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Construction of Blowup Solutions for the Nonlinear Schrödinger Equation with Critical Nonlinearity

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1. – Introduction and statement of results

In this paper we study the behavior of the blowup solution of the nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u + |u|^{\frac{4}{d}} u = 0 \quad (x \in R^d). \tag{1.1}$$

Recall that both the L^2-norm

$$\|\phi\|_2 = \left(\int_{\mathbb{R}^d} |\phi(x)|^2 dx \right)^{1/2} \tag{1.2}$$

and the Hamiltonian

$$H(\phi) = \frac{1}{2} \int |\nabla \phi|^2 - \frac{d}{2d+4} \int |\phi|^{\frac{4}{d}+2} \tag{1.3}$$

are conserved quantities for the flow of (1.1).

The NLS (1.1) has an important soliton solution

$$e^{it} R(x) \tag{1.4}$$

where R is the ground state of the equation

$$\Delta u - u + |u|^{\frac{4}{d}} u = 0. \tag{1.5}$$

It is known that equation (1.1) has no blowup solution in the class

$$\{u \in H^1(\mathbb{R}^d) \mid \|u\|_{L^2} < \|R\|_{L^2}\} \tag{1.6}$$

and, in the class

$$\{u \in H^1(\mathbb{R}^d) \mid \|u\|_{L^2} = \|R\|_{L^2}\} \tag{1.7}$$

(1.1) has a unique blowup solution

$$\tilde{R}(x, t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2-4}{4it}} R\left(\frac{x}{t}\right) \tag{1.8}$$

up to the invariances of the equation (see [G-V], [Wein], [M1]).

In the class $\{u \in H^1(\mathbb{R}^d) \mid \|u\|_{L^2} > \|R\|_2\}$, a general blowup criteria was established by Glassey [G], based on the viriel identity

$$\frac{d^2}{dt^2}\left[\int |x|^2 \, |u(x, t)|^2 dx\right] \sim H\big(u(t)\big). \tag{1.9}$$

However, a mathematically rigorous understanding of the blowup is largely open. It is known that if

$$\lim_{\substack{\rightarrow \\ t < T}} \|u(t)\|_{H^1} = \infty \tag{1.10}$$

there has to be a concentration of L^2-mass in the following sense: there exists points $z(t) \in \mathbb{R}^d$, $t < T$, such that

$$\lim_{t \to T} \int_{B(0,\,\delta)} |u(x + z(t), t)|^2 dx \geq \|R\|_{L^2}^2 \tag{1.11}$$

for any $\delta > 0$, see [Wein 2], [MT], [N], etc.

This statement is of course far from a complete description. Among the several conjectures (based in particular on numerics), one believes that the L^2-mass going into the blowup is quantized and the blowup solution looks like a superposition of finitely many solutions blowing up at single points and outside which it remains smooth. In the particular case of a solution satisfying (1.10) and

$$\|u\|_2 < \|R\|_2 + \delta \tag{1.12}$$

for small $\delta > 0$, it is expected that u decouples as superposition

$$u = u_0 + u_1 \tag{1.13}$$

where u_0 is a minimum norm blowup solution conformal to (1.8) and u_1 remains smooth after blowup. (Observe that these minimum norm blowup solutions are unstable). The purpose of our first result is to give at least examples of this phenomenon.

THEOREM 1. *Let $d = 1$ or $d = 2$. Let ϕ be a smooth function on $\mathbb{R}^d$ with fast decay at infinity and such that ϕ vanish at* 0 *of sufficiently high order. Then* (1.1) *has a blowup solution u on* $[-\delta,\ 0]$ *of the form*

$$u = \tilde{R} + u_1 \tag{1.14}$$

where $\tilde{R}$ is given by (1.8) *and u_1 extends to a smooth function on* $[-\delta,\ \delta]$ *solving the Cauchy problem*

$$\begin{cases} u_1(0) = \phi & (1.15)\\ i\dot{u}_1 + \Delta u_1 + u_1|u_1|^2 = 0 & (1.16)\end{cases}$$

on $[0,\ \delta]$.

Here $\delta > 0$ depends on an appropriate norm $\|\phi\|$ of ϕ and $\delta \to \infty$ for $\|\phi\| \to 0$. The reason of the restriction $d = 1, 2$ is the smoothness of the nonlinearity $u|u|^{4/d}$ in (1.1) (which seems insufficient in higher dimension for our purpose).

