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## **Maximum principles and minimal surfaces**

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## Maximum Principles and Minimal Surfaces

MARIO MIRANDA

In memory of Ennio De Giorgi

**Abstract.** It is well known the close connection between the classical Plateau problem and Dirichlet problem for the minimal surface equation.

Paradoxically in higher dimensions that connection is even stronger, due to the existence of singular solutions for Plateau problem.

This fact was emphasized by Fleming's remark [15] about the existence of such singular solutions, as a consequence of the existence of non-trivial entire solutions for the minimal surface equation.

The main goal of this article is to show how *generalized solutions* [26] apply to the study of both, singular and regular minimal surfaces, with particular emphasis on Dirichlet and Bernstein problems, and the problem of removable singularities.

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### 1. – The maximum principle in the calculus of variations

Let  $F : \mathbb{R}^n \rightarrow [0, +\infty)$  be a  $C^2$ -strictly convex function, i.e.

$$(1) \quad \sum_{i,j=1}^n D_i D_j F(p) \lambda_i \lambda_j > 0, \quad \forall p \in \mathbb{R}^n, \lambda \in \mathbb{R}^n \setminus \{0\}.$$

Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $K$ -Lipschitz function, and  $\Omega \subset \mathbb{R}^n$  an open and bounded set.

DEFINITION 1.1.  $u$  is a minimum for  $F$  in  $\text{Lip}_K(\Omega)$ , if

$$(2) \quad \int_{\Omega} F(Du(x)) \, dx \leq \int_{\Omega} F(Dv(x)) \, dx,$$

for any  $K$ -Lipschitz function  $v$  such that  $v|_{\partial\Omega} = u|_{\partial\Omega}$ .

The following **elementary Maximum Principle** can be easily proved:

THEOREM 1.2. *If  $u$  and  $w$  are two minima for  $F$  in  $\text{Lip}_K(\Omega)$ , then*

$$(3) \quad u|_{\partial\Omega} \geq w|_{\partial\Omega} \Rightarrow u|_{\Omega} \geq w|_{\Omega}.$$

John von Neumann [35] derived from Theorem 1.2, the following **Maximum Principle for the gradient** of a minimizing function:

THEOREM 1.3. *If  $u$  is a minimum for  $F$  in  $\text{Lip}_K(\Omega)$ , then*

$$(4) \quad \sup_{x \in \Omega, y \in \partial\Omega} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|}.$$

David Hilbert (see [17] and [18]) knew the identity (4), in the 2-dimensional case, for

$$F(p) = |p|^2,$$

and was able to prove the **Dirichlet Principle** under special conditions for the data.

THEOREM 1.4 (Hilbert). *If  $\Omega \subset \mathbb{R}^2$  is open, bounded and strictly convex, if  $\phi \in C^2(\mathbb{R}^2)$  then there exists a unique Lipschitz function  $u$  such that*

$$(5) \quad u|_{\partial\Omega} = \phi|_{\partial\Omega},$$

$$(6) \quad \int_{\Omega} |Du(x)|^2 dx \leq \int_{\Omega} |Dv(x)|^2 dx, \quad \forall \text{ Lipschitz } v \text{ satisfying (5)}$$

$$(7) \quad u|_{\Omega} \text{ is analytic and } \Delta u|_{\Omega} = 0.$$

Henri Lebesgue [20] used the elementary Maximum Principle proved in Theorem 1.2, and Hilbert's solutions to establish the following Existence result

THEOREM 1.5 (Lebesgue). *If  $\Omega \subset \mathbb{R}^2$  is open, bounded and strictly convex, if  $\phi \in C(\partial\Omega)$  then there exists a unique  $u \in C(\overline{\Omega})$ , satisfying (5) and (7).*

Alfred Haar [16] remarked that Hilbert's method could be applied to a general integrand  $F$ . The hypothesis  $n = 2$  is unnecessary, so we have the following

THEOREM 1.6 (Hilbert-Haar). *If  $\Omega \subset \mathbb{R}^n$  is open, bounded and strictly convex, if  $\phi \in C^2(\mathbb{R}^n)$  then there exists a unique Lipschitz function  $u$  satisfying (5) and*

$$(8) \quad \int_{\Omega} F(Du(x)) dx \leq \int_{\Omega} F(Dv(x)) dx, \quad \forall \text{ Lipschitz } v \text{ satisfying (5)}.$$

