GIUSEPPE DA PRATO

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Poincaré Inequality for Some Measures in Hilbert Spaces and Application to Spectral Gap for Transition Semigroups

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1. – Introduction and setting of the problem

Let H be a separable Hilbert space (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), and let ν be a Borel measure on H. This paper is devoted to prove, under suitable assumptions on ν, an estimate of this kind (Poincaré inequality):

$$\int_H \left|\varphi(x) - \int_H \varphi(y)\nu(dy)\right|^2 \nu(dx) \le C \int_H |D\varphi(x)|^2 \nu(dx)\,, \tag{1.1}$$

where C is a suitable positive constant.

Estimate (1.1) can be used to study the *spectral gap* for a transition semigroup corresponding to a differential stochastic equation:

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + Q^{1/2}dW(t),\ t \ge 0\,, \\ X(0) = x\,, \end{cases} \tag{1.2}$$

Here $A : D(A) \subset H \to H$ and $Q : H \to H$, are linear operators, $F : H \to H$ is nonlinear, and $W(t)$, $t \ge 0$ is an H-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [5].

Assume that problem (1.2) has unique solution $X(t,x)$, then the corresponding transition semigroup P_t, $t \ge 0$, is defined by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))],\ \varphi \in B_b(H)\,, \tag{1.3}$$

where $B_b(H)$ is the Banach space of all bounded and Borel functions from H into $\mathbb{R}$. We want to prove, under suitable assumptions, an estimate

$$\int_H \left|P_t\varphi(x) - \int_H \varphi(y)\nu(dy)\right|^2 \nu(dx) \le Ce^{-\omega t} \int_H |\varphi(x)|^2 \nu(dx)\,, \tag{1.4}$$

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for all $\varphi \in L^2(H, \nu)$, where ν is an invariant measure for the semigroup, and C, ω are positive constants.

Estimate (1.4) implies that the spectrum $\sigma(\mathcal{L})$ of the infinitesimal generator $\mathcal{L}$ of P_t in $L^2(H, \nu)$ has the following property

$$\sigma(\mathcal{L})\backslash\{0\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < \omega\}. \tag{1.5}$$

This *spectral gap* property is important in the applications, it has been studied in the literature, mainly when the semigroup P_t is symmetric, see [8], [9], [5].

The content of the paper is the following. In Section 2 we prove a Poincaré inequality when $\nu = \mu_R$ is a Gaussian measure of mean 0 and covariance operator $R \in \mathcal{L}_1^+(H)$, the space of all nonnegative, symmetric, linear operators from H into H of trace class. In this case estimate (1.1) is a natural generalization of a well known result when H is finite dimensional. Then we consider in Section 3 the case when ν is absolutely continuous with respect to a Gaussian measure μ_R. Finally section Section 4 is devoted to the spectral gap property.

2. – Poincaré inequality for Gaussian measures

We are given a Gaussian measure μ_R, on H with mean 0 and covariance operator $R \in \mathcal{L}_1^+(H)$. We denote by $\{e_k\}$ a complete orthonormal system in H consisting of eigenvectors of R and by $\{\lambda_k\}$ the corresponding sequence of eigenvalues:

$$\mathrm{R}e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We shall assume that sequence $\{\lambda_k\}$ is nonincreasing and that $\lambda_k > 0$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ we shall denote by D_k the derivative in the direction of e_k, and we shall set $x_k = \langle x, e_k \rangle$ for any $x \in H$. It is well known that D_k is a closable operator on $L^2(H, \mu)$, see e.g. [7]. The Sobolev space $W^{1,2}(H, \mu_R)$ is the Hilbert space of all $\varphi \in L^2(H, \mu_R) \cap$ dom(D_k), $k \in \mathbb{N}$, such that

$$\|\varphi\|^2_{W^{1,2}(H,\mu_R)} := \int_H |\varphi(x)|^2 \mu_R(dx) + \sum_{k=1}^{\infty} \int_H |D_k \varphi(x)|^2 \mu_R(dx) < +\infty.$$

We denote by $\mathcal{E}(H)$ the linear space spanned by all exponential functions $\psi(x) = e^{\langle h, x \rangle}$, $x \in H$. Obviously

$$\mathcal{E}(H) \subset C^\infty(H) \cap L^2(H, \mu_R).$$

and $\mathcal{E}(H)$ is dense in $L^2(H, \mu_R)$.

