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A Convex Darboux Theorem

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In memory of Ennio de Giorgi

Abstract. We give a necessary and sufficient condition for a vector field to be a linear combination of gradients of concave functions with positive coefficients.

Let ω be a smooth 1-form, defined on a neighbourhood $\mathcal{U}$ of some point $\bar{x}$ in $\mathbb{R}^N$. When is ω a linear combination of K differentials? In other words, given a number $K \leq N$, when can we find $2K$ functions a^k and u_k, with $1 \leq k \leq K$, such that:

$$(1) \qquad \omega = \sum_k a^k du_k \, .$$

The case $K = 1$ was solved by Frobenius, and the general case by Darboux: a necessary and sufficient condition for such a decomposition to hold on some neighbourhood $\mathcal{V}$ of $\bar{x}$ is that $\omega \wedge d\omega^K = 0$ on $\mathcal{U}$, where $d\omega^K$ denotes the k-fold wedge product $d\omega \wedge d\omega \ldots \wedge d\omega$. See [1] or [2] for a proof of Frobenius' theorem, and [2] for a proof of Darboux' theorem.

This paper addresses a further question, which arises from certain applications to economic theory: when can the functions u_k be taken to be concave, and the a^k positive? In the problems we have in mind (see [8] for the state of the art until 1971, and [4], [5], [6], [7], for recent developments), the u_k are to be understood as (direct or indirect) utility fuctions, and the a^k as Lagrange multipliers. The concavity and positivity requirements are then essential for the mathematical results to have an economic interpretation.

Necessary conditions are easy to find. The first one, of course, is the Darboux condition $\omega \wedge d\omega^K = 0$. Let us retrace it quickly: taking the exterior derivative of equation (1), we get:

$$(2) \qquad d\omega = \sum_k da^k \wedge du_k$$

so that:

$$(3) \quad \omega \wedge d\omega^K = \left(\sum_k a^k du_k\right) \wedge (da^1 \wedge du_1 \wedge \dots da^K \wedge du_K)K! = 0.$$

A second necessary condition follows from differentiating at $\bar{x}$. Define the matrix Ω by:

$$(4) \quad \Omega_{i,j} = \frac{\partial \omega_i}{\partial x^j}(\bar{x}).$$

Then, rewrite equation (1) as follows:

$$(5) \quad \omega_i = \sum_k a^k \frac{\partial u_k}{\partial x^i}$$

and differentiate at $\bar{x}$. We get:

$$(6) \quad \Omega_{i,j} = \sum_k a^k \frac{\partial^2 u_k}{\partial x^i \partial x^j} + \sum_k \frac{\partial a^k}{\partial x^j} \frac{\partial u_k}{\partial x^i}$$

which we rewrite as:

$$(7) \quad \Omega = \sum_k a^k(\bar{x})Q_k + \sum_k X^k \otimes Y_k$$

where $X \otimes Y$ denotes the rank one matrix $X^i Y_j$ and we have set:

$$(8) \quad Q_{k,i,j} = \frac{\partial^2 u_k}{\partial x^i \partial x^j}(\bar{x})$$

$$(9) \quad X^k_j = \frac{\partial a^k}{\partial x^j}(\bar{x})$$

$$(10) \quad Y_{k,i} = \frac{\partial u_k}{\partial x^i}(\bar{x}).$$

Since the u_k are concave, the Q_k are negative semi-definite; recall also that the a^k are assumed to be positive. It then follows from equation (7) that Ω is the sum of a symmetric, negative semi-definite matrix, and of a matrix of rank K.

We will now prove that these necessary conditions are also sufficient.

THEOREM 1. *Let ω be an analytic 1-form such that:*

$$(11) \qquad \omega \wedge d\omega^K = 0$$

$$(12) \qquad \omega \wedge d\omega^{K-1} \neq 0$$

on a neighbourhood $\mathcal{U}$ of $\overline{x}$. Define a matrix Ω by:

$$(13) \qquad \Omega_{i,j} = \frac{\partial \omega_i}{\partial x^j}(\overline{x})$$

and assume that

$$(14) \qquad \Omega = Q + R$$

where the matrix Q is symmetric and negative definite, and the matrix R has rank K.