The question "are the Hilbert-Haar minima smoother than Lipschitz?" was answered by Ennio De Giorgi [9] and John Nash [29]:

THEOREM 1.7 (De Giorgi-Nash). *The Hilbert-Haar minima have Hölder continuous first derivatives.*

De Giorgi's proof was a consequence of the following regularity result

**THEOREM 1.8 (De Giorgi).** *If  $\Omega \subset \mathbb{R}^n$  is open and  $(a_{ij}(x))$  is a  $(n \times n)$ -symmetric matrix with bounded measurable coefficients in  $\Omega$ , satisfying*

$$(9) \quad \sum_{i,j=1}^n a_{ij}(x)\lambda_i\lambda_j \geq |\lambda|^2, \quad \forall x \in \Omega, \lambda \in \mathbb{R}^n .$$

*If  $u \in L^2(\Omega)$  has first derivatives  $D_i u \in L^2(\Omega)$  and satisfying*

$$(10) \quad \int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j \psi(x) \, dx = 0, \quad \forall \psi \in C_0^1(\Omega) ,$$

*then  $u$  is Hölder continuous.*

Jürgen Moser [28], using De Giorgi's method, proved the following Harnack type result

**THEOREM 1.9 (Moser).** *If  $u \geq 0$  satisfies (10),  $\forall \Omega' \subset\subset \Omega \exists c = c(\Omega, \Omega', \Lambda)$  where*

$$(11) \quad \Lambda = \sup_{i,j} \sup_{x \in \Omega} |a_{ij}(x)| ,$$

*such that*

$$(12) \quad \sup_{\Omega'} u \leq c \inf_{\Omega'} u .$$

Moser's result implies the following strong Maximum Principle for Hilbert-Haar minima.

**THEOREM 1.10.** *If  $u \geq w$  are Hilbert-Haar minima for  $F$  in  $\Omega$ , if  $\Omega$  is connected and there exists  $x_0 \in \Omega$  with  $u(x_0) = w(x_0)$ , then  $u(x) = w(x) \forall x \in \Omega$ .*

**PROOF.**  $u$  and  $w$  satisfy

$$(13) \quad \int_{\Omega} \sum_{i=1}^n D_i F(Du(x)) D_i \phi(x) \, dx = 0, \quad \forall \phi \in C_0^1(\Omega) ,$$

$$(14) \quad \int_{\Omega} \sum_{i=1}^n D_i F(Dw(x)) D_i \phi(x) \, dx = 0, \quad \forall \phi \in C_0^1(\Omega) .$$

Therefore

$$\int_{\Omega} \sum_{i=1}^n [D_i F(Du(x)) - D_i F(Dw(x))] D_i \phi(x) \, dx = 0$$

and also

$$\int_{\Omega} \sum_{i,j=1}^n \left\{ \int_0^1 D_i D_j F [Dw + tD(u-w)] dt \right\} D_j (u-w) D_i \phi \, dx = 0 .$$

This is an equation of (10)-type, then Theorem 1.9 applies to the function  $u-w$ , what implies  $u-w=0$ . □



## 2. – Classical results for the minimal surface equation

The minimal surface equation is the equation

$$(15) \quad \operatorname{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0,$$

that can also be written

$$(16) \quad (1 + |Du|^2)\Delta u - \sum_{i,j=1}^n D_i u D_j u D_i D_j u = 0$$

and in two dimensions

$$(17) \quad (1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

where  $p = D_1 u$ ,  $q = D_2 u$ ,  $r = D_1 D_1 u$ ,  $s = D_1 D_2 u$ ,  $t = D_2 D_2 u$ .