Theorem 1 may be formulated in a more precise way. Define the spaces

$$X_A = \{\phi \in H^A(\mathbb{R}^d) \big| \, (1 + |x|)^A \phi \in L^2(\mathbb{R}^d)\} \tag{1.17}$$

endowed with the natural norm

$$\|\phi\|_{X_A} = \|\phi\|_{H^A} + \|(1 + |x|)^A \phi\|_{L^2}. \tag{1.18}$$

Denote also

$$\mathcal{P}_A = \left\{\phi \in X_A \big| \, D^\alpha \phi(0) = 0 \text{ for all } |\alpha| < A - \left[\frac{d}{2}\right]\right\}. \tag{1.19}$$

Recall also that if $\phi \in X_A$, then the IVP

$$\begin{cases} iu_t + \Delta u + u|u|^{4/d} = 0\\ u(0) = \phi\end{cases} \tag{1.20}$$

has a local solution z_ϕ in a neighborhood $[-\delta,\ \delta]$ of 0, satisfying

$$z_\phi \in \mathcal{C}([-\delta,\ \delta] : X_A). \tag{1.21}$$

If moreover $\|\phi\|_{L^2}$ is sufficiently small, then this local solution extends to a global one and it may be shown that (cf. [Bo])

$$\|z_\phi(t)\|_{X_A} \leq C(1 + |t|)^{C_A}. \tag{1.22}$$

Theorem 1 is then a consequence of the following Theorem 1′

THEOREM 1′. *Given A, there is A_1 such that if $\phi \in \mathcal{P}_{A_1}$, then* (1.1) *has a blowup solution u on $[-\delta, 0]$ of the form*

$$u = \tilde{R} + z_\phi + w \tag{1.23}$$

where

$$w \in \mathcal{C}([-\delta, 0]; X_A) \tag{1.24}$$

and

$$\|w(t)\|_{X_A} \le |t|^A \textit{ for } |t| \to 0. \tag{1.25}$$

To obtain (1.14), define

$$\begin{aligned} u_1 &= z_\phi + w \text{ for } t \in [-\delta, 0] \\ &= z_\phi \text{ for } t \in [0, \delta]. \end{aligned} \tag{1.26}$$

We treat Theorem 1′ basically as a perturbative problem. Our first tool is the standard pseudo-conformal invariance, i.e. the formula

$$Cu(x, t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2}{4it}} u\left(\frac{x}{t}, -\frac{1}{t}\right) \tag{1.27}$$

transforming a solution of (1.1) in another one (equation (1.1) has critical nonlinearity).

Applying this transformation in the setting of Theorem 1′, we basically sends $t = 0$ to $t = \infty$ and get an equivalent problem concerning perturbations of the ground state solution (1.4).

Writing in (1.1)

$$u(x, t) = e^{it} \big(R(x) + v(x, t)\big) \tag{1.28}$$

the function v has to satisfy the equation

$$iv_t + \Delta v - v + [|R(x) + v(x, t)|^{4/d}\big(R(x) + v(x, t)\big) - R(x)^{4/d+1}] = 0 \tag{1.29}$$

i.e.

$$iv_t + \Delta v - v + \left(\frac{2}{d} + 1\right) R^{4/d} v + \frac{2}{d} R^{4/d}\, \overline{v} + 0(|v|^2) = 0 \tag{1.30}$$

One second main tool is the result [Wein 1] of M. Weinstein (for $d = 1$, $d = 3$ and completed in the $d = 2$ case using subsequent work [Kw]) according to which the linearized equation

$$iv_t - Lv = 0 \tag{1.31}$$

where

$$Lv = -\Delta v + v - \left(\frac{2}{d} + 1\right) R^{4/d} v - \frac{2}{d} R^{4/d} \overline{v} \tag{1.32}$$

has only algebraic instabilities (a precise statement will appear in the next section).

REMARK. This fact holds in subcritical and critical cases; in the supercritical case, one has exponential instability (M. Weinstein - personal communication).

This result is obviously fundamental to our perturbative analysis and permits us to go beyond certain perturbative constructions as considered for instance in [M2], where this property was not exploited. As an application, one may indeed construct blowup solutions, of (1.1) with distinct blowup times.

