ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

GIAN-CARLO ROTA
JIANHONG SHEN
BRIAN D. TAYLOR

All polynomials of binomial type are represented by Abel polynomials

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 25, nº 3-4 (1997), p. 731-738
<http://www.numdam.org/item?id=ASNSP_1997_4_25_3-4_731_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXV (1997), pp. 731-738

All Polynomials of Binomial Type Are Represented by Abel Polynomials

GIAN-CARLO ROTA - JIANHONG SHEN

BRIAN D. TAYLOR

1. - Introduction

The umbral calculus began in the Nineteenth Century as a heuristic device whereby a sequence of scalars a_0, a_1, $a_2, \ldots$ would be treated as a sequence of powers $1, \alpha, \alpha^2, \ldots$ of a variable α called an umbra. Numerical and combinatorial identities were guessed by this method, whose foundations remained mysterious. One did not feel the need for justifying the method rigorously, since all identities guessed by umbral method could later be verified directly.

In several previous papers, ending with the work of Rota and Taylor, a rigorous foundation for the umbral calculus was given. The obvious idea was to consider a linear functional on polynomials, called **eval**, so that $\mathbf{eval}(\alpha^n) = a_n$. One says that the umbra α represents the sequence $a_0, a_1, a_2, \ldots$

This obvious idea must be supplemented by the notion of exchangeable umbrae. Two umbrae α and α' are said to be exchangeable when they represent the same sequence.

One realizes the need for exchangeable umbrae when considering the sequence

$$\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} .$$

This sequence cannot be represented using only the umbra α above. It requires two exchangeable umbrae α and α'. In terms of two such umb rae representing the same sequence $a_0, a_1, a_2, \ldots$ we have

$$\mathbf{eval}((\alpha + \alpha')^n) = \sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} .$$

This is the basic idea. The details are easily worked out, and they are given below.

A variety of computations can be carried out using the umbral calculus. The umbral interpretation of functional composition, however, remained unknown — even in the last century.

The purpose of the present work is to give an umbral interpretation of functional composition of formal power series. The basic theory is developed below; plenty of examples can be found in the literature.

We are honored to dedicate this paper to the memory of Ennio De Giorgi.

2. – The classical umbral calculus

In this section we review the *classical umbral calculus* in one variable over a commutative ring $\mathbf{k}$. The ring $\mathbf{k}$ will be assumed to contain the ring of rational numbers Q. The presentation is self contained; further details on the definition and use of the umbral calculus are given in the items listed in the bibliography. Some of the proofs of known results are skipped, but they are easily reconstructed or read from the references.

A classical umbral calculus consists of three types of data:

1. a polynomial ring $\mathbf{k}[\mathcal{A}, x, y]$. The variables belonging to the set $\mathcal{A}$ will be denoted by Greek letters, and are called *umbrae*.
2. a linear functional $\mathbf{eval} : k[\mathcal{A}, x, y] \to k[x, y]$ such that
   (a) $\mathbf{eval}(1) = 1$,
   (b) $\mathbf{eval}(\alpha^i \beta^j \cdots \gamma^k x^n y^r) = x^n y^r \mathbf{eval}(\alpha^i)\mathbf{eval}(\beta^j) \cdots \mathbf{eval}(\gamma^k)$, for distinct umbrae $\alpha, \beta, \dots, \gamma \in \mathcal{A}$ and $i, j, \cdots, k, n, r$ are nonnegative integers.
3. A distinguished umbra, $\varepsilon \in \mathcal{A}$, satisfying $\mathbf{eval}(\varepsilon^i) = \delta_{0,i}$ where δ is the Kronecker delta.

A polynomial $p \in \mathbf{k}[A]$ is called an *umbral polynomial*. Two umbral polynomials $p, q \in \mathbf{k}[\mathcal{A}]$ are said to be *equivalent*, written $p \simeq q$, when $\mathbf{eval}(p) = \mathbf{eval}(q)$.

A sequence $a_0, a_1, a_2, \dots$ in $\mathbf{k}[x]$ is said to be *umbrally represented* by an umbra α when $\alpha^i \simeq a_i$ for all $i \geq 0$. This implies $a_0 = 1$. More generally, an umbral polynomial p, is said to *represent* the sequence $\mathbf{eval}(p^0), \mathbf{eval}(p^1), \mathbf{eval}(p^2), \dots$

Two umbral polynomials p, q are said to be *exchangeable*, written $p \equiv q$, when p and q represent the same sequence in $k[x]$.