Then there exists a convex neighbourhood $\mathcal{V} \in \mathcal{U}$ of $\overline{x}$, and real analytic functions $u_1, \ldots, u_K$ and $a^1, \ldots, a^K$, defined on $\mathcal{V}$, such that the u_k are strictly convex, the a^k are positive, and:

$$(15) \qquad \omega = \sum_{k=1}^{K} a^k du_k .$$

It will follow from the proof that the du_k can be chosen arbitrarily close to ω, and the Hessians $\left(\frac{\partial^2 v_k}{\partial x_i \partial x_j} \right)$ arbitrarily close to Q, which implies, of course, that the λ_k will be arbitrarily close to $1/K$.

The proof itself relies on the Cartan-Kähler theorem (see [3], [2], or [5]). This is a theorem of Cauchy-Kowalewska type, so everything has to be real analytic. In contrast, the standard (non-convex) Darboux theorem only requires ω to be C^1 (or weaker; see for instance the work of Hartman: [9], [10]). We conjecture that our result holds true with weaker regularity, C^∞ for instance, but we have no idea how to prove it. However, V. M. Zakaljukin has proved the following result [11], which is another kind of convex Darboux theorem

THEOREM 2. *Let ω be a C^1 1-form on a neighbourhood $\mathcal{U}$ of $\overline{x}$ such that:*

$$(16) \qquad \omega \wedge d\omega^K \neq 0 .$$

Then there exists a convex neighbourhood $\mathcal{V} \in \mathcal{U}$ of $\overline{x}$, C^2 functions $u_1, \ldots, u_{K+1}$ and C^1 functions $a^1, \ldots, a^{K+1}$, defined on $\mathcal{V}$, such that the u_k are strictly convex, the a^k are positive, and:

$$(17) \qquad \omega = \sum_{k=1}^{K+1} a^k du_k .$$

Let us now prove Theorem 1. We will take $K = 2$; the proof in the general case is basically the same, but the notations obscure the argument. So we have, on a neighbourhood $\mathcal{U}$ of $\bar{x}$:

$$(18) \qquad\qquad \omega \wedge d\omega \wedge d\omega = 0$$

$$(19) \qquad\qquad \omega \wedge d\omega \neq O$$

and we are seeking concave functions u and v, positive functions a and b, all analytic, and such that:

$$(20) \qquad\qquad \omega = adu + bdv.$$

Let us point out some algebraic consequences of equation (18):

LEMMA 1. *There are analytic* 1-*forms* $\omega, \omega', \sigma, \sigma'$, *such that, in some neighbourhhood of* $\bar{x}$:

$$(21) \qquad\qquad d\omega = \omega \wedge \omega' + \sigma \wedge \sigma'.$$

PROOF. Let us work in the cotangent space to $\mathbb{R}^N$ at $\bar{x}$. By Proposition I.1.4 in [2], we have:

$$(22) \qquad\qquad d\omega(\bar{x}) \wedge d\omega(\bar{x}) \wedge d\omega(\bar{x}) = 0$$

and by Theorem I.1.5 in the same reference, there exists linear forms $\bar{\pi}, \bar{\pi}', \bar{\gamma}, \bar{\gamma}'$ such that:

$$(23) \qquad\qquad d\omega(\bar{x}) = \bar{\pi} \wedge \bar{\pi}' + \bar{\gamma} \wedge \bar{\gamma}'.$$

Writing this into equation (18) (at $\bar{x}$), we see that:

$$(24) \qquad\qquad \omega(\bar{x}) \wedge \bar{\pi} \wedge \bar{\pi}' \wedge \bar{\gamma} \wedge \bar{\gamma}' = 0.$$

So the five linear forms $\omega(\bar{x}), \bar{\pi}, \bar{\pi}', \bar{\gamma}, \bar{\gamma}'$ belong to the same four-dimensional subspace of $T_{\bar{x}}^* \mathbb{R}^N$. Expressing $\bar{\pi}$, for instance, in terms of $\omega(\bar{x})$ and the other vectors, we get:

$$(25) \qquad d\omega(\bar{x}) = (s\omega(\bar{x}) + t\bar{\pi}' + u\bar{\gamma} + v\bar{\gamma}') \wedge \bar{\pi}' + \bar{\gamma} \wedge \bar{\gamma}'$$

$$(26) \qquad\qquad = s\omega(\bar{x}) \wedge \bar{\pi}' + u\bar{\gamma} \wedge \bar{\pi}' + v\bar{\gamma}' \wedge \bar{\pi}' + \bar{\gamma} \wedge \bar{\gamma}'$$

$$(27) \qquad\qquad = s\omega(\bar{x}) \wedge \bar{\pi}' + (\bar{\gamma} - v\bar{\pi}')(\bar{\gamma}' + u\bar{\pi}')$$

$$(28) \qquad\qquad = \omega(\bar{x}) \wedge \bar{\omega}' + \bar{\sigma} \wedge \bar{\sigma}'$$

for suitable linear forms $\bar{\omega}', \bar{\sigma}, \bar{\sigma}'$ in $T_{\bar{x}}^* \mathbb{R}^N$.

