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Harmonic Maps on Planar Lattices

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Abstract. We show that a sequence of harmonic maps from a square, two-dimensional lattice of mesh-size h to a compact Riemannian manifold N with uniformly bounded energy as $h \to 0$ weakly accumulates at a harmonic map $u: T^2 \to N$ on the flat two-dimensional torus.

1. – Introduction

Let N be a smooth, compact Riemannian manifold without boundary of dimension k. By Nash's embedding theorem we may assume that $N \subset \mathbb{R}^n$ isometrically for some n. We are interested in understanding the relation between (smooth) harmonic maps $u: T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to N \subset \mathbb{R}^n$ on the torus, characterized as critical points of the energy

$$E(v) = \int_{T^2} e(v)\, dx \tag{1}$$

with density

$$e(v) = \frac{1}{2}|\nabla v|^2 \tag{2}$$

subject to the "target constraint" $v(T^2) \subset N$, and their counterparts on a discrete domain.

Our main result in this paper states that harmonic maps with uniformly bounded energy on a lattice, as the mesh-size $h \to 0, h^{-1} \in \mathbb{N}$, weakly accumulate at a harmonic map on the torus.

The above result may be of relevance for numerical purposes and may have implications for questions regarding existence of harmonic maps under constraints.

In fact, in a sequel to this paper [12] we will use a spatially discrete ansatz to give an alternative proof for the existence of global weak solutions to the Cauchy problem for wave maps on $(1+2)$-dimensional Minkowski space, established in Müller-Struwe [11] by a different method.

A core ingredient in the analysis - both in the stationary and in the time-dependent case - are the weak compactness results of Freire-Müller-Struwe [7], [8] which make contact with work of Bethuel [1], [2], Evans [5], Hélein [9] and stress the importance of Hardy space estimates for Jacobians, due to Coifman-Lions-Meyer-Semmes [3], and the $\mathcal{H}^1$-BMO-duality, due to Fefferman-Stein [6].

2. – Notation

Denote as $T = \mathbb{R}^2/\mathbb{Z}^2$ the flat 2-dimensional torus, and let $x = (x^1, x^2)$ denote a generic point in T; also let $\underline{e}_\alpha, \alpha = 1, 2$, denote the standard basis for $\mathbb{R}^2$.

For $h > 0$ with $h^{-1} \in \mathbb{N}$ consider the lattice $T_h = (h\mathbb{Z})^2/\mathbb{Z}^2$ with generic point $x_h = (x_h^1, x_h^2)$. For $l \in \mathbb{N}$ and $x_h \in T_h$ also let

$$Q_{lh}(x_h) = \{x \in T; x_h^\alpha \leq x^\alpha < x_h^\alpha + lh, \alpha = 1, 2\}$$

denote the square of edge-length lh with lower left corner x_h and for $x \in T$ denote as $[x]_h$ the unique point $[x]_h \in T_h$ such that $x \in Q_h([x]_h)$.

Given a discretely defined map $u^h : T_h \to \mathbb{R}^n$, we may extend u^h to T by letting

$$u^h(x) = u^h([x]_h) \quad \text{for} \quad x \in T. \tag{3}$$

The forward and backward difference quotients in direction $\underline{e}_\alpha$, defined as

$$\partial_\alpha^{\pm h} u^h(x) = \frac{u^h(x \pm h\underline{e}_\alpha) - u^h(x)}{\pm h}$$

for $x \in T_h, \alpha = 1, 2$, then also trivially satisfy the relation

$$\partial_\alpha^{\pm h} u^h(x) = \partial_\alpha^{\pm h} u^h([x]_h) \tag{4}$$

for $x \in T$.

Finally, we also introduce forward and backward means

$$m_\alpha^{\pm h} u^h(x) = \frac{u^h(x \pm h\underline{e}_\alpha) + u^h(x)}{2} \equiv m_\alpha^{\pm h} u^h([x]_h), \tag{5}$$

and translates

$$\tau_\alpha^{\pm h} u^h(x) = u^h(x \pm h\underline{e}_\alpha) \equiv \tau_\alpha^{\pm h} u^h([x]_h) \tag{6}$$

for $x \in T, \alpha = 1, 2$.

Observe that, in particular, there holds

$$\partial_\alpha^{-h}\left(\tau_\alpha^h u^h\right) = \tau_\alpha^h\left(\partial_\alpha^{-h} u^h\right) = \partial_\alpha^h u^h; \tag{7}$$

moreover, we have

$$\partial_\alpha^h \partial_\alpha^{-h} = \partial_\alpha^{-h} \partial_\alpha^h \tag{8}$$

for any $\alpha = 1, 2$.

3. – Difference calculus

For completeness, we quickly review the basic formulas from difference calculus that we will need.

3.1. – Product rule

Let $u^h, v^h : T_h \to \mathbb{R}$. Then

$$\begin{aligned}\partial_\alpha^h \left(u^h v^h\right) &= \partial_\alpha^h u^h \tau_\alpha^h v^h + u^h \partial_\alpha^h v^h \\ &= \partial_\alpha^h u^h v^h + \tau_\alpha^h u^h \partial_\alpha^h v^h \\ &= \partial_\alpha^h u^h m_\alpha^h v^h + m_\alpha^h u^h \partial_\alpha^h v^h,\end{aligned} \tag{9}$$

and similarly for $\partial_\alpha^{-h}, \alpha = 1, 2$.

In particular, by (7) we have

$$\partial_\alpha^h \left(\partial_\alpha^{-h} u^h v^h\right) = \partial_\alpha^h \partial_\alpha^{-h} u^h v^h + \partial_\alpha^h u^h \partial_\alpha^h v^h = \partial_\alpha^{-h} \left(\partial_\alpha^h u^h \tau_\alpha^h v^h\right). \tag{10}$$

In view of (10), later we will be able to avoid unnecessary shifting of arguments by working with backward differences.