The following result, stated by Sergei Bernstein in 1912, was proved by Tibor Rado [32]:

**THEOREM 2.1 (Rado).** *If  $\Omega \subset \mathbb{R}^2$  is bounded and convex, if  $\phi \in C(\partial\Omega)$  then there exists a unique  $u \in C(\overline{\Omega}) \cap C^\omega(\Omega)$  satisfying (17) in  $\Omega$  and the boundary condition*

$$(18) \quad u|_{\partial\Omega} = \phi.$$

This is a good solution of the Dirichlet problem for (17) because Robert Finn [14] proved the following

**THEOREM 2.2 (Finn).** *The equation (17) admits a solution in a domain  $\Omega$  for arbitrary continuous boundary data if and only if  $\Omega$  is convex.*

Lipman Bers [4] and Finn [13] proved independently another remarkable property of (17):

**THEOREM 2.3 (Bers-Finn).** *If  $u \in C^\omega(\Omega \setminus \{x_o\})$  is a solution of (17), then there exists a unique extension  $u^* \in C^\omega(\Omega)$  of  $u$ .*

Johannes C. C. Nitsche [31] improved Bers-Finn theorem

**THEOREM 2.4 (Nitsche).** *If  $u \in C^\omega(\Omega \setminus K)$  is a solution of (17), if  $K$  is closed in  $\Omega$  and  $H^1(K) = 0$ , where  $H^1$  is the Hausdorff 1-dimensional measure, then there exists a unique extension  $u^* \in C^\omega(\Omega)$  of  $u$ .*

Sergei Bernstein [3] proved

**THEOREM 2.5 (Bernstein).** *If  $u \in C^\omega(\mathbb{R}^2)$  is a solution of (17) then  $u$  is either a polynomial of degree 1 or a constant.*

Nitsche [30] was able to show that Bernstein's theorem is a corollary of Liouville's theorem for holomorphic functions.

**3. – The minimal surface equation in higher dimensions:  
Hilbert-Haar’s approach**

The minimal surface equation (15) in all dimensions is the Euler equation of the integrand

$$(19) \quad F(p) = \sqrt{1 + |p|^2}, \quad p \in \mathbb{R}^n,$$

therefore Hilbert-Haar’ and De Giorgi-Nash’ theorems imply the following

**THEOREM 3.1.** *If  $\Omega \subset \mathbb{R}^n$  is open, bounded and strictly convex, if  $\phi \in C^2(\mathbb{R}^n)$ , there exists a unique  $u \in \text{Lip}(\Omega) \cap C^\omega(\Omega)$  satisfying (15) in  $\Omega$  and the boundary condition (18).*

A natural question was asking whether Lebesgue remark for the Laplace equation, would apply to the minimal surface equation. The essential property of harmonic functions, that made Lebesgue method work, is the following:

**LEMMA 3.2.** *If  $u \in C(\Omega)$  is the uniform limit of a sequence  $u_j \in C^\omega(\Omega)$  of harmonic functions, then  $u$  is harmonic.*

The extension of Lemma 3.2 to the solutions of (15) is not trivial. This extension is a consequence of the gradient estimate proved by Enrico Bombieri, Ennio De Giorgi and Mario Miranda [6]:

**THEOREM 3.3 (Bombieri-De Giorgi-Miranda).** *If  $u \in C^\omega(\Omega)$  is a solution of (15), if  $x_0 \in \Omega$  then*

$$(20) \quad |Du(x_0)| \leq c_1 \exp\left(c_2 \frac{\sup_{\Omega} u - u(x_0)}{d}\right)$$

where  $d = \text{dist}(x_0, \partial\Omega)$ ,  $c_1$  and  $c_2$  are real numbers depending only on the dimension of  $\Omega$ .

Theorems 3.1 and 3.3 imply

**THEOREM 3.4.** *If  $\Omega \subset \mathbb{R}^n$  is open, bounded and strictly convex, if  $\phi \in C(\partial\Omega)$ , then there exists a unique  $u \in C(\overline{\Omega}) \cap C^\omega(\Omega)$  satisfying (15) in  $\Omega$  and the boundary condition (18).*

This statement is weaker than Rado’s two dimensional one, and far away from being a good solution for the Dirichlet problem.