The notions of equivalence and exchangeability are extended coefficientwise to formal power series $\mathbf{k}[\mathcal{A}][[t]]$ with coefficients in $\mathbf{k}[\mathcal{A}]$.

For any polynomial or formal power series $f = \sum_i a_i t^i$ with coefficients a_i in $\mathbf{k}[\mathcal{A}]$, we define the *support*, $\mathbf{supp}(f)$, of f to be the set of all umbrae α for which there exists an a_i in which α appears to a positive power in a monomial with nonzero coefficient.

Two umbral polynomials (or formal power series) are said to be *unrelated* when their supports have empty intersection.

The following fact is easily established; a proof is given in [4].

PROPOSITION 1. *Given $f(t) \in \mathbf{k}[\mathcal{A}, x, t]$, if p, q are umbral polynomials unrelated to $f(t)$, then $f(p) \simeq f(q)$.* □

An important (and easy) application of the preceding proposition is the following. If $p, r \in \mathbf{k}[\mathcal{A}, x]$ are unrelated and if $q, s \in \mathbf{k}[\mathcal{A}, x]$ are unrelated, then $p \simeq q$ and $r \simeq s$ implies $pr \simeq qs$ and similarly, $p \equiv q$ and $r \equiv s$ implies $pr \equiv qs$ and $p + r \equiv q + s$. These assertions are not, in general, true when the supports of the umbral polynomials are not disjoint. It is always true that $p + r \simeq q + s$.

A *sequence* of polynomials $p_0(x), p_1(x), \ldots$ will always denote a sequence of polynomials with coefficients in $\mathbf{k}$ such that $p_0(x) = 1$, $p_1(x) = x$ and $p_n(x)$ is of degree n for every positive integer n. An umbra χ such that $\chi^n \simeq p_n(x)$, where $p_0(x), p_1(x), \ldots$ is a polynomial sequence, is said to be a *polynomial umbra*. By contrast, an umbra α such that $\mathbf{eval}(\alpha^n)$ is an element of $\mathbf{k}$ for every non negative integer n will be said to be a *scalar* umbra.

Two umbrae α and β are said to be *inverse* when $\alpha + \beta \equiv \varepsilon$.

We will assume that the umbral calculus we deal with is *saturated*. A saturated umbral calculus satisfies the following requirements. Every sequence $a_0, a_1, a_2, \ldots$ in $\mathbf{k}[x]$ is represented by infinitely many distinct (and thus exchangeable) umbrae. Necessarily, in a saturated umbral calculus, any umbra α has infinitely many (exchangeable) inverses; it is sometimes useful to denote one such inverse umbra by $-1 \cdot \alpha$.

For any umbra α and every integer n, the *auxiliary umbra*, $n \cdot \alpha$ stands for one of the following umbrae:

1. when $n = 0$, the auxiliary umbra $0 \cdot \alpha$ denotes an umbra exchangeable with ε;
2. when $n > 0$, the auxiliary umbra $n \cdot \alpha$ denotes an umbra exchangeable with the sum $\alpha' + \alpha'' + \cdots + \alpha'''$ where $\alpha', \ldots, \alpha'''$ are n distinct umbrae each exchangeable with α;
3. when $n < 0$, the auxiliary umbra $n \cdot \alpha$ denotes the sum of n exchangeable umbrae $\beta + \beta' + \beta'' + \cdots + \beta'''$, each of them being an inverse umbra to α.

Expressions such as $m \cdot n \cdot \alpha$ will not be used. It may be shown that saturated umbral calculi exist.

3. – Operators and polynomial sequences

We summarize some known facts of finite operator calculus, rewritten in the language of umbrae.

We denote by D the derivative operator with respect to the variable x on the polynomial rings $\mathbf{k}[\mathcal{A}, x]$ and $\mathbf{k}[x]$. Thus, for $a \in \mathbf{k}$, the exponential e^{aD} is the linear operator

$$e^{aD} p(x) = p(x + a)$$

defined on the module $\mathbf{k}[x]$.

A linear operator T on $k[x]$ which commutes with the operator e^{aD} for all $a \in \mathbf{k}$ is said to be *shift-invariant*. If in addition the operator T has the property that $T1 = 1$, then the operator is said to be *unital*. Every shift-invariant operator S on $k[x]$ is of the form $S = cD^k T$, where c is an arbitrary constant and where T is a unital operator.