3.2. – Discrete integration

For $u^h : T_h \to \mathbb{R}$ we define the integral of u^h as

$$\int_{T_h} u^h = h^2 \sum_{x \in T_h} u^h(x) = \int_T u^h \, dx,$$

where in the latter integral u^h denotes the piecewise constant extension of u^h as in (3). Obviously, we have

$$\int_{T_h} \partial_\alpha^h u^h = 0, \int_{T_h} \tau_\alpha^h u^h = \int_{T_h} u^h$$

for any $u^h : T_h \to \mathbb{R}$, $\alpha = 1, 2$. Hence from (9) and (7) we obtain the following formula for integrating by parts

$$\begin{aligned}0 = \int_{T_h} \partial_\alpha^h \left(u^h v^h\right) &= \int_{T_h} \partial_\alpha^h u^h \tau_\alpha^h v^h + \int_{T_h} u^h \partial_\alpha^h v^h \\ &= \int_{T_h} \partial_\alpha^{-h} u^h v^h + \int_{T_h} u^h \partial_\alpha^h v^h;\end{aligned} \tag{11}$$

in particular,

$$\int_{T_h} \partial_\alpha^h u^h \partial_\alpha^h v^h = -\int_{T_h} \partial_\alpha^{-h} \partial_\alpha^h u^h v^h.$$

3.3. – Dirichlet integral

For u^h as above, denote its energy density at a point $x \in T_h$ as

$$e_h(u^h)(x) = \frac{1}{4} \sum_{\alpha=1,2} \left\{ |\partial_\alpha^h u^h(x)|^2 + |\partial_\alpha^{-h} u^h(x)|^2 \right\} \tag{12}$$

and define

$$E_h(u^h) = \int_{T_h} e_h(u^h). \tag{13}$$

We compute the first variation of E_h at u^h in direction v^h as

$$\begin{aligned} \langle dE_h(u^h), v^h \rangle &= \frac{d}{d\varepsilon} E_h(u^h + \varepsilon v^h)_{|\varepsilon=0} \\ &= \frac{1}{2} \sum_\alpha \int_{T_h} \left\{ \partial_\alpha^h u^h \partial_\alpha^h v^h + \partial_\alpha^{-h} u^h \partial_\alpha^{-h} v^h \right\} \\ &= \sum_\alpha \int_{T_h} \partial_\alpha^h u^h \partial_\alpha^h v^h = - \int_{T_h} \Delta^h u^h v^h, \end{aligned} \tag{14}$$

where $\Delta^h = \sum_\alpha \partial_\alpha^{-h} \partial_\alpha^h$ is the discrete (5-point) Laplacian.

3.4. – Exterior calculus

Differential forms on T_h can be most easily expressed in terms of the standard basis $dx^\alpha, \alpha = 1, 2, dx^1 \wedge dx^2$, respectively. A 1-form φ^h on T_h then can be identified with a pair of functions $\varphi_\alpha^h, \alpha = 1, 2$, such that $\varphi^h = \sum_\alpha \varphi_\alpha^h dx^\alpha$; similarly a 2-form b^h on T_h can be identified with a function β^h such that $b^h = \beta^h dx^1 \wedge dx^2$.

Two 1-forms $\varphi^h = \sum_\alpha \varphi_\alpha^h dx^\alpha$ and $\psi^h = \sum_\alpha \psi_\alpha^h dx^\alpha$ are contracted by letting

$$\varphi^h \cdot \psi^h = \sum_\alpha \varphi_\alpha^h \psi_\alpha^h;$$

also let

$$\varphi^h \cdot \varphi^h = |\varphi^h|^2.$$

The Hodge $*$-operator acts on the basis elements as

$$*1 = dx^1 \wedge dx^2, *dx^1 = dx^2, *dx^2 = -dx^1, *dx^1 \wedge dx^2 = 1,$$

and $*$ is linear with respect to multiplication by functions. In particular, there holds

$$\varphi^h \wedge (*\varphi^h) = |\varphi^h|^2 dx^1 \wedge dx^2$$

and

$$**\varphi^h = (-1)^p \varphi^h$$

for any p-form φ^h on T_h.

For a function $u^h\colon T_h \to \mathbb{R}$ or a 1-form $\varphi^h = \sum_\alpha \varphi^h_\alpha dx^\alpha$ the discrete differential is defined as

$$d^{\pm h} u^h = \sum_\alpha \partial^{\pm h}_\alpha u^h dx^\alpha,$$
$$d^{\pm h} \varphi^h = \sum_{\alpha,\beta} \partial^{\pm h}_\alpha \varphi^h_\beta dx^\alpha \wedge dx^\beta = \left(\partial^{\pm h}_1 \varphi^h_2 - \partial^{\pm h}_2 \varphi^h_1\right) dx^1 \wedge dx^2.$$

The co-differential operator is given by

$$\delta^{\pm h} = * \circ d^{\mp h} \circ *.$$

Note that for any $u^h\colon T_h \to \mathbb{R}$ and any 1-form φ^h on T_h we have

$$\int_{T_h} d^{\pm h} u^h \cdot \varphi^h = -\int_{T_h} u^h \delta^{\pm h} \varphi^h;$$

that is, $-\delta^{\pm h}$ is the adjoint of $d^{\pm h}$ with respect to the inner product on forms defined by contraction and the L^2-inner product on T_h.

Moreover, a direct computation shows that

$$d^h \circ d^h = 0, \delta^h \circ \delta^h = 0.$$

The Laplacian on p-forms (with the analysts' sign convention) is defined as

$$\Delta^h = d^h \circ \delta^h + \delta^h \circ d^h.$$

By (8) there holds $\Delta^h = \Delta^{-h}$. Moreover, it is easy to check that Δ^h acts as a diagonal operator with respect to the standard basis on forms; in particular, for a 1-form $\varphi^h = \sum_\alpha \varphi^h_\alpha dx^\alpha$ we have

$$\Delta^h \varphi^h = \sum_\alpha (\Delta^h \varphi^h_\alpha) dx^\alpha.$$

3.5. – Hodge decomposition

As in the continuous case, the following result holds.