We denote by T_t, $t \geq 0$, the Ornstein-Uhlenbeck semigroup:

$$T_t \varphi(x) = \int_H \varphi(e^{-t/2}x + y)\mu_{(1-e^{-t})R}(dy), \ t \geq 0, \ \varphi \in L^2(H, \mu_R). \tag{2.1}$$

It is well known that T_t, $t \geq 0$, is a strongly continuous semigroup of contractions on $L^2(H, \mu_R)$ having as unique invariant measure μ_R:

$$\int_H T_t\varphi(x)\mu_R(dx) = \int_H \varphi(x)\mu_R(dx), \ t \geq 0, \ \varphi \in L^2(H, \mu_R). \tag{2.2}$$

We denote by $\mathcal{L}$ the infinitesimal generator of T_t, $t \geq 0$. $\mathcal{L}$ is defined as the closure of the linear operator $\mathcal{L}_0$:

$$\mathcal{L}_0\varphi(x) = \frac{1}{2}\operatorname{Tr}[RD^2\varphi(x)] - \frac{1}{2}\langle x, D\varphi(x)\rangle, \ \varphi \in \mathcal{E}(H), \ x \in H. \tag{2.3}$$

We recall also that, for any $\varphi \in D(\mathcal{L})$ we have, see [1], [6],

$$\int_H \mathcal{L}\varphi(x)\varphi(x)\mu_R(dx) = -\frac{1}{2}\int_H |D\varphi(x)|^2\mu_R(dx). \tag{2.4}$$

Now we prove the result

THEOREM 2.1. *The following estimate holds*

$$\int_H |\varphi(x) - \overline{\varphi}|^2\mu_R(dx) \leq \int_H |R^{1/2}D\varphi(x)|^2\mu_R(dx), \ \varphi \in W^{1,2}(H, \mu_R), \tag{2.5}$$

where

$$\overline{\varphi} = \int_H \varphi(x)\mu_R(dx). \tag{2.6}$$

PROOF. For any $\varphi \in D(\mathcal{L})$ we have, in view of (2.4)

$$\begin{aligned} \frac{d}{dt}\int_H |T_t\varphi(x)|^2\mu(dx) &= 2\int_H \mathcal{L}T_t\varphi(x)T_t\varphi(x)\mu(dx) \\ &= -\int_H |R^{1/2}DT_t\varphi(x)|^2\mu(dx). \end{aligned} \tag{2.7}$$

To estimate $|R^{1/2}DT_t\varphi(x)|^2$ note that, in view of (2.1),

$$\langle Q^{1/2}DT_t\varphi(x), h\rangle = e^{-t/2}\int_H \langle D\varphi(e^{-t/2}x + y), h\rangle\mu_{R(1-e^{-t})}(dy),$$

for all $h \in H$. It follows, using Hölder's inequality

$$|\langle R^{1/2}DT_t\varphi(x), h\rangle|^2 \leq e^{-t}|h|^2T_t(|R^{1/2}D\varphi|^2)(x), \ h \in H.$$

Therefore, due to the arbitrariness of h,

$$|R^{1/2}DT_t\varphi(x)|^2 \leq e^{-t}T_t(|R^{1/2}D\varphi|^2)(x). \tag{2.8}$$

By integra ing on H with respect to μ_R, and taking into account the invariance of μ_R, we have

$$\int_H |R^{1/2} D T_t\varphi(x)|^2 \mu_R(dx) \le e^{-t} \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Now, comparing with (2.7) we find

$$\frac{d}{dt} \int_H |T_t\varphi(x)|^2 \mu_R(dx) \ge -e^{-t} \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Integrating in t find

$$\int_H |T_t\varphi(x)|^2 \mu_R(dx) \ge \int_H |\varphi(x)|^2 \mu_R(dx) - (1-e^{-t}) \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx).$$

Finally, letting t tend to $+\infty$, and using the fact that, as easily checked,

$$\lim_{t\to+\infty} P_t\varphi(x) = \overline{\varphi},\ x \text{ a.e. in } H,$$

we get

$$(\overline{\varphi})^2 \ge \int_H |\varphi(x)|^2 \mu_R(dx) - \int_H |R^{1/2} D\varphi(x)|^2 \mu_R(dx),$$

that is equivalent to (2.5). □

3. – Poincaré inequality for non Gaussian measures

Here we are given, besides a Gaussian measure $\mu = \mu_R$, with $R \in \mathcal{L}_1^+(H)$ and $\ker R = \{0\}$, a function $U : H \to \mathbb{R}$, such that

HYPOTHESIS 1.

