq C_5 \inf_{a^{-1}r \leq |x| \leq ar} w_n(x).$$

Hence, Step 3 is proved.

STEP 4. For any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $|u_n(x) - u_n(y)| \leq \varepsilon$ for $|x| = |y| \geq R_\varepsilon$ and large n .

Step 4 will be proved by contradiction. Suppose there exist a positive number $\varepsilon_0 > 0$ and x_n, y_n with $r_n = |x_n| = |\bar{x}_n| \rightarrow +\infty$ such that $u_n(\bar{x}_n) - u_n(x_n) \geq \varepsilon_0$. Let

$$v_n(y) = u_n(r_n y) - u_n(x_n).$$

Then v_n satisfies

$$\Delta v_n + K_n(y)e^{2v_n} = 0,$$

where $K_n(y) = e^{2u_n(x_n)} K(r_n y) r_n^2$. By Step 2,

$$(3.10) \quad K_n(y) \leq C_1 e^{2u_n(x_n)} r_n^{2-b} |y|^{-b} \leq C_2 |y|^{-b}.$$

For $|y| \geq 1$, we have

$$(3.11) \quad K_n(y) \geq C_3 e^{2u_n(x_n)} r_n^{2-b} |y|^{-b}.$$

By Step 3 and the Harnack inequality (3.9), $v_n(y)$ is bounded in $L_{\text{loc}}^\infty(\mathbb{R}^2)$. By the elliptic estimates, we may assume $v_n(y) \rightarrow v_0(y)$ in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$. Suppose there exists a subsequence of x_n (still denoted by x_n) such that $\lim_{n \rightarrow +\infty} e^{2u_n(x_n)} r_n^{2-b} = S > 0$, then by (3.10) and (3.11), we may assume $K_n(y) \rightarrow K_0(y)$ weakly in $L_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{0\})$, where $K_0(y)$ satisfies

$$C_1 |y|^{-b} \leq K_0(y) \leq C_2 |y|^{-b}$$

for some positive constants C_1 and C_2 , and $v_0(y)$ satisfies

$$\Delta v_0(y) + K_0(y)e^{2v_0(y)} = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\}.$$

For any $0 < r_0 < r_1$, we have

$$(3.12) \quad \begin{aligned} \int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_0(y)} dy &= \lim_{n \rightarrow +\infty} \int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_n(y)} dy \\ &= \int_{r_0 r_n \leq |y| \leq r_1 r_n} K(x)e^{2u_n(x)} dx \\ &\leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx \\ &\rightarrow \left(\frac{2-b}{2}\right) 2\pi \end{aligned}$$

Thus, the total curvature

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y) e^{2v_0(y)} dy \leq \frac{2-b}{2}.$$

Applying Corollary 1.4 in [CLn], $v_0(y)$ in fact satisfies

$$(3.13) \quad \Delta v_0(y) + K_0(y) e^{2v_0(y)} = 2\pi\beta\delta(0) \quad \text{in } \mathbb{R}^2$$

for some $\beta \in \mathbb{R}$, where $\delta(0)$ is the Dirac measure at the origin, and the function $v_1(y) = v_0(y) - \beta \log |y|$ satisfies

$$(3.14) \quad \Delta v_1(y) + K_0(y) |y|^{2\beta} e^{2v_1(y)} dy = 0 \quad \text{in } \mathbb{R}^2.$$

It is easy to see that

$$\begin{aligned} o(1) + \beta &= \frac{1}{2\pi} \int_{|y|=r} \frac{\partial v_0}{\partial \nu}(y) d\sigma \\ &= \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{|y|=r} \frac{\partial v_n}{\partial \nu}(y) d\sigma \\ &= \lim_{n \rightarrow +\infty} \frac{-1}{2\pi} \int_{|y| \leq r} K_n(y) e^{2v_n(y)} dy \\ &= \lim_{n \rightarrow +\infty} \frac{-1}{2\pi} \int_{|x| \leq rnr} K(x) e^{2u_n(x)} dx, \end{aligned}$$

where $o(1)$ denotes $o(1) \rightarrow 0$ as $r \rightarrow 0$. Thus, putting (3.12) and the above together, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y) |y|^{2\beta} e^{2v_1(y)} dy &= \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y) e^{2v_0(y)} dy \\ &\leq \frac{2-b}{2} + \beta = \frac{2-b+2\beta}{2}. \end{aligned}$$

Obviously, $2-b+2\beta > 0$. Since $K_0(y) |y|^{2\beta} \sim |y|^{-b+2\beta}$ at ∞ , by Remark 2.2, there exists no entire solution (3.14) with the total curvature equal to $\frac{2-b+2\beta}{2}$. Thus, it yields a contradiction. Hence we have proved $\lim_{n \rightarrow +\infty} e^{2u_n(x_n)} r_n^{2-b} = 0$.