PROPOSITION 3.1. *Any 1-form* $\varphi^h = \sum_\alpha \varphi_\alpha dx^\alpha$ *may be decomposed uniquely as*

$$\varphi^h = d^h a^h + \delta^h b^h + c^h,$$

where

$$\int_{T_h} a^h = 0, \int_{T_h} b^h = 0, d^h c^h = 0, \delta^h c^h = 0. \tag{15}$$

PROOF. Letting a^h, b^h be the unique solutions to the equations

$$\Delta^h a^h = \delta^h \varphi^h,\ \Delta^h b^h = d^h \varphi^h, \tag{16}$$

normalized by (15), for the remainder $c^h = \varphi^h - d^h a^h - \delta^h b^h$ we obtain

$$d^h c^h = d^h \varphi^h - \Delta^h b^h = 0,\ \delta^h c^h = \delta^h \varphi^h - \Delta^h a^h = 0,$$

as claimed. □

REMARK 3.2. Solving (16) for a^h can be achieved, for instance, by minimizing the functional

$$F(a^h) = \int_{T_h} \left\{ e_h(a^h) + a^h \delta^h \varphi^h \right\},$$

confer (14), and similarly for b^h.

4. – Interpolation and discretization

In addition to the trivial extension of a map $u^h\colon T_h \to \mathbb{R}$ to the torus, defined by 3, we introduce the bilinear extension of u^h, defined by letting

$$\overline{u}^h(x) = u^h(x) + \sum_\alpha \xi^\alpha \partial_\alpha^h u^h(x) + \xi^1 \xi^2 \partial_1^h \partial_2^h u^h(x),$$

for $x = [x]_h + \xi \in T$.

The following result is immediate from the definition.

LEMMA 4.1. *We have* $\overline{u}^h \in H^{1,2} \cap L^\infty(T)$*, and with a uniform constant* C *there holds*

i) $||\overline{u}^h - u^h||^2_{L^\infty\left(Q_h(x_h)\right)} \leq C \int_{Q_{2h}(x_h)} e_h(u^h),$ *for all* $x_h \in T_h$;

ii) $||\overline{u}^h - u^h||^2_{L^2(T)} \leq C h^2 E_h(u^h)$;

iii) $C^{-1} E_h(u^h) \leq E(\overline{u}^h) \leq C E(u^h)$.

In view of Lemma 4.1 iii) we will say that $u^h \rightharpoonup u$ weakly in $H^{1,2}(T)$ as $h \to 0$ if $\overline{u}^h \rightharpoonup u$ weakly in $H^{1,2}(T)$, and similarly for vector-valued maps.

Observe, however, that for $u^h: T_h \to N \subset \mathbb{R}^n$ the range of $\overline{u}^h$ in general will not lie in N. On the other hand, we have

LEMMA 4.2. *Suppose $u^h \rightharpoonup u$ weakly in $H^{1,2}(T;\mathbb{R}^n)$ as $h \to 0$, where $u^h: T_h \to N$. Then $u(x) \in N$ for almost every $x \in T$.*

PROOF. For $\delta > 0, h > 0$ let

$$\Sigma_h^\delta = \left\{ x \in T_h; \, \|u^h - \overline{u}^h\|_{L^\infty\left(Q_h(x)\right)} \geq \delta \right\}.$$

By Lemma 4.1 i) then there holds

$$\delta |\Sigma_h^\delta| \leq C E_h(u^h) \leq C < \infty,$$

and hence

$$\mathcal{L}^2\left(\left\{x \in T; \operatorname{dist}\left(\overline{u}^h(x), N\right) \geq \delta\right\}\right) \leq C\delta^{-1}h^2.$$

Here, $\mathcal{L}^2$ denotes 2-dimensional Lebesgue measure. Since $\overline{u}^h \rightharpoonup u$ weakly in $H^{1,2}(T;\mathbb{R})$ and hence strongly in $L^2(T;\mathbb{R}^n)$, after passing to a sub-sequence, if necessary, we may assume that $\overline{u}^h \to u$ almost everywhere as $h \to \infty$. Thus, for any $\delta > 0$ we infer

$$\mathcal{L}^2\left(\left\{x \in T; \operatorname{dist}\left(u(x), N\right) \geq \delta\right\}\right) = 0,$$

as claimed. □

In contrast to interpolating functions $u^h: T_h \to \mathbb{R}$, discretizing functions $\varphi \in H^{1,2}(T)$ is somewhat subtle, as such maps, for instance, are only defined pointwise almost everywhere. Moreover, interpolating the discretized map φ should recover the regularity properties of φ as much as possible.

Of the many possible choices we define as a discretized function φ the map

$$\varphi^h(x) = h^{-2} \int_{Q_h(x)} \varphi \, dx, x \in T_h.$$

Note that if φ is the trivial extension of a map $\psi^h: T_h \to \mathbb{R}$, defined by (3), then $\varphi^h = \psi^h$; however, in general, and even if φ is piecewise bilinear, $\varphi \neq \overline{\varphi^h}$, the bilinearly interpolated discretized map.

LEMMA 4.3. *Let $\varphi \in H^{1,2}(T)$. Then with a uniform constant C there holds*

i) $\|\varphi^h - \varphi\|^2_{L^2(T)} \leq Ch^2 E(\varphi)$;

ii) $\|\partial_\alpha^h \varphi^h - \partial_\alpha \varphi\|_{L^2(T)} \to 0$ *as* $h \to 0$.

PROOF. i) Estimating, for $x \in T$,

$$|\varphi^h(x) - \varphi(x)|^2 \leq h^{-2} \int_{Q_h\left([x]_h\right)} |\varphi(y) - \varphi(x)|^2 \, dy,$$

we obtain

$$\begin{aligned} \|\varphi^h - \varphi\|^2_{L^2(T)} &\leq h^{-2} \int_T \int_{Q_h\left([x]_h\right)} |\varphi(y) - \varphi(x)|^2 \, dy dx \\ &\leq \int_T \int_{Q_h\left([x]_h\right)} \int_0^1 |\nabla\varphi\big(x + \vartheta(y-x)\big)|^2 \, d\vartheta dy dx \leq Ch^2 E(\varphi). \end{aligned}$$

ii) By the Lebesgue differentiability theorem

$$\begin{aligned} \partial^h_\alpha \varphi^h(x) &= h^{-2} \int_{Q_h\left([x]_h\right)} \frac{\varphi(y + h\underline{e}_\alpha) - \varphi(y)}{h} \, dy \\ &= h^{-3} \int_0^h \int_{Q_h\left([x]_h\right)} \partial_\alpha \varphi(y + \vartheta \underline{e}_\alpha) \, d\vartheta dy \\ &\to \partial_\alpha \varphi(x) \quad \text{as} \quad h \to 0 \end{aligned}$$