For every unital operator T, there exists an umbra α, unique up to exhangeability, such that

$$Tp(x) \simeq p(x + \alpha)$$

for every polynomial $p(x) \in k[x]$. Such a unital operator satisfies $T = f(D)$, where $f(t)$ is the formal power series equivalent to $e^{\alpha t}$.

A shift-invariant operator of the form $Q = DT$, where T is a unital operator, is said to be a *delta* operator. We write $Q = Df(D)$.

To every delta operator Q there exists a unique sequence of polynomials $p_0(x), p_1(x), \dots$ such that $Qp_n(x) = np_{n-1}(x)$ and $p_n(0) = 0$ for $n > 0$. Such a sequence is called the *basic sequence* of the delta operator Q. A sequence of polynomials is said to be *of binomial type* whenever $p_0(x) = 1$, $p_1(x) = x$, and

$$p_n(x + a) = \sum_i \binom{n}{i} p_i(x) p_{n-i}(a) \tag{1}$$

for all $a \in k$.

Every basic sequence is of binomial type. Conversely, every sequence of polynomials $p_i(x)$ of binomial type such that $p_0(x) = 1$ and $p_1(x) = x$ is the basic sequence for a unique delta operator.

The most important sequence of binomial type is the sequence of Abel polynomials, namely, the sequence $p_n(x) = x(x + na)^{n-1}$ for $a \in Q$. The sequence of Abel polynomial is the basic sequence associated to the operator De^{-aD}. Because of the importance of this fact, we review the proof, which amounts to the following computation:

$$\begin{aligned} De^{-aD} x(x + na)^{n-1} &= \big(x + (n-1)a\big)^{n-1} + (n-1)(x - a)\big(x + (n-1)a\big)^{n-2} \\ &= \big(x + (n-1)a + (n-1)(x - a)\big)\big(x + (n-1)a\big)^{n-2} \\ &= nx\big(x + (n-1)a\big)^{n-2}. \end{aligned}$$

It follows from the above verification that the sequence of Abel polynomials is a sequence of polynomials of binomial type. The identity stating that the sequence of polynomials $p_n(x) = x(x + na)^{n-1}$ is of binomial type is due to Abel.

In closing this section, we mention the central notion in the present work. Let $p_n(x), q_n(x)$ be two sequences of polynomials in $\mathbf{k}[x]$, represented by the umbrae χ, ξ respectively. Their *umbral composition*, $p_n(\mathbf{q})$ is the sequence of polynomials umbrally defined as $p_n(\xi)$, that is, the sequence of polynomials $r_n(x)$ defined as $r_n(x) = \mathbf{eval}(p_n(\xi))$, or equivalently the polynomial sequence $r_n(x) \in \mathbf{k}[x]$ such that $r_n(x) \simeq p_n(\xi)$.

4. – Main result

We begin by proving two technical identities that will be used in the main theorem below.

PROPOSITION 2. *If α, γ are any two distinct umbrae, and if $p(t)$ is any polynomial in $k[t]$, then*

$$\left(\sum_{i=1}^{n}\alpha^{\{i\}}\right) p\left(\gamma+\sum_{i=1}^{n}\alpha^{\{i\}}\right) \simeq (n\alpha)p(\gamma+\alpha+(n-1)\cdot\alpha'),$$

where α' and each $\alpha^{\{i\}}$, for $1 \le i \le n$, is an umbra exchangeable with α.

PROOF. We have

$$\begin{aligned}\alpha^{\{j\}}p\left(\gamma+\sum_{i=1}^{n}\alpha^{\{i\}}\right) &= \alpha^{\{j\}}p\left(\gamma+\alpha^{\{j\}}+\sum_{i=1,i\neq j}^{n}\alpha^{\{i\}}\right)\\ &\simeq \alpha^{\{j\}}p\left(\gamma+\alpha^{\{j\}}+(n-1)\cdot\alpha'\right)\\ &\simeq \alpha\, p(\gamma+\alpha+(n-1)\cdot\alpha')\,.\end{aligned}$$

Summing over the variable j, we obtain

$$\left(\sum_{i=1}^{n}\alpha^{\{i\}}\right) p\left(\gamma+\sum_{i=1}^{n}\alpha^{\{i\}}\right) \simeq (n\alpha)p(\gamma+\alpha+(n-1)\cdot\alpha'),$$

as desired. □

PROPOSITION 3. *For any polynomial $p(t)$ in $k[t]$, for any umbra α, and for any umbra γ inverse to α, the following identity holds,*

$$(n\gamma+n\cdot\alpha)p(\gamma+n\cdot\alpha)\simeq 0\,.$$

PROOF. The following calculation verifies the identity, using the preceding proposition.