(i) U is co vex and of class C^2.

(ii) DU is . ipschitz continuous.

We set

$$\alpha(x) = k e^{-2U(x)},\ x \in H, \tag{3.1}$$

where k is chosen such that

$$\int_H \alpha(x)\mu(dx) = 1.$$

Finally we consider the Borel probability measure on H

$$\nu(dx) = \alpha(x)\mu(dx).$$

We are going to prove a Poincaré estimate for measure ν. We notice that assumptions on α could be considerably weakned. It will be enough to assume convexity of U (that implies dissipativity of $-DU$), and some additional properties similar to [5]. But we prefer to make Hypothesis 1 for the sake of simplicity.

It is useful to introduce a differential stochastic equation having ν as invariant measure:

$$\begin{cases} dZ = (AZ - DU(Z))dt + dW(t) \\ Z(0) = x \in H, \end{cases} \tag{3.2}$$

where A is the negative self-adjoint operator in H defined as

$$A = -\frac{1}{2} R^{-1},$$

and W is a cylindrical H-valued Wiener process in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Problem (3.2) has a unique solution $Z(t, x)$, and measure ν is invariant, see [5]. The corresponding transition semigroup is defined in $L^2(H, \nu)$ by

$$N_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \ \varphi \in L^2(H, \nu), \ t \geq 0. \tag{3.3}$$

Its infinitesimal generator $\mathcal{N}$ is defined by, see [4]

$$\begin{aligned} D(\mathcal{N}) = \Big\{ &\varphi \in W^{2,2}(H; \nu) \cap W_A^{1,2}(H; \nu) : \\ &\int_H \langle D^2U(x)D\varphi(x), D\varphi(x)\rangle \nu(dx) < +\infty \Big\}, \end{aligned} \tag{3.4}$$

where $W_A^{1,2}(H; \nu)$ is the linear space of all $\varphi \in W^{1,2}(H, \nu)$ such that $\langle AD\varphi, D\varphi\rangle \in L^2(H, \nu)$.

Finally in [5] it is proved that ν is strongly mixing

$$\lim_{t\to\infty} N_t\varphi(x) = \int_H \varphi(y)\nu(dy), \ \varphi \in L^2(H, \nu). \tag{3.5}$$

We can now prove

THEOREM 3.1. *The following estimate holds*

$$\int_H |\varphi(x) - \overline{\varphi}|^2 \nu(dx) \leq \frac{1}{\|R\|} \int_H |D\varphi(x)|^2 \nu(dx), \ \varphi \in W^{1,2}(H, \nu), \tag{3.6}$$

where

$$\overline{\varphi} = \int_H \varphi(x)\nu(dx). \tag{3.7}$$

PROOF. For any $\varphi \in D(\mathcal{N})$ we have, see [4],

$$\begin{aligned}\frac{d}{dt}\int_H |N_t\varphi(x)|^2\nu(dx) = {} & 2\int_H \mathcal{N}N_t\varphi(x)N_t\varphi(x)\nu(dx)\\ = {} & -\int_H |DN_t\varphi(x)|^2\nu(dx)\,.\end{aligned} \tag{3.8}$$

We want now to estimate $|DN_t\varphi(x)|^2$. To this purpose we note that $X(t,x)$ is differentiable with respect to x and

$$\|X_x^*(t,x)\| \le e^{-\frac{1}{2\|R\|}t},\ t \ge 0\,. \tag{3.9}$$

It follows

$$DN_t\varphi(x) = \mathbb{E}[X_x^*(t,x)D\varphi(X(t,x))]\,. \tag{3.10}$$

Now by (3.10) and the Hölder's estimate, it follows,

$$|DN_t\varphi(x)|^2 \le e^{-\frac{1}{\|R\|}t}N_t(|D\varphi|^2(x)\,. \tag{3.11}$$