for almost every $x \in T$. Moreover, for any $\Omega \subset T$ we can estimate

$$\begin{aligned} \int_\Omega |\partial^h_\alpha \varphi^h(x)|^2 \, dx &\leq \int_\Omega \left\{ h^{-3} \int_0^h \int_{Q_h([x]_h)} |\partial_\alpha \varphi(y + \vartheta \underline{e}_\alpha)|^2 \, d\vartheta dy \right\} dx \\ &\leq h^{-3} \int_0^h \int_{Q_{2h}((-h,-h))} \left\{ \int_\Omega |\partial_\alpha \varphi(x + y + \vartheta \underline{e}_\alpha)|^2 \, dx \right\} dy d\vartheta \\ &\leq 4 \sup_{y \in T} \left\{ \int_{\Omega + y} |\partial_\alpha \varphi|^2 \, dx \right\} < \delta \end{aligned}$$

if $\mathcal{L}^2(\Omega) < \mu_0(\delta)$, by absolute continuity of the Lebesgue integral. Thus, the family of indefinite integrals $\left(\int |\partial^h_\alpha \varphi^h|^2\right)_{h>0}$ is uniformly absolutely continuous, and the assertion follows from Vitali's convergence theorem. □

5. – Harmonic maps

A map $u^h : T_h \to N \subset \mathbb{R}^n$ by definition is *harmonic* if u^h is a critical point for E_h among maps $v^h : T \to N$; that is, if the first variation

$$\langle dE_h(u^h), \varphi^h \rangle = 0 \tag{17}$$

for all $\varphi^h \in (u^h)^{-1}TN$, where

$$(u^h)^{-1}TN = \{\varphi^h : T_h \to \mathbb{R}^n; \varphi^h(x) \in T_{u^h(x)}N \quad \text{for all} \quad x \in T_h\}$$

is the pullback tangent bundle, with T_pN denoting the tangent space to N at a point $p \in N$. By (14), equation (17) is equivalent to the relation

$$-\Delta^h u^h \perp T_{u^h}N, \tag{18}$$

where "$\perp$" means orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$ in the ambient $\mathbb{R}^n$.

Introducing a local frame $\nu_{k+1}, \dots, \nu_n$ for the normal bundle $TN^\perp$ near a point $p = u^h(x) \in N$, we can also locally express (18) in the form

$$-\Delta^h u^h = \sum_l \lambda^l \nu_l \circ u^h \tag{19}$$

The coefficient functions λ^l can be determined as

$$\begin{aligned} \lambda^l &= -\langle \Delta^h u^h, \nu_l \circ u^h \rangle \\ &= -\sum_\alpha \partial_\alpha^{\mp h} \langle \partial_\alpha^{\pm h} u^h, \nu_l \circ u^h \rangle + \sum_\alpha \langle \partial_\alpha^{\pm h} u^h, \partial_\alpha^{\pm h}(\nu_l \circ u^h) \rangle \end{aligned} \tag{20}$$

Also recall that a smooth map $u: T \to N \subset \mathbb{R}^n$ is harmonic if u is critical for E among maps $v: T \to N$, or, equivalently, if

$$-\Delta u = A(u)(\nabla u, \nabla u) \perp T_u N, \tag{21}$$

where $A(p): T_pN \times T_pN \to T_pN^\perp$ is the second fundamental form of $N \subset \mathbb{R}$. Locally, with respect to a smooth local frame $\nu_{k+1}, \dots, \nu_n$ for $TN^\perp$ we have

$$A(p)(\xi, \eta) = \sum_l A^l(p)(\xi, \eta)\, \nu_l(p),$$

where

$$A^l(p)(\xi, \eta) = \langle \xi, d\nu_l(p).\eta \rangle$$

denotes the second fundamental form of ν_l, $k < l \le n$.

Our main result then is the following.

THEOREM 5.1. *For a sequence of numbers $h \to 0$, $h^{-1} \in \mathbb{N}$, suppose $u^h: T_h \to N \subset \mathbb{R}^n$ is harmonic and $u^h \rightharpoonup u$ weakly in $H^{1,2}(T; N)$ as $h \to 0$. Then u is harmonic.*

6. - Equivalent Hodge system

As observed by Christodoulou-Tahvildar-Zadeh [4] and Hélein [9], we may assume that TN is parallelizable. Let $\underline{e}_1, \dots, \underline{e}_k$ be a smooth orthonormal frame field on N, such that $(\underline{e}_1(p), \dots, \underline{e}_k(p))$ is an orthonormal basis for T_pN at any $p \in N$.

Given $u^h: T_h \to N$, and a family of rotations $R^h: T_h \to SO(k)$, then

$$e_i^h = \sum_j R_{ij}^h (\underline{e}_j \circ u^h)$$

is a frame for $(u^h)^{-1}TN$.

Let

$$\begin{aligned} \vartheta_{i,\alpha}^h &= \langle \partial_\alpha^h u^h, \tau_\alpha^h e_i^h \rangle dx^\alpha, \\ \vartheta_{i,\alpha}^{-h} &= \tau_\alpha^{-h} \vartheta_{i,\alpha}^h = \langle \partial_\alpha^{-h} u^h, e_i^h \rangle dx^\alpha, \\ \omega_{ij,\alpha}^{\pm h} &= \langle \partial_\alpha^{\pm h} e_k^h, m_\alpha^{\pm h} e_j^h \rangle dx^\alpha. \end{aligned}$$

Then

$$\begin{aligned} \delta^h \vartheta_i^h &= \sum_\alpha \partial_\alpha^{-h} \vartheta_{i,\alpha}^h = \sum_\alpha \partial_\alpha^h \vartheta_{i,\alpha}^{-h} \\ &= \sum_\alpha \langle \partial_\alpha^h \partial_\alpha^{-h} u^h, e_i^h \rangle + \sum_\alpha \langle \partial_\alpha^h u^h, \partial_\alpha^h e_i^h \rangle. \end{aligned}$$