THEOREM 2. *Take $d = 1, 2$ and A a large number. Consider times*

$$0 = t_0 > t_1 > t_2 \text{ where } |t_1 - t_2| < \delta(|t_0 - t_1|),\ |t_0 - t_1| < \delta_0 \tag{1.33}$$

with δ_0 small enough and let $x_0,\ x_1 \in \mathbb{R}^d$ be such that

$$|x_0 - x_1| > 1. \tag{1.34}$$

Then (1.1) *has a solution u on $(t_2, t_1) \cup (t_1, t_0)$ satisfying*

$$u \in \mathcal{C}\big((t_2, t_1) \cup (t_1, 0); X_A\big) \tag{1.35}$$

$$u - \tilde{R}_0 \in \mathcal{C}([t_1, 0]; X_A) \tag{1.36}$$

$$u - \tilde{R}_1 \in \mathcal{C}([t_2, t_1]; X_A) \tag{1.37}$$

$$(u - \tilde{R}_1)(t_1 - 0) = u(t_1 + 0)\,. \tag{1.38}$$

We denote here

$$\tilde{R}_0(x, t) = R(x - x_0, t - t_0) \text{ and } \tilde{R}_1(x, t) = R(x - x_1, t - t_1). \tag{1.39}$$

REMARK.

(1) A similar statement may be formulated for an arbitrary number of times.

(2) In our setup in Theorem 2, we did not continue the blowup solutions $\tilde{R}_\alpha (\alpha = 0, 1)$ after the blowup time (thus there is no L^2-conservation) as one may do according to results of [M3].

(3) A result in the spirit of Theorem 1 may be proved also in $d = 3$, but technicalities are exceedingly more significant, due to lack of smoothness of nonlinearity.

(4) The results of this paper constitute a preliminary investigation in this direction.

There is no doubt that elaborating the techniques presented here and an appropriate scattering theory should also lead to general stability of solutions of (1.1) nearby $\tilde{R}$, provided we restrict data to appropriate finite codimensional spaces.

2. – Estimates on the linearized operator

In this section, we briefly recall a basic result from [Wein 1] on the linearized problem (1.31), (1.32)

$$\begin{cases} iv_t - Lv = 0 \\ v(0) = \phi \end{cases} \tag{2.1}$$

where

$$Lv = -\Delta v + v - \left(\frac{2}{d} + 1\right) R^{4/d} v - \frac{2}{d} R^{4/d} \bar{v}. \tag{2.2}$$

It is shown that one may decompose the space $H^1(\mathbb{R}^d)$ as

$$H^1 = M \oplus S \tag{2.3}$$

with both components invariant under the flow map e^{itL} and where the space S of "secular modes" (obtained as generalized null space of L) is spanned by $2d+4$ Schwartz functions

$$S = [e_1, \dots, e_{2d+4}] \tag{2.4}$$

satisfying

$$|e_j(x)| < e^{-c|x|} \text{ for } |x| \to \infty. \tag{2.5}$$

Moreover, for $\phi \in M$

$$\|e^{itL}\phi\|_{H^1} \leq C\|\phi\|_{H^1} \tag{2.6}$$

while, for $\phi \in S$

$$\|e^{itL}\phi\| < C(1+|t|^3) \int e^{-c|x|}|\phi(x)|dx. \tag{2.7}$$

We will also need for later purpose the following estimates on the behaviour of e^{itL} in X_A-norm.

Proposition 2.8. *Denote P_M (respectively P_S) the orthogonal projection on M (respectively S), then*

$$\|e^{itL}(P_M\phi)\|_{H^s} \leq C\|\phi\|_{H^s} \tag{2.9}$$

$$\|e^{itL}(P_S\phi)\|_{H^s} \leq C(1+|t|^3). \int |\phi| e^{-c|x|} dx \tag{2.10}$$

$$\| \, |x|^\alpha e^{itL}(P_M\phi)\|_{L^2} \leq C\| \, |x|^\alpha \phi\|_{L^2} + C(1+|t|^\alpha)\|\phi\|_{H^\alpha} \tag{2.11}$$

$$\| \, |x|^\alpha e^{itL}(P_S\phi)\|_{L^2} \leq C(1+|t|^3). \int |\phi| \, e^{-c|x|} dx\,. \tag{2.12}$$

PROOF. The statements (2.10), (2.12) are obvious from (2.7) and the fact that S is contained in the Schwartz space, satisfying (2.5).