The same construction extends analytically to neighbouring points, and formula (21) follows.   □

Introduce the $3N$-dimensional space E defined by:

$$(29) \qquad E = \{(\alpha_i, \beta_j, x^n) \mid 1 \le i, j, n \le N, \ x \in \mathcal{U}\}.$$

Define analytic 1-forms α and β on E by:

$$(30) \qquad \alpha = \sum_i \alpha_i dx^i$$

$$(31) \qquad \beta = \sum_j \beta_j dx^j.$$

In E, consider the subset $\mathcal{M}$ defined by the equations:

$$(32) \qquad \omega(x) \wedge \alpha \wedge \beta = 0$$

$$(33) \qquad d\omega(x) \wedge \alpha \wedge \beta = 0.$$

These are $C_N^3 + C_N^4 = C_{N+1}^4$ equations in the $3N$ variables α_i, β_j, x_n. Since there are many more equations than variables, our first task will be to show that $\mathcal{M}$ is non-empty.

Define $A = \{(\alpha, \beta, x) \mid \alpha \ne \omega, \beta \ne \omega, x \in \mathcal{U}\}$; it is an open subset of E. We set:

$$(34) \qquad \mathcal{M} = M \cap A.$$

LEMMA 2. $\mathcal{M}$ is a $(n + 5)$-dimensional submanifold of E defined by the equations:

$$(35) \qquad \omega \wedge \alpha \wedge \beta = 0$$

$$(36) \qquad d\omega \wedge \sigma \wedge \sigma' \wedge \alpha = 0$$

$$(37) \qquad d\omega \wedge \sigma \wedge \sigma' \wedge \beta = 0.$$

PROOF. Equation (32) means that ω, α, β are linearly dependent: there exists functions f and g such that

$$(38) \qquad \beta = f\alpha + g\omega.$$

Writing this into equation (33), we get:

$$(39) \qquad d\omega \wedge \alpha \wedge \omega = 0.$$

Applying Lemma 1, we get:

$$(40) \qquad \sigma \wedge \sigma' \wedge \alpha \wedge \omega = 0$$

so α must belong to the linear span of σ, σ', ω, which is three-dimensional. Once α is chosen, β must belong to the linear span of α, ω, which is two-dimensional. Conversely, if α and β are chosen in this way, they satisfy equations (32) and (33), which means that (α, β, x) belongs to M. □

In the manifold $\mathcal{M}$, we consider the exterior differential system:

$$(41) \qquad\qquad d\alpha = 0$$

$$(42) \qquad\qquad d\beta = 0$$

$$(43) \qquad\qquad dx^1 \wedge dx^2 \wedge \ldots \wedge dx^N \neq 0.$$

Recalling the definition of α and β, we can rewrite this system in a more explicit way:

$$(44) \qquad\qquad \sum_i d\alpha_i \wedge dx^i = O$$

$$(45) \qquad\qquad \sum_i d\beta_i \wedge dx^i = O$$

$$(46) \qquad\qquad dx^1 \wedge dx^2 \wedge \ldots \wedge dx^N \neq 0.$$

Any integral manifold of this system in $\mathcal{M}$ provides us with functions λ_k and V_k, $k = 1, 2$, satisfying equation (20). Indeed, because of relation (43), this submanifold will in fact be the graph of the map $x \to (\alpha_i(x), \beta_j(x))_{1 \leq i,j \leq N}$, and equations (41) and (42) will imply, by the Poincaré lemma, that there are functions U and V such that $\alpha = dU$ and $\beta = dV$. Going back to the definition of M (equation (32)) we see that:

$$(47) \qquad\qquad \omega \wedge dU \wedge dV = 0$$

which is the algebraic way of expressing the fact that ω is a linear combination of dU and dV:

$$(48) \qquad\qquad \omega(x) = \lambda(x)dU(x) + \mu(x)dV(x).$$

It is easy to see that the coefficients $\lambda(x)$ and $\mu(x)$ then depend analytically on x. Note that, if this integral manifold goes through $(\overline{\alpha}, \overline{\beta}, \overline{x})$, then the relations $\alpha = dU$ and $\beta = dV$ imply that:

$$(49) \qquad\qquad \frac{\partial U}{\partial x^i}(\overline{x}) = \overline{\alpha}_i(\overline{x})$$

$$(50) \qquad\qquad \frac{\partial V}{\partial x^i}(\overline{x}) = \overline{\beta}_i(\overline{x})$$

and:

$$(51) \qquad\qquad \frac{\partial^2 U}{\partial x^i \partial x^j}(\overline{x}) = \frac{\partial \alpha_i}{\partial x_j}(\overline{x})$$

$$(52) \qquad\qquad \frac{\partial^2 V}{\partial x^i \partial x^j}(\overline{x}) = \frac{\partial \beta_i}{\partial x_j}(\overline{x}).$$

Choose $(\overline{\alpha}, \overline{\beta}, \overline{x}) \in \mathcal{M}$. We will find such integral manifolds through this point by applying the Cartan-Kähler theorem: the system (41), (42), (43) has to be closed, there has to be an integral element, and the point $(\overline{\alpha}, \overline{\beta}, \overline{x})$ has to be ordinary.

The system is obviously closed, since $dd = 0$. We proceed to finding an integral element. To do this, we will write the system in an equivalent, but simpler, form

LEMMA 3. *There exists five linearly independent 1-forms* $\lambda(x)$, $\mu(x)$, $\nu(x)$, $\theta(x)$, $\gamma(x)$, *analytic and defined on a neighbourhood of* $\overline{x}$, *such that the exterior differential system* (41), (42), (43) *on* $\mathcal{M}$ *can be reformulated as follows:*

(53) $$0 = \lambda \wedge \omega + \mu \wedge \sigma + \nu \wedge \alpha$$

(54) $$0 = \theta \wedge \omega + \gamma \wedge \alpha$$

(55) $$0 \neq dx^1 \wedge dx^2 \wedge \ldots \wedge dx^N .$$

PROOF. By Lemma 2, the defining equations can be written as follows:

(56) $$\omega \wedge \alpha \wedge \beta = 0$$

(57) $$\sigma \wedge \sigma' \wedge \omega \wedge \alpha = 0$$

(58) $$\sigma \wedge \sigma' \wedge \omega \wedge \beta = 0$$

which, by formula (21), can again be rewritten:

(59) $$\omega \wedge \alpha \wedge \beta = 0$$

(60) $$d\omega \wedge \omega \wedge \alpha = 0$$

(61) $$d\omega \wedge \omega \wedge \beta = 0 .$$

By taking the exterior derivative of equation (60) and applying Lemma 1, we have:

(62) $$0 = d\omega \wedge d\omega \wedge \alpha + \omega \wedge d\omega \wedge d\alpha$$

(63) $$= 2\omega \wedge \omega' \wedge \alpha + \omega \wedge d\omega \wedge d\alpha$$

(64) $$= \omega \wedge d\omega \wedge d\alpha$$

(65) $$= \omega \wedge \sigma \wedge \sigma' \wedge d\alpha .$$

This means that there exists analytic 1-forms $\lambda(x)$, $\mu(x)$, $\mu'(x)$ such that:

(66) $$d\alpha = \lambda \wedge \omega + \mu \wedge \sigma + \mu' \wedge \sigma'$$

By relation (40), we know that σ' can be expressed as a linear combination of ω, σ and α. This gives us equation (53)

Taking the exterior derivative of equation (59), we can relate $d\alpha$ and $d\beta$. Namely:

(67) $0 = d\omega \wedge \alpha \wedge \beta + \omega \wedge d\alpha \wedge \beta + \omega \wedge \alpha \wedge d\beta$

(68) $= (\omega \wedge \omega' + \sigma \wedge \sigma') \wedge \alpha \wedge \beta + \omega \wedge d\alpha \wedge \beta + \omega \wedge \alpha \wedge d\beta$

(69) $= \sigma \wedge \sigma' \wedge \alpha \wedge \beta + \omega \wedge d\alpha \wedge \beta + \omega \wedge \alpha \wedge d\beta$.

But we know from Lemma 2 that α and β belong to the linear span of ω, σ, and σ', which is three-dimensional. So the first term in the last equation vanishes, and we are left with:

(70) $0 = \omega \wedge d\alpha \wedge \beta + \omega \wedge \alpha \wedge d\beta$.