It follows that $u^h: T_h \to N$ is discretely harmonic, if and only if

$$\delta^h \vartheta_i^h = \sum_\alpha \langle \partial_\alpha^h u^h, \partial_\alpha^h e_i^h \rangle = \sum_\alpha \tau_\alpha^h \langle \partial_\alpha^{-h} u^h, \partial_\alpha^{-h} e_i^h \rangle.$$

Letting $\nu_{k+1}, \dots, \nu_n$ be a smooth local frame for the normal bundle, we can expand

$$\begin{aligned} \langle \partial_\alpha^{-h} u^h, \partial_\alpha^{-h} e_i^h \rangle &= \sum_j \langle \partial_\alpha^{-h} u^h, e_j^h \rangle \langle e_j^h, \partial_\alpha^{-h} e_i^h \rangle \\ &\quad + \sum_l \langle \partial_\alpha^{-h} u^h, \nu_l \circ u^h \rangle \langle \nu_l \circ u^h, \partial_\alpha^{-h} e_i^h \rangle \\ &= \sum_j \vartheta_{j,\alpha}^{-h} \omega_{ij,\alpha}^{-h} + \sum_j \vartheta_{j,\alpha}^{-h} \left\langle \frac{e_j^h - \tau_\alpha^{-h} e_j^h}{2}, \partial_\alpha^{-h} e_i^h \right\rangle \\ &\quad + \sum_l \langle \partial_\alpha^{-h} u^h, \nu_l \circ u^h \rangle \langle \nu_l \circ u^h, \partial_\alpha^{-h} e_i^h \rangle \\ &=: \sum_j \vartheta_{j,\alpha}^{-h} \omega_{ij,\alpha}^{-h} + \eta_{1i,\alpha}^{-h}. \end{aligned}$$

The presence of the error term $\eta_{1i,\alpha}^h$ marks one major difference between the discrete and continuous cases.

LEMMA 6.1. *There is a constant $C = C(N)$ such that*

$$|\eta_{1i,\alpha}^{-h}| \le C|u^h - \tau_\alpha^{-h} u^h| \left(|\partial_\alpha^{-h} u^h|^2 + \sum_j |\partial_\alpha^{-h} e_j^h|^2 \right).$$

PROOF. Since $u^h : T_h \to N$ and since for $p, q \in N$, with $C = C(N)$ we have

$$|\langle p - q, \nu \rangle| \le C|p - q|^2, \quad \text{for all} \quad \nu \in T_p N^\perp,$$

it follows that

$$\begin{aligned} |\langle \partial_\alpha^{-h} u^h, \nu_l \circ u^h \rangle| &\le C h^{-1} |u^h - \tau_\alpha^{-h} u^h|^2 \\ &= C|u^h - \tau_\alpha^{-h} u^h| |\partial_\alpha^{-h} u^h|. \end{aligned}$$

Hence the second term defining η_1^{-h} can be estimated as claimed. For the first it suffices to note that

$$|\partial_\alpha^{-h} u| |e_j^h - \tau_\alpha^{-h} e_j^h| = |u^h - \tau_\alpha^{-h} u^h| |\partial_\alpha^{-h} e_j^h|. \qquad \square$$

In the following we denote

$$\eta_{1i,\alpha}^h = \tau_\alpha^h \eta_{1i,\alpha}^{-h}, \; \eta_1^h = (\eta_{1i,\alpha}^h)_{1 \le i \le k, 1 \le \alpha \le 2}.$$

With this notation then u^h is discretely harmonic if and only if there holds

$$\delta^h \vartheta_i^h = \sum_j \vartheta_j^h \cdot \omega_{ij}^h + \eta_{1i}^h. \tag{22}$$

Observe that $\tau_\alpha^h \left(\vartheta_{j\alpha}^{-h} \omega_{ij,\alpha}^{-h} \right) = \vartheta_{j,\alpha}^h \cdot \omega_{ij,\alpha}^h$.

Similarly, if $u \in H^{1,2}(T; N)$ is weakly harmonic and if $e_i = R_{ij} \bar{e}_j \circ u$ is a frame for $u^{-1} TN$ with connection 1-forms

$$\omega_{ij} = \langle e_j, de_i \rangle, \quad 1 \le i, j \le k,$$

then, letting

$$\vartheta_i = \langle du, e_i \rangle, \quad 1 \le i \le k,$$

the equation holds

$$\delta \vartheta_i = \sum_j \vartheta_j \cdot \omega_{ij}, \tag{23}$$

and conversely.

7. – Coulomb gauge

Equations (22), (23) involve an arbitrary choice of frame for the pull-back bundle. We may use this gauge freedom of rotating the frames to fix a frame with particularly nice analytic properties. Following Hélein [9], we impose Coulomb gauge, as follows.

For each h choose $R^h\colon T_h \to SO(k)$ such that

(24) $$E_h\left(R^h(\underline{e} \circ u^h)\right) := \sum_i E_h\big(\sum_j R^h_{ij}(\underline{e}_j \circ u^h)\big) = \inf_{S^h: T_h \to SO(k)} E_h\big(S^h(\underline{e} \circ u^h)\big).$$

Observe that trivially

$$E_h\big(R^h(\underline{e} \circ u^h)\big) \le E_h(\underline{e} \circ u^h) \le C E_h(u^h).$$

Lemma 7.1. *Let $R^h\colon T_h \to SO(k)$ satisfy (24). Then there holds the equation $\delta^h \omega^h_{ij} = \sum_\alpha \partial^{-h}_\alpha \omega^h_{ij,\alpha} = 0$ for the connection 1-form ω^h_{ij} associated with the frame $e^h_i = \sum_j R^h_{ij}(\underline{e}_j \circ u^h)$, $1 \le i \le k$.*

Proof. For $r^h = r^h_{ij}\colon T_h \to T_{id}SO(k) = so(k)$, letting $e^h_i = \sum_j R^h_{ij}(\underline{e}_j \circ u^h)$ and using (9), (11), and (10) we compute