By integrating on H with respect to ν, and taking into account the invariance of ν, we have

$$\int_H |DN_t\varphi(x)|^2\nu(dx) \le e^{-\frac{1}{\|R\|}t}\int_H |D\varphi(x)|^2\nu(dx)\,.$$

By substituting in (3.8) we find

$$\frac{d}{dt}\int_H |N_t\varphi(x)|^2\nu(dx) \ge -e^{-\frac{1}{\|R\|}t}\int_H |D\varphi(x)|^2\nu(dx)\,.$$

Integrating in t we have

$$\int_H |N_t\varphi(x)|^2\nu(dx) \ge \int_H |\varphi(x)|^2\nu(dx) - \|R\|(1-e^{-\frac{1}{\|R\|}t})\int_H |D\varphi(x)|^2\nu(dx)\,.$$

Finally, letting t tend to $+\infty$, and using (3.5) we get

$$(\overline{\varphi})^2 \ge \int_H |\varphi(x)|^2\nu(dx) - \|R\|\int_H |D\varphi(x)|^2\nu(dx)\,,$$

and the conclusion follows. □

4. – Spectral gap

4.1. – Gaussian case

We are here concerned with the Ornstein-Uhlenbeck process $X(\cdot, x)$ solution of the following differential stochastic equation

$$\begin{cases} dX(t) = AX(t)dt + Q^{1/2}dW(t), \ t \geq 0, \\ X(0) = x, \end{cases} \tag{4.1}$$

under the following assumptions.

HYPOTHESIS 2.

(i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on H.

(ii) Q is bounded, symmetric, and nonnegative.

(iii) For all $t > 0$ the operator $e^{tA}Qe^{tA^*}$ is of trace class and its kernel is equal to $\{0\}$. Moreover

$$\int_0^{+\infty} \mathrm{Tr}[e^{tA}Qe^{tA^*}]dt < +\infty .$$

If Hypothesis 2 holds the linear operator

$$Q_\infty x = \int_0^{+\infty} e^{tA}Qe^{tA^*}x\, dt, \ x \in H,$$

is well defined and it is of trace-class. Moreover problem (4.1) has a unique mild solution given by, see [5]

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s). \tag{4.2}$$

The corresponding transition semigroup $P_t, t \geq 0$, is defined by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x)] = \int_H \varphi(e^{tA}x + y)\mu_{Q_t}(dy), \ \varphi \in B_b(H), \tag{4.3}$$

where

$$Q_t x = \int_0^t e^{sA}Qe^{sA^*}x\, ds, \ x \in H.$$

Finally the measure μ_{Q_∞} is invariant, and so the semigroup $P_t, t \geq 0$, can be uniquely extended to a strongly continuous semigroup of contractions on $L^2(H, \mu)$, that we still denote by $P_t, t \geq 0$. Its infinitesimal generator will be denoted by $\mathcal{L}$.

THEOREM 4.1. *Assume, besides Hypothesis 2 that*

$$Q^{1/2}(H) \subset Q_\infty^{1/2}(H). \tag{4.4}$$

Then for any $\varphi \in W^{1,2}(H,\mu)$ we have

$$\int_H |P_t\varphi(x) - \overline{\varphi}|^2\,\mu(dx) \le e^{-\frac{1}{\|Q^{-1/2}Q_\infty^{1/2}\|}t} \int_H |\varphi(x)|^2\mu(dx), \tag{4.5}$$

where

$$\overline{\varphi} = \int_H \varphi(y)\mu(dy)$$

PROOF. By the Poincaré inequality (2.5), with $R = Q_\infty$, it follows

$$\int_H |\varphi(x) - \overline{\varphi}|^2\mu(dx) \le \|Q^{-1/2}Q_\infty^{1/2}\| \int_H |Q^{1/2}D\varphi(x)|^2\mu(dx).$$