Howard Jenkins and James Serrin [19] proved that convexity of  $\Omega$  is no longer a necessary condition in the case  $n > 2$ ;

**THEOREM 3.5 (Jenkins-Serrin).** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  whose boundary is of class  $C^2$ . Then the Dirichlet problem for the minimal surface equation in  $\Omega$  is well posed for  $C^2$  boundary data if and only if the mean curvature of  $\partial\Omega$  is everywhere non-negative.*

REMARK 3.6. Theorem 3.3 proves that Jenkins-Serrin's solutions exist for all continuous boundary data.

REMARK 3.7. The  $C^2$ -regularity of  $\partial\Omega$  is an essential assumption for Jenkins-Serrin's argument. These authors establish an a priori estimate for the gradient of solutions of the Dirichlet problem in terms of the boundary data and the geometry of  $\Omega$ . This estimate in turn depends on a classical barrier argument.

Therefore Jenkins-Serrin's method does not apply to all convex domains.

#### 4. – The minimal surface equation in higher dimensions: De Giorgi's approach

Ennio De Giorgi [11] studied the existence and regularity problem for codimension one minimal surfaces, in the class of Caccioppoli sets of  $\mathbb{R}^n$  [8].

De Giorgi gave the following definition

DEFINITION 4.1. The set  $E \subset \mathbb{R}^n$  has minimal boundary in the open set  $\Omega \subset \mathbb{R}^n$  if

$$(21) \quad P(E, \Omega) < +\infty$$

$$(22) \quad P(E, \Omega) = \inf\{P(B, \Omega) : (B \setminus E) \cup (E \setminus B) \subset\subset \Omega\}.$$

De Giorgi proved the following, non-difficult, existence result

THEOREM 4.2. For any open set  $A \subset\subset \mathbb{R}^n$ , for any Caccioppoli set  $E \subset \mathbb{R}^n$ , there exists a Caccioppoli set  $M \subset \mathbb{R}^n$  such that

$$(23) \quad P(M) = \inf\{P(B) : B \setminus A = E \setminus A\}.$$

REMARK 4.3. The set  $M$  has minimal boundary in  $A$ .

Ennio De Giorgi and John Nash visited Trento on March 6, 1996. In that occasion De Giorgi declared "the regularity theorem for minimal boundaries is the most difficult of all my results".

THEOREM 4.4 (De Giorgi's Regularity). If  $E$  has minimal boundary in the open  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , then the reduced boundary  $\partial^*E$  is analytic in  $\Omega$  and

$$(24) \quad H^{n-1}((\partial E \setminus \partial^*E) \cap \Omega) = 0.$$

To study the boundary behavior of solutions  $M$ , whose existence was proved in Theorem 4.2, I introduced the following definition (see [23] and [22]);

DEFINITION 4.5 (Pseudoconvexity). The open set  $A \subset \mathbb{R}^n$  is pseudoconvex if its boundary  $\partial A$  is Lipschitz and  $\forall x \in \partial A$  there exists a ball  $B_\rho(x)$  such that

$$(25) \quad P(A, B_\rho(x)) \leq P(A \cup L, B_\rho(x)), \quad \forall L \subset\subset B_\rho(x).$$

A strong Maximum Principle type Theorem can be proved for minimal boundaries contained in a pseudoconvex set:

THEOREM 4.6. *If  $M$  has minimal boundary in  $B_\rho(x)$ , if  $A$  is a pseudoconvex set and  $M \cap B_\rho(x) \subset A \cap B_\rho(x)$ , if  $x \in \partial A \cap \partial M \cap B_\rho(x)$ , then*

$$(26) \quad \partial A \cap B_\rho(x) = \partial M \cap B_\rho(x).$$

An easy consequence of Theorem 4.6 is the following

THEOREM 4.7. *If  $A \subset\subset \mathbb{R}^n$  is pseudoconvex, if  $E \subset \mathbb{R}^n$  is a Caccioppoli set and  $M$  satisfies (23), then*

$$(27) \quad \partial A \cap \partial M = \partial A \cap \partial E.$$

We want to show how De Giorgi's approach to the Plateau problem, can be used to solve the Dirichlet problem for the minimal surface equation.