Since $e^{2u_n(x_n)} r_n^{2-b} \rightarrow 0$ as $n \rightarrow +\infty$, then $v_0(y)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$. By Step 3,

$$|\nabla v_n(y)| = r_n |\nabla u_n(r_n y)| \leq C |y|^{-1}.$$

By Liouville's theorem, we have

$$v_0(y) = \alpha_0 \log |y| + C,$$

where both α_0 and C are constant. Since $v_0(y)$ is radially symmetric, it obviously yields a contradiction to the assumption. Hence, Step 4 is proved.

STEP 5. Set

$$(3.15) \quad F_n(r) = \int_{B_r} K(x) e^{2u_n(x)} dx,$$

and

$$(3.16) \quad \bar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) ds.$$

Define $\bar{K}_n(r)$ by

$$\bar{K}_n(r) = (2\pi r)^{-1} e^{-2\bar{u}_n(r)} \int_{|x|=r} K(x) e^{2u_n(x)} ds.$$

Differentiating (3.15) and (3.16) with respect to r , we have

$$(3.17) \quad F'_n(r) = (2\pi r) \bar{K}_n(r) e^{2\bar{u}_n(r)}$$

$$(3.18) \quad \bar{u}'_n(r) = \frac{-1}{2\pi r} \int_{B_r} K(x) e^{2u_n(x)} dx = \frac{-F_n(r)}{2\pi r}.$$

Thus, we have

$$(3.19) \quad \begin{aligned} \left(\frac{r^{1-b} F'_n(r)}{\bar{K}_n(r)} \right)' &= (2\pi r^{2-b} e^{2\bar{u}_n(r)})' \\ &= 2\pi [(2-b)r^{1-b} e^{2\bar{u}_n(r)} + 2r^{2-b} e^{2\bar{u}_n(r)} \bar{u}'_n(r)] \\ &= \frac{(2-b)F'_n}{r^b \bar{K}_n} - \frac{F_n F'_n(r)}{\pi r^b \bar{K}_n} \\ &= \frac{-F'_n(r)}{r^b \bar{K}_n} \left[\frac{F_n(r)}{\pi} - (2-b) \right]. \end{aligned}$$

Since $F_n(\infty) > \pi(2-b)$, set r_n to satisfy $F_n(r_n) = \pi(2-b)$. Obviously, $\lim_{n \rightarrow +\infty} r_n = +\infty$. For any $\varepsilon > 0$, by Step 4, there exists $R = R(\varepsilon) > 0$ such that

$$Ae^{2\varepsilon} r^{-b} \leq \bar{K}_n(r) \leq e^{-2\varepsilon} B r^{-b} \quad \text{for } r \geq R_\varepsilon.$$

Hence,

$$\left(\frac{r^{1-b} F'_n(r)}{\bar{K}_n(r)} \right)' \geq \begin{cases} \frac{e^{2\varepsilon}}{B} \left((2-b) - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for } R \leq r \leq r_n, \\ \frac{e^{-2\varepsilon}}{A} \left((2-b) - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for } r \geq r_n. \end{cases}$$

Since $\lim_{r \rightarrow +\infty} \frac{rF'_n(r)}{r^b \bar{K}_n(r)} = 0$ for any n , we have

$$\begin{aligned} -\frac{r^{1-b}F'_n(r)}{\bar{K}_n(r)} \Big|_{r=R} &\geq \frac{e^{-2\varepsilon}}{B} \int_R^{r_n} \left[(2-b) - \frac{F_n(r)}{\pi} \right] F'_n(r) dr \\ &\quad + \frac{e^{2\varepsilon}}{A} \int_{r_n}^{\infty} \left((2-b) - \frac{F_n(r)}{\pi} \right) F'_n(r) dr \\ &= -e^{-2\varepsilon} B^{-1} \left[(2-b)F_n(r) - \frac{F_n^2(r)}{2\pi} \right] \Big|_{r=R} \\ &\quad + \left(\frac{e^{2\varepsilon}}{B} - \frac{e^{-2\varepsilon}}{A} \right) \frac{\pi(2-b)^2}{2} + e^{-2\varepsilon} A^{-1} \left((2-b)F_n(\infty) - \frac{F_n^2(\infty)}{2\pi} \right). \end{aligned}$$