$$\begin{aligned}(n\gamma+n\cdot\alpha)p(\gamma+n\cdot\alpha) &= n\gamma p(\gamma+n\cdot\alpha)+(n\cdot\alpha)p(\gamma+n\cdot\alpha)\\ &\simeq n\gamma p(\gamma+n\cdot\alpha)+\left(\sum_{i=1}^{n}\alpha^{\{i\}}\right) p\left(\gamma+\sum_{i=1}^{n}\alpha^{\{i\}}\right)\\ &\simeq n\gamma p(\gamma+n\cdot\alpha)+(n\alpha)p(\gamma+\alpha+(n-1)\cdot\alpha')\\ &\simeq n(\gamma+\alpha)p(\gamma+\alpha+(n-1)\cdot\alpha'),\end{aligned}$$

where α' is an umbra exchangeable with α. But

$$n(\gamma+\alpha)p(\gamma+\alpha+(n-1)\cdot\alpha')$$

is a polynomial in $\gamma+\alpha$ (its coefficeints lie in $\mathbf{k}[(n-1)\cdot\alpha']$) with zero constant term. Since $\gamma+\alpha\equiv 0$, the conclusion follows. □

THEOREM 4. *If α is a scalar umbra, then the polynomial sequence $p_n(x) \simeq x(x+n.\alpha)^{n-1}$ is of binomial type.*

Conversely, for any sequence $p_n(x)$ of binomial type, there exists a scalar umbra α such that $p_n(x) \simeq x(x+n\cdot\alpha)^{n-1}$ for all $n \geq 0$.

PROOF. Let γ be an inverse umbra to the umbra α. Define a delta operator Q by setting $Q \simeq De^{\gamma D}$. Clearly

$$Qp(x) \simeq De^{\gamma D}p(x) = p'(x+\gamma),$$

for all $p(x) \in \mathbf{k}[A, x]$ unrelated to γ. We show that

$$De^{\gamma D}x(x+n.\alpha)^{n-1} \simeq nx\big(x+(n-1).\alpha\big)^{n-2}.$$

Indeed, by the preceding proposition we have

$$\begin{aligned}
De^{\gamma D}x(x+n.\alpha)^{n-1} &= D\,(x+\gamma)(x+n\cdot\alpha+\gamma)^{n-1}\\
&= (n-1)(x+\gamma)(x+n\cdot\alpha+\gamma)^{n-2}+\\
&\quad+(x+n\cdot\alpha+\gamma)^{n-1}\\
&= [n(x+\gamma)+n\cdot\alpha]\,(x+n\cdot\alpha+\gamma)^{n-2}\\
&\simeq nx\big(x+(n-1)\cdot\alpha\big)^{n-2}+\\
&\quad+(n\gamma+n\cdot\alpha)(x+\gamma+n\cdot\alpha)^{n-2}\\
&\simeq nx\big(x+(n-1)\cdot\alpha\big)^{n-2}.
\end{aligned}$$

The converse is immediate. □

COROLLARY 5. *Let α be a scalar umbra. Let χ be a polynomial umbra, representing a sequence of polynomials of binomial type in $k[x]$. Then the sequence of polynomials umbrally represented by $\chi(\chi+n\cdot\alpha)^{n-1}$ is also a sequence of polynomials of binomial type.*

PROOF. The verification that the sequence of polynomials $\chi(\chi+n\cdot\alpha)^{n-1}$ is of binomial type is carried out as in the preceding theorem, replacing the derivative relative to x by the partial derivative D_χ relative to χ. □

COROLLARY 6. *In the notation of the preceding corollary, suppose that the sequence of polynomials umbrally represented by χ^n ($n = 0, 1, 2, \dots$) is the basic sequence of the delta operator $f(D)$, and say that $De^{-1\cdot\alpha D} \simeq g(D)$. Then the sequence of polynomials umbrally represented by $\chi(\chi+n\cdot\alpha)^{n-1}$ is the basic sequence of the delta operator $g(f(D))$.* □

COROLLARY 7. *Suppose that $p_n(x)$ is a sequence of binomial type associated to a delta operator $P \in \mathbf{k}[[D]]$. If $Q \in \mathbf{k}[[D]]$ is equivalent (as a formal power series) to $Pe^{-1\cdot\alpha D}$, and if*

$$q_n(x) \simeq \frac{x}{x+n\cdot\alpha}p_n(x+n\cdot\alpha)\,,$$

then $q_n(x)$ is a sequence of binomial type and it is the basic sequence associated to the operator Q.