Let us go back to the solutions  $u$  found in Theorem 3.4. They satisfy the following inequalities

$$(28) \quad \int_{\Omega} \sqrt{1 + |Du|^2} \, dx \leq \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx$$

for any  $v \in C(\bar{\Omega})$ , satisfying (18). We can forget condition (18) and claim that

$$(29) \quad \int_{\Omega} \sqrt{1 + |Du|^2} \, dx \leq \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx + \int_{\partial\Omega} |v - \phi| \, dH^{n-1}$$

$\forall v \in C(\bar{\Omega})$ .

In (28) and (29), and for any  $v \in L^1(\Omega)$  by

$$\int_{\Omega} \sqrt{1 + |Dv|^2} \, dx$$

we mean the total variation in  $\Omega$  of the vector distribution

$$(H^n, Dv)$$

where  $H^n$  is the Lebesgue measure and  $Dv$  is the gradient of  $v$  in the sense of distributions.

Therefore, it makes sense to say that the inequality (29) is satisfied for all  $v \in L^1(\Omega)$ , because if

$$\int_{\Omega} \sqrt{1 + |Dv|^2} \, dx < +\infty,$$

i.e.  $v \in BV(\Omega)$ , then

$$\lim_{\Omega \ni x \rightarrow y} v(x)$$

does exist, in the sense of Lebesgue, for  $H^{n-1}$ -almost all  $y \in \partial\Omega$ , and (29) is satisfied by meaning

$$v(y) = \lim_{\Omega \ni x \rightarrow y} v(x), \quad y \in \partial\Omega.$$

The following result was proved in [23]

**THEOREM 4.8.** *For any  $\Omega \subset\subset \mathbb{R}^n$  with Lipschitz boundary,  $\forall \phi \in C(\partial\Omega)$ ,  $\exists u \in C^\omega(\Omega) \cap BV(\Omega)$  such that*

$$(30) \quad \begin{aligned} & \int_{\Omega} \sqrt{1 + |Du|^2} \, dx + \int_{\partial\Omega} |u - \phi| \, dH^{n-1} \\ & \leq \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx + \int_{\partial\Omega} |v - \phi| \, dH^{n-1} \quad \forall v \in L^1(\Omega). \end{aligned}$$

This theorem was precised in the following way

**THEOREM 4.9.** *For any  $\Omega \subset\subset \mathbb{R}^n$  pseudoconvex,  $\forall \phi \in C(\partial\Omega)$ , there exists a unique  $u \in C^\omega(\Omega) \cap BV(\Omega) \cap C(\overline{\Omega})$  such that*

$$(31) \quad u|_{\partial\Omega} = \phi$$

$$(32) \quad \int_{\Omega} \sqrt{1 + |Du|^2} \, dx \leq \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx + \int_{\partial\Omega} |v - \phi| \, dH^{n-1},$$

for any  $v \in L^1(\Omega)$ .

**REMARK 4.10.** The functions  $u$ , that appear in Theorems 4.8 and 4.9 solve the minimal surface equation in  $\Omega$ . Therefore, Theorem 4.9 is an extension of Theorems 2.1 and 3.5.

## 5. – Removable singularities

There is only one joint paper by Ennio De Giorgi and Guido Stampacchia [12]. This paper contains an extension of Theorem 2.4.

**THEOREM 5.1** (De Giorgi-Stampacchia). *If  $\Omega \subset \mathbb{R}^n$  is open and  $K \subset\subset \Omega$  is compact with  $H^{n-1}(K) = 0$ , if  $u \in C^\omega(\Omega \setminus K)$  solves the minimal surface equation, then there exists a unique extension  $u^* \in C^\omega(\Omega)$  of  $u$ .*

De Giorgi-Stampacchia's proof is based on the Maximum Principle proved in Theorem 1.2.

**6. – The Bernstein theorem in higher dimensions**

Wendell H. Fleming [15] proved that the existence of a non-trivial solution for the minimal surface equation in  $\mathbb{R}^n$  would imply the existence of a singular minimal boundary in  $\mathbb{R}^{n+1}$ . Fleming was able to prove the non-existence of singular minimal boundary in  $\mathbb{R}^3$ , getting so another proof of the classical Bernstein theorem.