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon} E_h\big((id + \varepsilon r^h) \cdot R^h(\underline{e} \circ u^h)\big)_{|\varepsilon=0} \\
&= \frac{1}{2} \sum_{\alpha,i,j} \int_{T_h} \left\{ \langle \partial^h_\alpha e^h_i, \partial^h_\alpha (r^h_{ij} e^h_j)\rangle + \langle \partial^{-h}_\alpha e^h_i, \partial^{-h}_\alpha (r^h_{ij} e^h_j)\rangle \right\} \\
&= \frac{1}{2} \sum_{\alpha,i,j} \int_{T_h} \left\{ \langle \partial^h_\alpha e^h_i, \partial^h_\alpha e^h_j\rangle + \langle \partial^{-h}_\alpha e^h_i, \partial^{-h}_\alpha e^h_j\rangle \right\} r^h_{ij} \\
&\quad + \frac{1}{2} \sum_{\alpha,i,j} \int_{T_h} \left\{ \langle \partial^h_\alpha e^h_i, \tau^h_\alpha e^h_j\rangle \partial^h_\alpha r^h_{ij} + \langle \partial^{-h}_\alpha e^h_i, \tau^{-h}_\alpha e^h_j\rangle \partial^{-h}_\alpha r^h_{ij} \right\} \\
&= -\frac{1}{2} \sum_{\alpha,i,j} \int_{T_h} \left\{ \partial^h_\alpha \langle \partial^{-h}_\alpha e^h_i, e^h_j\rangle + \partial^{-h}_\alpha \langle \partial^h_\alpha e^h_i, e^h_j\rangle \right\} r^h_{ij},
\end{aligned}$$

where we also used anti-symmetry $r^h_{ij} = -r^h_{ji}$ to obtain the last equation.

Recalling that

$$\partial^h_\alpha \langle \partial^{-h}_\alpha e^h_i, e^h_j\rangle = \partial^{-h}_\alpha \langle \partial^h_\alpha e^h_i, \tau^h_\alpha e_j\rangle,$$

we infer that

$$\sum_{\alpha,i,j} \int_{T_h} \partial^{-h}_\alpha \langle \partial^h_\alpha e^h_i, m^h_\alpha e^h_j\rangle r^h_{ij} = 0$$

for all $r^h \in so(k)$. Since by (9) we have

$$\omega^h_{ij,\alpha} = \langle \partial^h_\alpha e^h_i, m^h_\alpha e^h_j\rangle = -\omega^h_{ji,\alpha},$$

we conclude that

$$\delta^h \omega^h_{ij} = \sum_\alpha \partial^{-h}_\alpha \omega^h_{ij,\alpha} = 0,$$

as claimed. □

In view of the gauge condition we may assume that

$$e_i^h \rightharpoonup e_i \quad \text{weakly in } H^{1,2}, \tag{25}$$

$$\vartheta_i^h \rightharpoonup \vartheta_i \quad \text{weakly in } L^2, \tag{26}$$

$$\omega_{ij}^h \rightharpoonup \omega_{ij} \quad \text{weakly in } L^2. \tag{27}$$

Let

$$\omega_{ij}^h = d^h a_{ij}^h + \delta^h b_{ij}^h + c_{ij}^h$$

be the Hodge decomposition of ω_{ij}^h. In view of the gauge condition $\delta^h \omega_{ij}^h = 0$, it follows that

$$a_{ij}^h = 0.$$

Moreover, we have

$$E_h(b_{ij}^h) \le C \int_{T_h} |\omega_{ij}^h|^2 \le C, |c_{ij}^h| \le C \int_{T_h} |\omega_{ij}^h|^2 \le C.$$

Hence we may assume

$$b_{ij}^h \overset{w}{\rightharpoonup} b_{ij} \ \text{ in } \ H^{1,2}, c_{ij}^h \to c_{ij} (h \to 0),$$

where

$$\omega_{ij} = \delta b_{ij} + c_{ij}.$$

Let $\beta_{ij}^h = *b_{ij}^h, \beta_{ij} = *b_{ij}$. Then

$$\omega_{ij}^h = *d^{-h}\beta_{ij}^h + c_{ij}^h, \ \omega_{ij} = *d\beta_{ij} + c_{ij}.$$

Note that

$$d^h u^h \cdot *d^{-h}\beta_{ij}^h = *(d^h u^h \wedge d^{-h}\beta_{ij}^h), \text{ etc.}$$

Thus, we have

$$\vartheta_j^h \cdot \omega_{ij}^h = *\langle d^h u^h \wedge d^{-h}\beta_{ij}^h, e_j^h \rangle + \sum_\alpha \langle \partial_\alpha^h u^h, \tau_\alpha^h e_j^h - e_j^h \rangle \omega_{ij,\alpha}^h + \vartheta_j^h \cdot c_{ij}^h.$$

8. - Convergence

Passing to the limit $h \to 0$ in the distribution sense is no problem for the terms $\delta^h \vartheta_j^h$, $\vartheta_j^h \cdot c_{ij}^h$. The term

$$\eta_{2i,\alpha}^h = \langle \partial_\alpha^h u^h, \tau_\alpha^h e_j^h - e_j^h \rangle \omega_{ij,\alpha}^h$$

is easily dominated

$$|\eta_{2i,\alpha}^h| \leq C |\tau_\alpha^h u^h - u^h| \cdot \sum_j |\partial_\alpha^h e_j^h|^2$$

in the same way as $\eta_{1i,\alpha}^h$; see Lemma 6.1. These and similar error terms will be dealt with later.

Now we concentrate on the term

$$J^h(u^h, \beta_{ij}^h, e_j^h) = \langle d^h u^h \wedge d^{-h} \beta_{ij}^h, e_j^h \rangle .$$

This term has the structure of a Jacobian determinant. In the continuous limit $h = 0$ concentration-compactness arguments yield weak convergence results for such terms. Our aim in the following will be to reduce the discrete case to the continuous one.

Choose a smooth testing function $\varphi \in C^\infty(T), 0 \leq \varphi \leq 1$. Discretize φ to obtain a map $\varphi^h : T_h \to \mathbb{R}$. We intend to prove

$$\int_{T_h} J^h(u^h, \beta_{ij}^h, e_j^h) \varphi^h \to \int_T \langle du \wedge d\beta_{ij}, e_j \varphi \rangle$$

as $h \to 0$.