We also recall that, for any $\varphi \in D(\mathcal{L})$ we have, see [1], [6],

$$\int_H \mathcal{L}\varphi(x)\varphi(x)\mu(dx) = -\frac{1}{2}\int_H |Q^{1/2}D\varphi(x)|^2\mu(dx).$$

This implies

$$\int_H \mathcal{L}\varphi(x)\varphi(x)\mu(dx) \le \frac{1}{2\|Q^{-1/2}Q_\infty^{-1/2}\|}\int_H |\varphi(x) - \overline{\varphi}|^2\mu(dx). \tag{4.6}$$

Let now consider the space

$$Y = \left\{\varphi \in L^2(H,\mu) :\ \overline{\varphi} = 0\right\}.$$

Y is obviously an invariant subspace of P_t, $t \ge 0$; denote by $\mathcal{L}_Y$ the part of $\mathcal{L}$ in Y. By (4.6) it follows

$$\int_H \mathcal{L}_Y\varphi(x)\varphi(x)\mu(dx) \le \frac{1}{2\|Q^{-1/2}Q_\infty^{-1/2}\|}\int_H |\varphi(x)|^2\mu(dx),\ \varphi \in D(\mathcal{L}_Y). \tag{4.7}$$

It is easy to check that this inequality yields (4.5). □

Another condition implying the spectral gap property holds when the semigroup P_t, $t \geq 0$; is strong Feller.

HYPOTHESIS 3. For any $t > 0$ we have

$$e^{tA}(H) \subset Q_t^{1/2}(H).$$

When Hypothesis 3 is fulfilled we set

$$\Gamma(t) = Q_t^{1/2} Q_t^{-1/2} e^{tA}, \ t > 0.$$

We recall that $\|\Gamma(t)\|$ is nonincreasing in t and $\lim_{t\to 0} \|\Gamma(t)\| = +\infty$. Moreover for any $\varphi \in L^2(H, \mu)$ and any $t > 0$, one has $P_t\varphi \in W^{1,2}(H, \mu)$ and the following estimate holds, see [5],

$$\int_H |DP_t\varphi(x)|^2 \mu(dx) \leq \|\Gamma(t)\|^2 \int_H |\varphi(x)|^2 \mu(dx). \tag{4.8}$$

THEOREM 4.2. *Assume, besides Hypotheses* 2 *and* 3, *that there exist* $M, \omega > 0$ *such that*

$$\|Q_\infty^{1/2} e^{tA}\| \leq Me^{-\omega t}, \ t \geq 0.$$

The there exists $M_1 > 0$ *such that the following estimate holds*

$$\int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq M_1 e^{-2\omega t} \int_H |\varphi(x)|^2 \mu(dx). \tag{4.9}$$

PROOF. Replacing in (2.5) φ with $P_t\varphi$, and taking into account that $\overline{P_t\varphi} = \overline{\varphi}$ by the invariance of μ, we have

$$\int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) \leq \int_H |Q_\infty^{1/2} DP_t\varphi(x)|^2 \mu(dx), \ \varphi \in W^{1,2}(H, \mu).$$

Since

$$DP_t\varphi(x) = e^{tA^*} P_t D\varphi(x),$$

it follows

$$\begin{aligned}\int_H |P_t\varphi(x) - \overline{\varphi}|^2 \mu(dx) &\leq \|Q_\infty^{1/2} e^{tA^*}\|^2 \int_H |DP_t\varphi(x)|^2 \mu(dx) \\ &\leq M^2 e^{-2\omega t} \int_H |D\varphi(x)|^2 \mu(dx).\end{aligned}$$

By replacing φ with $P_1\varphi$, and taking into account (4.8), we find

$$\begin{aligned}\int_H |P_{t+1}\varphi(x) - \overline{\varphi}|^2 \mu(dx) &\leq M^2 e^{-2\omega t} \int_H |DP_1\varphi(x)|^2 \mu(dx) \\ &\leq M^2 e^{-2\omega t} \|\Gamma(1)\|^2 \int_H |\varphi(x)|^2 \mu(dx).\end{aligned}$$

By replacing $t + 1$ with t the conclusion follows. □

4.2. – Non Gaussian case

We are here concerned with the solution $X(\cdot, x)$ of the following differential stochastic equation

$$\left\{ \begin{array}{l} dX(t) = (AX(t) + F(X))dt + dW(t),\ t \geq 0, \\ X(0) = x, \end{array} \right. \tag{4.10}$$

under the following assumptions.