Ennio De Giorgi [10] improved Fleming’s remark; the existence of a non-trivial solution for the minimal surface equation in  $\mathbb{R}^n$  would imply the existence of a singular minimal boundary in  $\mathbb{R}^n$ . Therefore De Giorgi got the first extension of Bernstein theorem to solutions in  $\mathbb{R}^3$ . Fred J. Almgren [1] and James Simons [34] pushed the validity of Bernstein theorem to solutions in  $\mathbb{R}^7$ .

The proof of Bernstein theorem in higher dimensions, by De Giorgi-Fleming method was stopped by the proof of the existence in  $\mathbb{R}^8$  of singular minimal boundaries, such as

$$\{(x, y) : x \in \mathbb{R}^k, y \in \mathbb{R}^k, x^2 = y^2\}$$

for  $k \geq 4$ . See the famous paper by Bombieri-De Giorgi-Giusti [5], or [21]. In [5] the existence of non-trivial solutions for the minimal surface equation in  $\mathbb{R}^n$ ,  $n \geq 8$ , was proved (see also [2]).

**7. – Harnack type inequalities on minimal boundaries and generalized solutions of the minimal surface equation**

Fundamental remarks about Maximum Principles and Minimal Surfaces can be found in a paper by Enrico Bombieri and Enrico Giusti [7].

One of the most interesting results presented there, is the following

**THEOREM 7.1 (Bombieri-Giusti).** *If  $E$  has minimal boundary in the open set  $\Omega$  of  $\mathbb{R}^{n+1}$ , if  $B_\rho$  is an open ball of  $\mathbb{R}^{n+1}$ , contained in  $\Omega$  and  $u$  is a positive supersolution of a strongly elliptic linear equation in  $\partial E \cap B_\rho$ , i.e.*

$$(33) \quad \int_{\partial E \cap B_\rho} \sum_{i,j=1}^{n+1} a_{ij}(x) \delta_i u \delta_j \phi \, dH^n, \quad \phi \geq 0, \phi \in C_0^1(B_\rho),$$

where  $\delta_i$  are the tangential derivatives on  $\partial E$ , then the following inequality holds

$$(34) \quad \int_{\partial E \cap B_{\varepsilon\rho}} u \, dH^n \leq c \sqrt{\Lambda} \inf_{\partial E \cap B_{\varepsilon\rho}} u \cdot H^n(\partial E \cap B_{\varepsilon\rho}),$$

where

$$(35) \quad \Lambda = \sup_{i,j} \sup_x |a_{ij}(x)|,$$

and  $\varepsilon, c$  are real numbers, depending only on the dimension  $n$ .

This result plays an important role in the analysis of **generalized solutions**, introduced in [26]:

DEFINITION 7.2 (generalized solutions). A Lebesgue measurable function

$$f : \Omega \rightarrow [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\},$$

is a **generalized solution** of the minimal surface equation, iff the set

$$E = \{(x, t) : x \in \Omega, t < f(x)\}$$

has minimal boundary in all open  $A \subset \subset \Omega \times \mathbb{R}$ .

REMARK 7.3.  $f$  remains a generalized solution, when its values are changed on sets of zero measure. We can assume that they are chosen in order to obtain, for the set  $E$ , the following property

$$(36) \quad (x, t) \in \partial E \quad \text{iff} \quad 0 < H^{n+1}(E \cap B_\rho(x, t)) < H^{n+1}(B_\rho), \quad \forall \rho > 0.$$

For the set  $E$ , associated with a generalized solution  $f$ , we can consider the exterior normal unit vector

$$\nu : \partial^* E \cap (\Omega \times \mathbb{R}) \rightarrow \mathbb{R}^{n+1}.$$

$\nu$  is an analytic function, defined  $H^n$ -almost everywhere on  $\partial E \cap (\Omega \times \mathbb{R})$ . Its last component  $\nu_{n+1}$  verifies

$$(37) \quad \nu_{n+1} \geq 0, \quad \sum_{i=1}^{n+1} \delta_i \delta_i \nu_{n+1} \leq 0.$$

Therefore Theorem 7.1 applies to  $\nu_{n+1}$  and we get,

$$\forall x \in \Omega, \quad \rho = \varepsilon \operatorname{dist}(x, \partial\Omega), \quad t \in \mathbb{R} :$$

either

$$(38) \quad \nu_{n+1}|_{\partial^* E \cap B_\rho(x, t)} = 0,$$

or

$$(39) \quad \inf\{\nu_{n+1} : \partial^* E \cap B_\rho(x, t)\} > 0.$$

From (38) one gets

$$(40) \quad \partial^* E \cap B_\rho(x, t) \neq \emptyset,$$

and  $\partial E \cap B_\rho(x, t)$  is a vertical cylinder.