By Step 1, we note that the boundary term at R tends to 0 as $n \rightarrow +\infty$. By letting $n \rightarrow +\infty$ first and then $\varepsilon \rightarrow 0$ the above yields

$$\begin{aligned} 0 &\geq \left(\frac{1}{B} - \frac{1}{A} \right) \frac{\pi(2-b)^2}{2} + \frac{1}{A} \left[(2-b) \lim_{n \rightarrow +\infty} F_n(\infty) - \frac{\lim_{n \rightarrow +\infty} F_n^2(\infty)}{2\pi} \right] \\ &= \frac{\pi(2-b)^2}{2B}, \end{aligned}$$

a contradiction, where $\lim_{n \rightarrow +\infty} F_n(\infty) = (2-b)\pi$ is used. Therefore, the proof of Theorem 1.2 is completely finished. \square

PROOF OF THEOREM 1.3. Suppose u_n is a sequence of solution of (1.1) and satisfies the assumption of Theorem 1.3. By Remark 3.2, we may assume that either u_n is uniformly bounded in any compact set or u_n uniformly converges to $-\infty$ in any compact set of \mathbb{R}^2 . By the the same reasoning of Step 1 and Step 2 of Theorem 1.2, there exists a constant $C > 0$ such that inequalities

$$(3.20) \quad K(x)e^{2u_n(x)} \leq C|x|^{-2},$$

$$(3.21) \quad |\nabla u_n(x)| \leq C|x|^{-1},$$

$$(3.22) \quad |u_n(x) - u_n(y)| \leq C \quad \text{whenever} \quad |x| = |y|$$

hold.

First, we want to prove u_n is bounded in $L_{\text{loc}}^\infty(\mathbb{R}^2)$. Suppose the claim is not true. As before, we want to prove the asymptotic symmetry of u_n , i.e. for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for $|y| = |x| \geq R$, $|u_n(x) - u_n(y)| \leq \varepsilon$. Assume the conclusion is not true. Then there exists $r_n \rightarrow +\infty$ such that $u_n(\bar{x}_n) \geq u_n(x_n) + \varepsilon_0$ with $|\bar{x}_n| = |x_n| = r_n$ for some positive constant $\varepsilon_0 > 0$. Let

$$v_n(y) = u_n(r_n y) - u_n(x_n).$$

Then v_n is bounded in $L_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{0\})$ by Harnack inequality and satisfies

$$\Delta v_n + K_n(y)e^{2v_n} = 0 \quad \text{in } \mathbb{R}^2,$$

where $K_n(y) = e^{2u_n(x_n)}K(r_n y)r_n^2$. By the assumption on K and (3.20) for any $r_0 > 0$, we have for $|y| \geq r_0$,

$$K_n(y) \leq 2e^{2u_n(x_n)}r_n^{2-b}|y|^{-b}$$

for large n . If $\lim_{n \rightarrow +\infty} e^{u_n(x_n)}r_n^{2-b} = 0$, then using (3.21) and the same argument of Step 4 of Theorem 1.2, $v_n(y)$ converges to $v_0(y) = \alpha_0 \log |y| + C_0$ in $L_{\text{loc}}^\infty(\mathbb{R}^2 \setminus \{0\})$ where α_0 and C_0 are constant. Since $v_0(y)$ is radially symmetric, it yields a contradiction.

If $\lim_{n \rightarrow +\infty} e^{2u_n(x_n)}r_n^{2-b} = s > 0$, then $K_n(y) \rightarrow s|y|^{-b}$ uniformly in any compact set of $\mathbb{R}^2 \setminus \{0\}$. Then $v_0(y)$ satisfies

$$\begin{cases} \Delta v_0(y) + s|y|^{-b}e^{2v_0(y)} = \beta\delta(0) & \text{in } \mathbb{R}^2, \\ v_0(y) = \frac{\beta}{2\pi} \log |y| + O(1) & \text{as } y \rightarrow 0, \end{cases}$$

where $\beta \in \mathbb{R}$ and $\delta(0)$ is the Dirac measure. For any $r_0 > 0$,

$$\int_{|y|=r_0} \frac{\partial v_0(y)}{\partial \nu} d\sigma = \lim_{n \rightarrow +\infty} \int_{|y|=r_0} \frac{\partial v_n}{\partial \nu} d\sigma = - \lim_{n \rightarrow +\infty} \int_{|y| \leq r_0} K_n(y)e^{2v_n} dy \leq 0.$$

Thus either $v_0(y)$ is regular at 0 or $v_0(y) \rightarrow +\infty$ as $|y| \rightarrow +\infty$. Since $|y|^{-b}e^{2v_0(y)} \in L^1(\mathbb{R}^2)$ and $v_0(y) = \alpha \log |y| + O(1)$ as $|y| \rightarrow +\infty$, for some $\alpha \in \mathbb{R}$, we have $2\alpha - b < -2$, i.e. $|y|^{-b}e^{2v_0(y)} = o(1)|y|^{-2}$ as $|y| \rightarrow +\infty$. Hence, we can apply the method of moving planes as in [CL] and [CLn] to prove $v_0(y)$ is radially symmetric with respect to the origin, which obviously yields a contradiction. Hence the uniformly asymptotic symmetry of u_n is proved.