Let $v^h = \beta_{ij}^h, w^h = e_j^h \cdot \varphi^h$ for brevity. Note that $u^h \rightharpoonup u, v^h \rightharpoonup v = \beta_{ij}, w^h \rightharpoonup w = e_j \varphi$ weakly in $H^{1,2}$ as $h \to 0$.

LEMMA 8.1. *For any $u^h, v^h, w^h : T_h \to \mathbb{R}^n$ there holds*

$$\begin{aligned} \int_{T_h} J^h(u^h, v^h, w^h) &= -\int_{T_h} J^h(u^h, w^h, v^h) + \int_{T_h} \eta_3^h \\ &= -\int_{T_h} J^h(w^h, v^h, u^h) + \int_{T_h} \eta_4^h \end{aligned}$$

where

$$|\eta_{3,4}^h| \leq C \sum_\alpha |\tau_\alpha^h u^h - u^h| \left(|\partial_\alpha^{\pm h} v^h|^2 + |\partial_\alpha^{\pm h} w^h|^2 \right)$$

with an absolute constant C.

PROOF. It suffices to prove the first estimate; the second is obtained replacing forward by backward difference quotients. Moreover, considering each triple of components separately, we may assume that $n = 1$ to simplify the notation.

We have

$$\tau_2^h\left(\partial_1^h u^h \partial_2^{-h} v^h w^h\right) = \partial_2^h\left(\partial_1^h u^h v^h w^h\right) - \partial_2^h \partial_1^h u^h v^h w^h - \partial_1^h \tau_2^h u^h v^h \partial_2^h w^h,$$
$$\tau_1^h\left(\partial_2^h u^h \partial_1^{-h} v^h w^h\right) = \partial_1^h\left(\partial_2^h u^h v^h w^h\right) - \partial_1^h \partial_2^h u^h v^h w^h - \partial_2^h \tau_1^h u^h v^h \partial_1^h w^h.$$

Taking the difference of these two equations, since $\partial_1^h \partial_2^h u^h = \partial_2^h \partial_1^h u^h$ the middle terms on the right cancel. Integrating the resulting identity and shifting, we thus obtain

$$\int_{T_h} J(u^h, v^h, w^h) = -\int_{T_h} \left\{\partial_1^h u^h \tau_2^{-h} v^h \partial_2^{-h} w^h - \partial_2^h u^h \tau_1^{-h} v^h \partial_1^{-h} w^h\right\}$$
$$= -\int_{T_h} J(u^h, w^h, v^h) + \int_{T_h} \eta_3^h,$$

where

$$\eta_3^h = \partial_1^h u^h (v^h - \tau_2^{-h} v^h) \partial_2^{-h} w^h - \partial_2^h u^h (v^h - \tau_1^{-h} v^h) \partial_1^{-h} w^h$$
$$= (\tau_1^h u^h - u^h) \partial_2^{-h} v^h \partial_2^{-h} w^h + (\tau_2^h u^h - u^h) \partial_1^{-h} v^h \partial_1^{-h} w^h$$

can be estimated as claimed. □

Let $\overline{u}^h, \overline{v}^h, \overline{w}^h$ be the bilinearly interpolated functions u^h, etc. Observe the following:

$$\partial_1 \overline{u}^h (x_h + h\xi) = (1 - \xi_2) \partial_1^h u^h (x_h) + \xi_2 \partial_1^h u^h (x_h + h\underline{e}_2)$$
$$\partial_2 \overline{u}^h (x_h + h\xi) = (1 - \xi_1) \partial_2^h u^h (x_h) + \xi_1 \partial_2^h u^h (x_h + h\underline{e}_1)$$

for $x_h \in T_h, \xi \in Q_1(0)$, and similarly for $\overline{v}^h$ and $\overline{w}^h$. In particular,

$$\partial_1 \overline{u}^h (x_h + \xi) = \partial_1 \overline{u}^h (x_h + \xi_2 \underline{e}_2),$$
$$\partial_2 \overline{u}^h (x_h + \xi) = \partial_2 \overline{u}^h (x_h + \xi_1 \underline{e}_1)$$

for all $x_h \in T_h, \xi \in Q_h(0)$.

Thus, for instance,

$$\int_{Q_h(x_h)} \langle \partial_1 \overline{u}^h \partial_2 \overline{v}^h, w^h \rangle d\xi_1 d\xi_2 = \int_{Q_h(x_h)} \langle m_2^h \partial_1^h u^h m_1^h \partial_2^h v^h, w^h \rangle$$
$$= \int_{Q_h(x_h)} \left\langle \partial_1^h (m_2^h u^h) \partial_2^h (m_1^h v^h), w^h \right\rangle.$$

In view of this identity,

$$\int_T \langle \partial_1 \overline{u}^h \partial_2 \overline{v}^h - \partial_2 \overline{u}^h \partial_1 \overline{v}^h, w^h \rangle dx_1 dx_2$$
$$= \int_{T_h} \left\langle \partial_1^h (m_2^h u^h) \partial_2^h (m_1^h v^h) - \partial_2^h (m_1^h u^h) \partial_1^h (m_2^h v_h), w^h \right\rangle$$

Shifting coordinates,

$$\begin{aligned}\int_{T_h} \langle \partial_1^h (m_2^h u^h) \partial_2^h (m_1^h v^h), w^h \rangle &= \int_{T_h} \langle \partial_1^h (m_2^{-h} u^h) \partial_2^{-h} (m_1^h v^h), w^h \rangle + \int_{T_h} \eta_5^h \\ &= \int_{T_h} \langle \partial_1^h (m_2^{-h} u^h), m_1^h (\partial_2^{-h} v^h w^h) \rangle + \int_{T_h} \eta_6^h \\ &= \int_{T_h} \langle \partial_1^h (m_1^{-h} m_2^{-h} u^h) \partial_2^{-h} v^h, w^h \rangle + \int_{T_h} \eta_7^h,\end{aligned}$$

with η_j^h bounded like $\eta_{3,4}^h$, and similarly with ∂_1^h and ∂_2^h exchanged.