Therefore

$$(41) \quad \nu_{n+1}|_{\partial^* E \cap B_\rho(x,t+s)} = 0, \quad \forall s \in \mathbb{R},$$

and

$$(42) \quad \partial E \cap (B_\rho(x) \times \mathbb{R}) \text{ is a vertical cylinder,}$$

and the same is true for  $E \cap (B_\rho(x) \times \mathbb{R})$ .

From (39) one gets

$$(43) \quad \inf \{ \nu_{n+1} : \partial^* E \cap (B_\rho(x) \times (t-s, t+s)) \} > 0, \quad \forall s > 0;$$

so, there exists an open subset  $G$  of  $B_\rho(x)$ , such that  $f|_G$  is a real analytic function and

$$(44) \quad \partial E \cap (B_\rho(x) \times \mathbb{R}) = \text{graph } f|_G.$$

When  $G = \emptyset$ ,  $f|_{B_\rho(x)}$  is either  $+\infty$  or  $-\infty$ .

A non-difficult argument proves the following

**THEOREM 7.4.** *If  $f$  is a generalized solution, then the sets  $P, N$  defined by*

$$(45) \quad P = \{x \in \Omega : f(x) = +\infty\}, \quad N = \{x \in \Omega : f(x) = -\infty\},$$

*have minimal boundaries in  $\Omega$ .*

The following result completes our description of generalized solutions (see [26]):

**THEOREM 7.5.** *For any generalized solution  $f$  in  $\Omega$ , there exists an open set  $G \subset \Omega$  such that  $f|_G$  is real analytic and a solution of the minimal surface equation. Moreover*

$$(46) \quad \Omega = G \cup P \cup N \cup (\partial P \cap \partial N \cap \Omega),$$

$$(47) \quad \partial E \cap (\Omega \times \mathbb{R}) = \{\text{graph } f|_G\} \cup ((\partial P \cap \partial N \cap \Omega) \times \mathbb{R}),$$

*and  $f|_{G \cup P \cup N}$  is a continuous function with values in  $[-\infty, +\infty]$ .*

**REMARK 7.6.** In the case

$$\Omega = \mathbb{R}^n, \quad n \geq 2$$

the description is much simpler.

The existence of a ball  $B \subset \mathbb{R}^{n+1}$  such that

$$(48) \quad \nu_{n+1}|_{\partial^* E \cap B} = 0,$$

implies the same identity for any other ball  $B^* \supset B$ . Therefore

$E$  and  $\partial E$  are vertical cylinders .

If (48) is never verified and  $\partial E \neq \emptyset$ , then there exists a non-empty open set  $G \subset \mathbb{R}^n$ , such that  $f|_G$  is real analytic and

$$(49) \quad \partial E = \text{graph } f|_G .$$

Moreover

$$(50) \quad \mathbb{R}^n = G \cup P \cup N$$

and  $f$  is continuous everywhere.

This last remark, clarifies De Giorgi-Fleming's remark about Bernstein theorem.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n \geq 2$  be a solution of the minimal surface equation. For any  $\rho > 0$ , let us consider

$$f_\rho : \mathbb{R}^n \rightarrow \mathbb{R}$$

defined by

$$f_\rho(x) = \rho^{-1} f(\rho x), \quad \forall x \in \mathbb{R}^n .$$

The family  $\{f_\rho\}_{\rho>0}$  being compact in the class of generalized solutions, we can assume the existence of  $\rho_h \rightarrow +\infty$  with

$$f_{\rho_h} \rightarrow F$$

where  $F$  is a generalized solution.