To finish the proof of Theorem 1.3, we set

$$F_n(r) = \int_{B_r} K(x)e^{2u_n(x)} dx,$$

and,

$$\bar{K}_n(r) = (2\pi r)^{-1} e^{-2\bar{u}_n(r)} \int_{|x|=r} K(x)e^{2u_n(x)} dx.$$

As in (3.19), we have

$$(3.23) \quad \left(\frac{r^{1-b} F_n'(r)}{\bar{K}_n(r)} \right)' = \frac{-F_n'(r)}{r^b \bar{K}_n(r)} \left[\frac{F_n(r)}{\pi} - (2-b) \right].$$

For any $\varepsilon > 0$, let $R = R(\varepsilon)$ be large such that

$$e^{-3\varepsilon} r^{-b} \leq \bar{K}_n(r) \leq e^{3\varepsilon} r^{-b}$$

holds for $r \geq R$. This immediately follows from the uniformly asymptotic symmetry of u_n and the assumption on K . Let r_n satisfy $F_n(r_n) = \pi(2-b)$. Suppose $\lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx < (2-b)$ first. Then we can follow the same proof as Step 5 in Theorem 1.2 to obtain

$$(2-b)F(\infty) - (2\pi)^{-1}F^2(\infty) \leq 0,$$

where $F(\infty) = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx$. Obviously, the above yields $F(\infty) \geq 2\pi(2-b)$, a contradiction.

Suppose $\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx > 2\pi(2-b)$. Then by (3.23), we have the reverse inequality

$$\left(\frac{r^{1-b} F'_n(r)}{\bar{K}_n(r)} \right)' \leq \begin{cases} e^{3\varepsilon} [(2-b) - F_n(r)/\pi] F'_n(r) & \text{for } R \leq r \leq r_n, \\ e^{-3\varepsilon} [(2-b) - F_n(r)/\pi] F'_n(r) & \text{for } r \geq r_n. \end{cases}$$

Integrating the above and letting $n \rightarrow +\infty$ first and then $\varepsilon \rightarrow 0$, we have

$$(2-b)F(\infty) - \frac{F^2(\infty)}{2\pi} \geq 0,$$

which implies

$$F(\infty) \leq 2\pi(2-b).$$

Obviously, it yields a contradiction. Hence the boundedness of u_n in $L_{\text{loc}}^\infty(\mathbb{R}^2)$ is proved.

To prove (1.9), we may assume $u_n \rightarrow u_0$ in $W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p > 1$. Obviously, u_0 satisfies (1.1) and has a finite total curvature. In particular,

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_0(x)} dx > \frac{2-b}{2}.$$

Hence, there exists $R_0 > 0$, $\varepsilon_0 > 0$ and n_0 such that

$$\frac{1}{2\pi} \int_{B_r} K(x) e^{2u_n(x)} dx > \left(\frac{2-b}{2} + \varepsilon_0 \right)$$

for all $r \geq R_0$ and $n \geq n_0$. Integrating (1.1), we have

$$\frac{d}{dr} \bar{u}_n(r) < - \left(\frac{2-b}{2} + \varepsilon_0 \right) r^{-1}$$

for all $r \geq R_0$ and $n \geq n_0$, where $\bar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) d\sigma$. Thus,

$$\bar{u}_n(r) \leq \bar{u}_n(R_0) - \left(\frac{2-b}{2} + \varepsilon_0 \right) \log r/R_0.$$

Applying the Harnack inequality, we have

$$u_n(x) \leq \bar{u}_n(|x|) + C_1 \leq C_2 - \left(\frac{2-b}{2} + \varepsilon_0 \right) \log r$$

for $r \geq R_0$ and $n \geq n_0$ where C_1 and C_2 are constants independent of n and r . In particular,

$$\int_{|y| \geq r} K(x) e^{2\bar{u}_n(x)} dx \leq C_3 \int_{|y| \geq r} |x|^{-(2+2\varepsilon_0)} dx$$

could be arbitrarily small provided that r is large. Thus, (1.9) follows immediately. And the proof of Theorem 1.3 is finished. \square

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