Thus

$$\int_T d\overline{u}^h \wedge d\overline{v}^h w^h = \int_{T_h} J^h(m_1^{-h} m_2^{-h} u^h, v^h, w^h) + \int_{T_h} \eta_8^h,$$

and by Lemma 8.1 the latter

$$\begin{aligned}&= -\int_{T_h} J^h(w^h, v^h, m_1^{-h} m_2^{-h} u^h) + \int_{T_h} \eta_9^h \\ &= -\int_{T_h} J^h(w^h, v^h, u^h) + \int_{T_h} \eta_{10}^h \\ &= \int_{T_h} J^h(u^h, v^h, w^h) + \int_{T_h} \eta_{11}^h.\end{aligned}$$

It follows that

$$\int_{T_h} J^h(u^h, v^h, w^h) = \int_T \langle d\overline{u}^h \wedge d\overline{\beta}_{ij}^h, \overline{e}_j^h \rangle \varphi + \int_{T_h} \eta_{12}^h,$$

where the η_j^h are all bounded as $\eta_{3,4}^h$ above.

We can now use the concentration-compactness argument from [8], proof of Theorem 1.1, based on [10], Lemma 4.3, to pass to the limit in (22). In fact, [10] implies that, as $h \to 0$,

$$\langle d\overline{u}^h \wedge d\beta_{ij}^h, e_j^h \rangle \rightharpoonup \langle du \wedge d\beta_{ij}, e_j \rangle + \sum_{k \in K} \nu_k \delta_{\{\overline{y}_k\}}$$

in the sense of distributions, where K is at most countable and $\sum_{k\in K} |\nu_k| < \infty$.

For the error terms η_j^h we have a similar result. We combine all these terms in a single term η^h, satisfying the estimate

$$|\eta^h| \le C \sum_\alpha \left(h ||\nabla\varphi||_{L^\infty} + |\varphi| \cdot |\tau_\alpha^h u^h - u^h| \right) \left(|\partial_\alpha^h u^h|^2 + \sum_{i,j} |\partial_\alpha^h e_j^h|^2 + |\partial_\alpha^h \beta_{ij}^h|^2 \right) \tag{28}$$

with a constant $C = C(N)$ independent of φ.

LEMMA 8.2. *There exists an at most countable set $\{\overline{x}_j\}_{j\in J}$ in T and numbers $\eta_j > 0$ such that, as $h \to 0$ suitably, $\eta^h \rightharpoonup \sum_j \eta_j \delta_{\{\overline{x}_j\}}$ in measure, where $\eta_j = \mu_j \varphi(\overline{x}_j)$ and $\sum_j |\mu_j|^{2/3} < \infty$.*

PROOF. Passing to a subsequence, if necessary, we may assume that $|\eta^h| \rightharpoonup \eta$ in measure. Let $\delta > 0$. As observed in Lemma 4.2 above, denoting

$$\Sigma_h^\delta = \left\{ x_h \in T_h;\ \sum_\alpha |(\tau_\alpha^h u^h - u^h)(x_h)|^2 > \delta \right\},$$

there holds

$$\delta |\Sigma_h^\delta| \le C E_h(u^h) \le C < \infty.$$

Possibly passing to a further subsequence, we may assume $\Sigma_h^\delta \to \Sigma^\delta$, where $|\Sigma^\delta| < \infty$.

Moreover, for $h > 0$ we have

$$\int_T |\eta^h| = \int_{T \setminus \cup_{x_h \in \Sigma_h^\delta} Q_h(x_h)} |\eta^h| + \sum_{x_h \in \Sigma_h^\delta} \int_{Q_h(x_h)} |\eta^h| = I_h^\delta + II_h^\delta,$$

and for fixed $\varphi \in C^\infty(T)$ we can estimate

$$\begin{aligned} I_h^\delta &\le C\left(h \|\nabla\varphi\|_{L^\infty} + \sqrt{\delta}\|\varphi\|_{L^\infty}\right) \int_{T_h} \sum_{\alpha,i,j} \left(|\partial_\alpha^h u^h|^2 + |\partial_\alpha^h e_j^h|^2 + |\partial_\alpha^h \beta_{ij}^h|^2\right) \\ &\le C\left(h + \sqrt{\delta}\right) \end{aligned}$$

while for each $x_h \in \Sigma_h^\delta$ in view of Lemma 4.1. i) we find

$$\int_{Q_h(x_h)} |\eta^h| \le C \left\{ h \|\nabla\varphi\|_{L^\infty} + |\varphi(x_h)| \left(\int_{Q_{2h}(x_h)} e_h(u^h) \right)^{1/2} \right\} \cdot \int_{Q_h(x_h)} \sum_{\alpha,i,j} \left(|\partial_\alpha^h u^h|^2 + |\partial_\alpha^h e_j^h|^2 + |\partial_\alpha^h \beta_{ij}^h|^2\right).$$

Thus, for $\delta > 0$ we may decompose $\eta = \eta_0^\delta + \sum_{\overline{x}_j \in \Sigma^\delta} \eta_j^\delta \delta_{\{\overline{x}_j\}}$, where

$$\int_T \eta_0^\delta \le \lim_{r\to 0} \lim_{h \to 0} \int_{T \setminus \cup_{\overline{x}_j \in \Sigma^\delta} B_r(\overline{x}_j)} \eta^h \le C\sqrt{\delta}$$

and where for each $\overline{x}_j \in \Sigma^\delta$ we have

$$\eta_j = \mu_j \varphi(\overline{x}_j)$$

with

$$\sum_j \mu_j^{2/3} \le C$$

independent of $\delta > 0$. (In the latter estimate we also used concavity of the function $t \mapsto t^{2/3}$ to obtain the desired bound in case different sequences (x_h) in Σ_h^δ should converge to the same limit $\overline{x}_j \in \Sigma^\delta$.) Passing to the limit $\delta \to 0$, the assertion follows. □

Passing to the limit $h \to 0$ in (22), thus we obtain the equation

$$\delta \vartheta_i - \sum_j \omega_{ij} \cdot \vartheta_j = \sum_{j \in J} \mu_j \delta_{\{\overline{x}_j\}} + \sum_{k \in K} \nu_k \delta_{\{\overline{y}_k\}}.$$

But the left hand side belongs to the space $H^{-1} + L^1$ which does not contain any atoms. Therefore all μ_j and ν_k must vanish and we find (23), as desired.

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