Fleming proved that if  $f$  is non-trivial, then  $F$  must be singular. Therefore, because of Remark 7.6,  $F$  must be a singular vertical cylinder, whose horizontal section is a singular boundary in  $\mathbb{R}^n$  (De Giorgi).

A simple application of Theorem 7.4 is the following improvement of De Giorgi-Stampacchia's result about removable singularities:

**THEOREM 7.7** ([25]). *For any  $\Omega \subset \mathbb{R}^n$  open, for any  $K$  closed in  $\Omega$  with*

$$(51) \quad H^{n-1}(K) = 0 ,$$

*for any solution  $u \in C^\omega(\Omega \setminus K)$  of the minimal surface equation, there exists a unique extension  $u^* \in C^\omega(\Omega)$ .*

**PROOF.** The function  $u$  is defined almost everywhere in  $\Omega$ , and (51) implies that it is a generalized solution in  $\Omega$ . The sets  $P$  and  $N$  must be contained in  $K$ , and have minimal boundaries in  $\Omega$ . Therefore

$$(52) \quad P = N = \emptyset ,$$

that implies

$$(53) \quad G = \Omega \text{ i.e. } u|_\Omega \text{ is real analytic.} \quad \square$$

**8. – Approximation of minimal boundaries by graphs of regular solutions**

Let  $E$  have minimal boundary in the open set  $\Omega$  of  $\mathbb{R}^n$ , and  $B \subset\subset \Omega$  be an open ball such that  $\partial E \cap B \neq \emptyset$ . Consider the sequence of Dirichlet problems for the minimal surface equation in  $B$ , with boundary data

$$(54) \quad \phi_j = j \cdot \varphi_E,$$

where  $\varphi_E$  is the characteristic function of  $E$ .

Each solution  $u_j$  takes the value  $j$  at all  $x \in \partial B \cap E^\circ$ , and the value 0 at all  $x \in \partial B \cap \bar{E}$ . Therefore the sequence  $\{u_j\}_{j \in \mathbb{N}}$  is increasing as  $j \rightarrow +\infty$  and converges to a generalized solution  $f$  in  $B$ .

For such an  $f$  there exists a non-empty open set  $G \subset B$ , such that

$$f|_G \text{ is real analytic.}$$

Moreover

$$(55) \quad f(x) = +\infty, \quad \forall x \in B \setminus G = \bar{E} \cap B.$$

Let us consider now the sequence

$$f_h = f - h, \quad h \in \mathbb{N}.$$

The sequence of analytic minimal surfaces

$$\text{graph } f_h|_G$$

converges to the vertical cylinder

$$(\partial E \cap B) \times \mathbb{R}.$$

**9. – The strong maximum principle for minimal boundaries**

Let  $E_1, E_2$  have minimal boundaries in the open set  $\Omega$  of  $\mathbb{R}^n$ . Assume

$$(56) \quad E_1 \cap \Omega \supset E_2 \cap \Omega, \quad x_0 \in \Omega \cap \partial E_1 \cap \partial E_2.$$

If  $x_0$  is a regular point for  $\partial E_1$ , i.e.  $x_0 \in \partial^* E_1$ , then  $x_0 \in \partial^* E_2$  and the classical Strong Maximum Principle implies

$$(57) \quad \exists \rho > 0 : \quad E_1 \cap B_\rho(x_0) = E_2 \cap B_\rho(x_0).$$

Maria Pia Moschen [27] proved (57), by using the properties of generalized solutions, in the singular case

$$(58) \quad x_0 \in \partial E_1 \setminus \partial^* E_1.$$

### 10. – Complete minimal graphs and entire solutions of the minimal surface equation

As we have already recalled, Bombieri-De Giorgi-Giusti proved the existence of non-trivial entire solutions of the minimal surface equation in  $\mathbb{R}^n$  for  $n \geq 8$ . The proof presented in [5] consisted in a very smart combination of elementary functions to define a couple of global barriers for the Dirichlet problem.

In [24] I presented a more direct method for proving the existence of non-trivial generalized solutions in  $\mathbb{R}^n$  for  $n \geq 8$ . My solutions were analytic complete graphs, but I was not able to prove for them

$$(59) \quad P = N = \emptyset.$$

This was done by Leon Simon in [33].

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