To verify (2.9), consider first $\|e^{itL}\phi\|_{H^{2s+1}}$, $\phi \in M$, where $s > 0$ is an integer. From the definition (2.2) of L, it is clear that

$$\|e^{itL}\phi\|_{H^{2s+1}} \leq \|L^s e^{itL}\phi\|_{H^1} + c\|e^{itL}\phi\|_{H^{2s-1}}. \tag{2.13}$$

Since L maps M into itself, one gets from (2.6)

$$\|L^s e^{itL}\phi\|_{H^1} = \|e^{itL}L^s\phi\|_{H^1} \leq C\|L^s\phi\|_{H^1} \leq C\|\phi\|_{H^{2s-1}}. \tag{2.14}$$

Hence, from (2.13), (2.14)

$$\|e^{itL}\phi\|_{H^{2s+1}} \leq C\|\phi\|_{H^{2s+1}} + C\|e^{itL}\phi\|_{H^{2s-1}}. \tag{2.15}$$

By induction, (2.6), (2.15), we get (2.9) for s of the form $2s+1$, $s \in \mathbb{Z}_+$. The result in general follows then by interpolation.

Next, we verify (2.11). Take $\phi \in M$ and denote $v = e^{itL}\phi$. Then the equation

$$iv_t - Lv = 0$$

yields

$$\begin{aligned} \frac{d}{dt}\left[\int |x|^{2\alpha}|v(x,t)|^2 dx\right] &= 2\mathrm{Re}\,\langle |x|^{2\alpha}\,v,\ \dot{v}\rangle \\ &= 2\,\mathrm{Im}\,\langle |x|^{2\alpha}v,\ Lv\rangle. \end{aligned} \tag{2.16}$$

Substituting L from (2.2) in (2.16), one obtains easily

$$2\,\mathrm{Im}\,\langle |x|^{2\alpha}v,\ \Delta v\rangle + 0\left(\int |v|^2\,e^{-c|x|}\right)$$

bounded by

$$\int |x|^{2\alpha-1}|v|\,|\nabla v|dx + \|v\|_2^2. \tag{2.17}$$

Interpolating between H^α and $L^2(|x|^\alpha,\,dx)$, estimate by Hölder's inequality

$$\begin{aligned} \int |x|^{2\alpha-1}|v|\,|\nabla v| &\leq \|\,|x|^\alpha\,|v|\,\|_2\,\|\,|x|^{\alpha-1}|\nabla v|\,\|_2 \\ &\leq \|\,|x|^\alpha|v|\,\|_2^{2-\frac{1}{\alpha}}\,\|v\|_{H^\alpha}^{\frac{1}{\alpha}}. \end{aligned} \tag{2.18}$$

Invoking (2.9), we conclude that

$$\frac{d}{dt}[\|\,|x|^\alpha\,|v(t)|\,\|_2^2] \leq C\|\,|x|^\alpha|v|\,\|_2^{2-\frac{1}{\alpha}}\,\|\phi\|_{H^\alpha}^{\frac{1}{\alpha}} + \|v\|_2^2 \tag{2.19}$$

from where (2.11) is deduced.

COROLLARY 2.20. *For all ϕ and $A \geq 3$*

$$\|e^{itL}\phi\|_{X_A} \leq (1+|t|)^A\,\|\phi\|_{X_A}.$$

3. – Proof of Theorem 1′

Recall first the conformal transformation

$$\tilde{u}(x,t) = Cu(x,t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2}{4it}} u\Big(\frac{x}{t}, -\frac{1}{t}\Big). \tag{3.1}$$

Thus

$$\begin{aligned} \|(Cu)(t)\|_{H^s} &\le C \max_{r+|\beta|\le s} \Big\| \frac{|y|^r}{|t|^{s-r}} |(D^\beta u)\Big(-\frac{1}{t}\Big) \Big\|_2 \\ &\le C\Big(1+\frac{1}{|t|^s}\Big) \Big\| u\Big(-\frac{1}{t}\Big)\Big\|_{X_s} \end{aligned} \tag{3.2}$$

and

$$\| \, |x|^\alpha (Cu)(t)\|_2 \le |t|^\alpha \Big\| |y|^\alpha u\Big(-\frac{1}{t}\Big)\Big\|_2. \tag{3.3}$$

Hence, from (3.2), (3.3)

$$\|(Cu)(t)\|_{X_A} \le C\Big(|t|^A + \frac{1}{|t|^A}\Big) \Big\| u\Big(-\frac{1}{t}\Big)\Big\|_{X_A}. \tag{3.4}$$

Transforming (1.23) by C and applying (3.4), it will clearly suffice to get a solution of (1.1) on $[\frac{1}{\delta}, \infty[$ of the form

$$u = \mathrm{Re}^{it} + C^{-1}(z_\phi) + we^{it} \tag{3.5}$$

where

$$w \in \mathcal{C}\Big([\frac{1}{\delta}, \infty[; X_A\Big) \tag{3.6}$$

satisfies

$$\|w(t)\|_{X_A} \le \frac{1}{t^{2A}}. \tag{3.7}$$

Recall that z_ϕ solves (1.1)

$$i\dot{z}_\phi + \Delta z_\phi + |z_\phi|^{4/d} z_\phi = 0 \tag{3.8}$$

with $\phi \in \mathcal{P}_{A_1}$.