HYPOTHESIS 4.

(i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on H and there exists $\omega > 0$ such that $\|e^{tA}\| \leq e^{-\omega t}, t \geq 0$.

(ii) For all $t > 0$ the operator $e^{tA}e^{tA^*}$ is of trace class, and $\int_0^\infty \mathrm{Tr}[e^{tA}e^{tA^*}]dt < +\infty$.

(iii) $F : H \to H$ is uniformly continuous and bounded together with its Fréchet derivative.

If Hypothesis 4 holds the linear operator

$$Q_\infty x = \int_0^{+\infty} e^{tA}e^{tA^*}x\, dt,\ x \in H,$$

is well defined and it is of trace-class. Moreover problem (4.10) has a unique mild solution, see [5]. The corresponding transition semigroup $P_t, t \geq 0$, is defined by as before by

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x)],\ \varphi \in B_b(H). \tag{4.11}$$

We set $\mu = \mu_{Q_\infty}$, and denote by $\mathcal{E}_A(H)$ the vector space generated by all functions of the form

$$\varphi(x) = e^{\langle h, x\rangle},\ h \in D(A^*).$$

We denote by $\mathcal{L}$ the infinitesimal generator of $P_t,\ t \geq 0$. $\mathcal{L}$ is defined as the closure of the linear operator $\mathcal{L}_0$:

$$\begin{aligned} \mathcal{L}\varphi(x) &= \frac{1}{2}\mathrm{Tr}[D^2\varphi(x)] \\ &+ \langle x, A^*D\varphi(x)\rangle + \langle F(x), D\varphi(x)\rangle,\ \varphi \in \mathcal{E}_A(H),\ x \in H. \end{aligned} \tag{4.12}$$

We need an integration by parts formula.

LEMMA 4.3. *Assume that Hypotheses* 1 *and* 4 *hold. Let α be defined by* (3.1), *and let $\varphi, \psi \in \mathcal{E}_A(H)$. Then the following identity holds.*

$$\begin{aligned} &\int_H [D_k\varphi(x)\psi(x) + \varphi(x)D_k\psi(x)]\nu(dx) \\ &= \int_H \left(\frac{x_k}{\lambda_k} - D_k \log\alpha(x)\right)\varphi(x)\psi(x)\nu(dx). \end{aligned} \tag{4.13}$$

PROOF. Denote by J the left hand side of (4.13). Taking into account a well known result on Gaussian measures, we have

$$
\begin{aligned}
J &= \int_H [D_k\varphi(x)\psi(x)\alpha(x) + \varphi(x)D_k\psi(x)\alpha(x)]\mu(dx) \\
&= \int_H [-\varphi(x)D_k(\psi(x)\alpha(x))\varphi(x)D_k\psi(x)\alpha(x)]\mu(dx) \\
&\quad + \int_H \frac{x_k}{\lambda_k}\,\alpha(x)\varphi(x)\psi(x)\mu(dx) \\
&= \int_H \left(\frac{x_k}{\lambda_k} - D_k\alpha(x)\right)\varphi(x)\psi(x)\mu(dx)\,.
\end{aligned}
$$

The conclusion follows. □

PROPOSITION 4.4. *Assume that Hypotheses* 1 *and* 4 *hold. Let* α *be defined by* (3.1) *and* $\mathcal{L}$ *by* (4.12). *Then for any* $\varphi, \psi \in \mathcal{E}(H)$ *we have*

$$
\begin{aligned}
\int_H \mathcal{L}\varphi(x)\psi(x)\nu(dx) &= \int_H \langle AQ_\infty D\psi(x), D\varphi(x)\rangle\nu(dx) \\
&\quad + \int_H \langle AQ_\infty D\log\alpha(x) + F(x), D\psi(x)\rangle\varphi(x)\nu(dx)\,,
\end{aligned} \tag{4.14}
$$

and

$$
\begin{aligned}
\int_H \mathcal{L}\varphi(x)\varphi(x)\nu(dx) &= -\frac{1}{2}\int_H |D\varphi(x)|^2\nu(dx) \\
&\quad + \int_H \langle AQ_\infty D\log\alpha(x) + F(x), D\varphi(x)\rangle\varphi(x)\nu(dx)\,.
\end{aligned} \tag{4.15}
$$

Notice that

$$Q_\infty(H) \subset D(A)\,,$$

see [3], so that AQ_∞ is a well defined bounded operator.