Since, by relation (35), β, α and ω are linearly dependent, we can write $\beta = s\alpha + t\omega$ in the preceding equation, which becomes:

(71) $0 = \omega \wedge \alpha \wedge (d\beta + s \, d\alpha)$.

So there exists 1-forms θ and γ such that:

(72) $d\beta = d\alpha + \theta \wedge \omega + \gamma \wedge \alpha$.

Writing $d\alpha = O$ in this equation gives us equation (54). Hence the lemma.   □

To find an integral element, we differentiate equations (53) and (54), and we substitute:

(73) $d\alpha_i = \sum_j A_{i,j} dx^j$

(74) $d\beta_i = \sum_j B_{i,j} dx^j$.

We complete ω, α and σ into a basis of the cotangent space, and we develop the exterior products on that basis. Equation (53) then gives us $(3N - 6)$ linearly independent conditions on $A_{i,j}$ and $B_{i,j}$, while (54) gives us $(2N - 3)$ additional ones. To see this, consider for instance equation (53): expanding the first monomial on the right-hand side gives us $(N - 1)$ terms (there is no term in $\omega \wedge \omega$), the second one gives us $(N - 2)$ more (because the term in $\omega \wedge \alpha$ has already appeared in the first expansion), and the third one $(N - 3)$ more (because the terms in $\omega \wedge \sigma$ and $\alpha \wedge \sigma$ have already appeared in the two preceding expansions).

So integral elements are defined by $(5N-9)$ linearly independent conditions. In other words, the codimension of the submanifold of integral elements in the full Grassmannian of tangent N-planes to $\mathcal{M}$ is $C = 5N - 9$. Since $\mathcal{M}$ has

dimension $(N + 5)$, this Grassmannian has dimension $5N$, and the manifold of integral elements has dimension 9.

It remains to check that the point $(\overline{\alpha}, \overline{\beta}, \overline{x})$ is ordinary. Applying the procedure described in [2] or [5], we find the values:

(75) $\qquad\qquad\qquad\qquad c_0 = 0$

(76) $\qquad\qquad\qquad\qquad c_1 = 2$

(77) $\qquad\qquad\qquad\qquad c_2 = 4$

(78) $\qquad\qquad\qquad\qquad c_k = 5 \text{ for } k \geq 3$

which gives:

(79) $\qquad c_0 + c_1 + \ldots + c_{N-1} = 2 + 4 + 5(N - 3)$

(80) $\qquad\qquad\qquad\qquad\qquad\quad = 5N - 9 .$

This coincides with C, so the point $(\overline{\alpha}, \overline{\beta}, \overline{x})$ is ordinary, and we can apply the Cartan-Kähler theorem. Note that the Cartan characters are:

(81) $\qquad\qquad\qquad\qquad s_0 = c_0 = 0$

(82) $\qquad\qquad\qquad\qquad s_1 = c_1 - c_0 = 2$

(83) $\qquad\qquad\qquad\qquad s_2 = c_2 - c_1 = 2$

(84) $\qquad\qquad\qquad\qquad s_3 = c_3 - c_2 = 1$

(85) $\qquad\qquad\qquad\qquad s_k = 0 \text{ for } 4 \leq k \leq N - 1$

(86) $\qquad\qquad\qquad\qquad s_N = (N + 5) - N - c_{N-1} = 0$

so that every integral manifold will be determined by two functions of one variables, two functions of two variables, and one function of three variables.

By the Cartan-Kähler theorem, for every integral element at $(\overline{\alpha}, \overline{\beta}, \overline{x})$, there will be an integral manifold going through $(\overline{\alpha}, \overline{\beta}, \overline{x})$ and tangent to that integral element. In other words, for every choice of symmetric matrices $A_{i,j}$ and $B_{i,j}$ such that $d\alpha_i = \sum_j A_{i,j} dx^j$ and $d\beta_i = \sum_j B_{i,j} dx^j$ satisfy equations (53) and (54), there will be an integral manifold going through $(\overline{\alpha}, \overline{\beta}, \overline{x})$ and such that $\frac{\partial \alpha_i}{\partial x_j} = A_{i,j}$ and $\frac{\partial \beta_i}{\partial x_j} = B_{i,j}$. So we know, for instance, that ω can be decomposed into the form (48):

(87) $\qquad\qquad \omega(x) = a(x) du(x) + b(x) dv(x) .$

But this is not enough for our purposes: we want a and b to be positive, and u and v to be concave. By relations (51) and (52), the latter amounts to showing that we can take A and B to be negative definite.