PROOF. Immediate from Theorem 4. If $p_n(x) \simeq x(x+n\cdot\beta)^{n-1}$ for some umbra β, then the sequence associated to Q is $x(x+n\cdot\beta+n\cdot\alpha)^{n-1}$. □

COROLLARY 8. *Suppose that $\chi^n \simeq x(x+n\cdot\alpha)^{n-1}$ for all n. If $x^n \simeq \chi(\chi+n\cdot\beta)^{n-1}$, then*

$$n(-1\cdot\alpha)^{n-1} \simeq (n\cdot\beta)^{n-1}\,.$$

PROOF. If $P \simeq De^{-1.\alpha D}$ is the delta operator of which χ^n is the basic sequence, apply the operator P to the left-hand side and D_χ to the right-hand side of $x^n \simeq \chi(\chi+n\cdot\beta)^{n-1}$. Then set $x=0$. □

The last corollary is the Lagrange inversion formula. More explicitly, if $p(t), q(t) \in \mathbf{k}[[t]]$ with $q(0)=0$ and $p(q(t))=t$, we have

$$[t^{n-1}]\left(\frac{t}{q(t)}\right)^n = n[t^n]p(t)\,. \tag{2}$$

Indeed, suppose $p, q \in \mathbf{k}[[t]]$ are defined by $q(t) \simeq te^{-1\cdot\beta t}$ and $p(t) \simeq te^{-1\cdot\alpha t}$ and with α, β as in the preceding corollary. We have

$$(n\cdot\beta)^{n-1} \simeq (n-1)![t^{n-1}]\left(\frac{t}{q(t)}\right)^n \text{ and } (-1\cdot\alpha)^{n-1} \simeq (n-1)![t^{n-1}]\frac{p(t)}{t}\,,$$

and it suffices to show that if $p(q(t)) = t$, then $x^n \simeq \chi(\chi+n\cdot\beta)^{n-1}$. By Corollary 6 we know that $p(q(D))$ is the delta operator associated to the sequence of binomial type whose nth element is equivalent to $\chi(\chi+n\cdot\beta)^{n-1}$. But since this operator is D, $\chi(\chi+n\cdot\beta)^{n-1} \simeq x^n$.

Other versions of the Lagrange inversion formula are similarly proved.

EXAMPLE. The sequence of polynomials $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is of binomial type. By the preceding theorem, there exists an umbra β such that

$$(x)_n \simeq x(x+n.\beta)^{n-1}$$

for $n=0,1,2,\ldots$. Furthermore, the sequence $(x)_n$ is the basic sequence for the difference operator $\Delta = e^D - I$, i.e. $\Delta p(x) = p(x+1) - p(x)$. By the proof of the above theorem, $\Delta \simeq De^{-1.\beta D}$. Therefore,

$$De^{-1.\beta D} \simeq e^D - I\,,$$

$$e^{\beta D} \simeq \frac{D}{e^D - I}\,,$$

$$e^{\beta t} \simeq \frac{t}{e^t - 1}\,.$$

Hence $\beta^n \simeq B_n$ for all n, where B_n is the n-th Bernoulli number.

REFERENCES

[1] N. RAY, *All binomial sequences are factorial*, unpublished manuscript.

[2] S. ROMAN – G.-C. ROTA, *The umbral calculus*, Adv. Math. **27** (1978), 95-188.

[3] G.-C. ROTA – D. KAHANER – A. ODLYZKO, *Finite operator calculus*, J. Math. Anal. Appl. **42** (1973), 685-760.

[4] G.-C. ROTA – B. D. TAYLOR, *An introduction to the umbral calculus*, in *Analysis, Geometry and Groups: A Riemann Legacy Volume* Hadronic Press, Palm Harbor, FL, 1993, 513-525.

[5] G.-C. ROTA – B. D. TAYLOR, *The classical umbral calculus*, SIAM J. Math. Anal. **25** (1994), 694-711.

[6] B. D. TAYLOR, *Difference equations via the classical umbral calculus*, in *Proceedings of the Rotafest and Umbral Calculus Workshop* Birkhauser, Boston, to appear.

MIT, Cambridge MA 02139, USA