PROOF. We first compute the integral

$$J = \int_H \langle Ax, D\varphi(x)\rangle\psi(x)\nu(dx)\,.$$

We denote by $\{e_k\}$ a complete orthonormal system in H consisting of eigenvectors of Q_∞ and by $\{\lambda_k\}$ the corresponding sequence of eigenvalues:

$$Q_\infty e_k = \lambda_k e_k, \quad k \in \mathbb{N}\,.$$

We assume for simplicity that $\{e_k\} \subset D(A)$, this extra assumption can be easily removed by approximating A with its Yosida approximations. We have

$$\langle Ax, D\varphi(x)\rangle = \sum_{h,k=1}^{\infty} a_{h,k}x_k D_h\varphi(x)\,,$$

where $a_{h,k} = \langle Ae_k, e_h\rangle$, and $x_k = \langle x, e_k\rangle$. We proceed here as in [6]. By integration by parts formula (4.11) we have

$$\int_H x_k D_h\varphi(x)\psi(x)\nu(dx) = \int_H \lambda_k D_h D_k\varphi(x)\psi(x)\nu(dx) + \int_H \lambda_k D_h\varphi(x) D_k\psi(x)\nu(dx) + \int_H \lambda_k D_k \log\alpha(x) D_h\varphi(x)\psi(x)\nu(dx)\,.$$

It follows

$$J = \int_H \mathrm{Tr}[AQ_\infty D^2\varphi(x)]\psi(x)\nu(dx) + \int_H \langle AQ_\infty D\psi(x), D\varphi(x)\rangle\nu(dx) + \int_H \langle AQ_\infty D\log\alpha(x), D\varphi(x)\rangle\psi(x)\nu(dx)\,.$$

Now, taking into account (4.12), a simple computation yields (4.14). Finally (4.15) follows as in [6], recalling the Lyapunov equation

$$AQ + QA^* + Q_\infty = 0\,.$$ □

THEOREM 4.5. *Assume that Hypotheses* 1 *and* 4 *hold. Assume in addition that* α*, defined by* (3.1)*, can be chosen such that*

$$F(x) = -AQ_\infty D\log\alpha(x),\ x\in H\,. \tag{4.16}$$

Then ν *is an invariant measure for* P_t*,* $t \ge 0$*, and for all* $\varphi \in L^2(H,\mu)$ *we have*

$$\int_H |P_t\varphi(x) - \overline{\varphi}|^2\,\nu(dx) \le e^{-\frac{1}{Q_\infty}t}\int_H |\varphi(x)|^2\nu(dx)\,, \tag{4.17}$$

where

$$\overline{\varphi} = \int_H \varphi(y)\nu(dy)$$

PROOF. First notice that if (4.16) holds, then setting $\psi(x) = 1$, $x \in H$, we have by (4.14)

$$\int_H \mathcal{L}\varphi(x)\nu(dx) = 0,\quad \varphi\in D(\mathcal{L})\,.$$

This implies that ν is invariant for P_t, $t \ge 0$. Now by (4.15) it follows

$$\int_H \mathcal{L}\varphi(x)\varphi(x)\nu(dx) = -\frac{1}{2}\int_H |D\varphi(x)|^2\nu(dx),\quad \varphi\in D(\mathcal{L})\,.$$

Consequently, by (3.6) we have

$$\int_H \mathcal{L}\varphi(x)\varphi(x)\nu(dx) \le -\frac{1}{2\|Q_\infty\|}\int_H |\varphi(x) - \overline{\varphi}\nu(dx)\,.$$

Arguing as in the proof of Theorem 4.1, we arrive at (4.17). □

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Scuola Normale Superiore di Pisa
Piazza dei Cavalieri, 7
56100 Pisa, Italy