Choose any real-valued functions $f(x), g(x), h(x), k(x)$, and set:

(88) $\qquad\qquad \alpha(x) = f(x)\omega(x) + g(x) du(x)$

(89) $\qquad\qquad \beta(x) = h(x)\omega(x) + k(x) du(x) .$

Clearly, $\alpha(x)$ and $\beta(x)$ satisfy equations (35), (36) and (37), so that the matrices $A_{i,j} = \frac{\partial \alpha_i}{\partial x_j}$ and $B_{i,j} = \frac{\partial \beta_i}{\partial x_j}$ define an N-plane in the tangent space to $\mathcal{M}$ at $(\overline{\alpha}, \overline{\beta}, \overline{x})$. This tangent space will be an integral element if and only if A and B are symmetric. Differentiating relations (88) and (89), we have:

$$(90) \qquad A = f(\overline{x})\Omega + g(\overline{x})U + {}^t F \omega(\overline{x}) + {}^t G du(\overline{x})$$

$$(91) \qquad B = h(\overline{x})\Omega + k(\overline{x})U + {}^t H \omega(\overline{x}) + {}^t K du(\overline{x})$$

where

$$(92) \qquad \Omega_{i,j} = \frac{\partial \omega_i}{\partial x^j}(\overline{x})$$

$$(93) \qquad U_{i,j} = \frac{\partial^2 u}{\partial x^i \partial x^j}(\overline{x})$$

$$(94) \qquad F_i = \frac{\partial f}{\partial x_i}(\overline{x})$$

$$(95) \qquad G_i = \frac{\partial g}{\partial x_i}(\overline{x})$$

$$(96) \qquad H_i = \frac{\partial h}{\partial x_i}(\overline{x})$$

$$(97) \qquad K_i = \frac{\partial k}{\partial x_i}(\overline{x}).$$

By assumption (see equation (14) in Theorem 1), Ω is the sum of a symmetric, positive definite matrix Q, and a matrix R of rank 2. The antisymmetric part of this matrix R can be computed:

$$(98) \qquad \sum_{i,j} R_{i,j} dx_i \wedge dx_j = \sum_{i,j} (\Omega_{i,j} - \Omega_{j,i}) dx^i \wedge dx^j$$

$$(99) \qquad = d\omega.$$

On the other hand, by equation (87), we have:

$$(100) \qquad d\omega = da \wedge du + db \wedge dv$$

$$(101) \qquad = da \wedge du + \frac{db}{b} \wedge (\omega - adu)$$

$$(102) \qquad = \frac{db}{b} \wedge \omega + (da - a\frac{db}{b}) \wedge du.$$

Comparing the two values of $d\omega$, we get:

$$(103) \qquad R = {}^t(\gamma + s\omega(\overline{x})) \otimes \omega(\overline{x}) + {}^t(\zeta + tdu(\overline{x})) \otimes du(\overline{x})$$

where $\gamma = db/b$, and $\zeta = da - adb/b$, and the numbers s and t are arbitrary.

Taking $F = H = -\gamma$ and $G = K = -\zeta$, we kill the antisymmetric part of equations (90) and (91), and since U and V are symmetric, we get

$$(104) \qquad A = f(\overline{x})Q + g(\overline{x})U$$

$$(105) \qquad B = h(\overline{x})Q + k(\overline{x})U \,.$$

Setting $f(\overline{x}) = h(\overline{x}) = 1$ and $g(\overline{x}) = -k(\overline{x}) = \epsilon$, this becomes:

$$(106) \qquad A = Q + \epsilon U$$

$$(107) \qquad B = Q - \epsilon U \,.$$

Since Q is negative definite, so will be A and B if ϵ is small enough. On the other hand, writing these values into equations (88) and (89), we get:

$$(108) \qquad \alpha(\overline{x}) = \omega(\overline{x}) + \epsilon du(\overline{x})$$

$$(109) \qquad \beta(\overline{x}) = \omega(\overline{x}) - \epsilon du(\overline{x})$$

which implies that $\alpha(\overline{x})$ and $\beta(\overline{x})$ are linearly independent, and that

$$(110) \qquad \omega(\overline{x}) = \frac{1}{2}(\alpha(\overline{x}) + \beta(\overline{x})) \,.$$

This concludes the proof of Theorem 1.

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