Denote

$$v_0 = C^{-1}(z_\phi)e^{-it} \tag{3.9}$$

and substitute

$$v = v_0 + w$$

in the difference equation (1.29). Since v_0 satisfies

$$i\dot{v}_0 - v_0 + \Delta v_0 + |v_0|^{4/d} v_0 = 0 \tag{3.10}$$

we get

$$i w_t + \Delta w - w + [|R + v_0 + w|^{4/d}(R + v_0 + w) - R^{4/d+1} - |v_0|^{4/d} v_0] = 0. \tag{3.11}$$

Define

$$f_0 = |R + v_0|^{4/d}(R + v_0) - R^{4/d+1} - |v_0|^{4/d} v_o \tag{3.12}$$

$$a = \left(\frac{2}{d} + 1\right)(|R + v_0|^{4/d} - R^{4/d}) \tag{3.13}$$

$$b = \frac{2}{d}\big((R + v_0)^{2/d+1}(R + \overline{v}_0)^{2/d-1} - R^{4/d}\big) \tag{3.14}$$

$$\begin{aligned} G(w) = {} & |R + v_0 + w|^{4/d}(R + v_0 + w) - (R + v_0)^{4/d}(R + v_0) \\ & - aw - b\overline{w} \end{aligned} \tag{3.15}$$

and rewrite (3.11) as

$$i w_t - Lw + aw + b\overline{w} + G(w) + f_0 = 0. \tag{3.16}$$

Here $G(w)$ is at least quadratic in w.

In order to produce a solution of (3.16) on $[\frac{1}{\delta}, \infty[$, we solve the equivalent integral equation

$$w(t) = -i \int_t^\infty e^{+i(\tau - t)L}[f_0 + aw + b\overline{w} + G(w)](\tau) d\tau. \tag{3.17}$$

This procedure is reminiscent of the wave map construction in scattering theory, except that here the reference equation is the nonlinear equation (1.1).

Our aim is to derive the bound (3.7) from (3.17).

We first establish some bounds on v_0. From (3.1), (3.9)

$$v_0(x, t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2}{4it} - it} z_\phi\left(\frac{x}{t}, -\frac{1}{t}\right). \tag{3.18}$$

Since $\phi \in \mathcal{P}_{A_1} \subset X_{A_1}$, we have by (1.21)

$$\|z_\phi(t)\|_{X_{A_1}} < C \text{ for } |t| < \delta. \tag{3.19}$$

We assume here A_1 sufficiently large with respect to A.

From the equation (3.8), it follows from definition of $\mathcal{P}_{A_1}$, cf. (1.19), that

$$\partial_x^\alpha \partial_t^\ell z_\phi\big|_{x=0=t} = 0 \text{ for } |\alpha| + 2\ell < A_1 - \left[\frac{d}{2}\right]. \tag{3.20}$$

Thus, by Taylor's theorem

$$|z_\phi(x, t)| < C(|x| + |t|)^{A'} \text{ for } A' \le \frac{A_1}{2} - 1 \text{ and } |t| < \delta. \tag{3.21}$$

From (3.18)

$$\begin{aligned} D^\alpha v_0(x, t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2}{4it} - it} \bigg\{ &0\bigg(\Big(\frac{|x|}{t}\Big)^{|\alpha|} \Big| z_\phi\Big(\frac{x}{t}, -\frac{1}{t}\Big)\Big|\bigg) \\ &+ 0\bigg(\frac{1}{t^{|\alpha|}} \Big| D^\alpha z_\phi\Big(\frac{x}{t}, -\frac{1}{t}\Big)\Big|\bigg) + \text{others}\bigg\} \end{aligned} \tag{3.22}$$

and thus

$$|D^\alpha v_0(x, t)| \le \frac{C}{t^{d/2}} \max_{r+|\beta|\le|\alpha|} \left\{ \frac{|x|^r}{t^{|\alpha|}} \left| D^\beta z_\phi\Big(\frac{x}{t}, -\frac{1}{t}\Big)\right| \right\}. \tag{3.23}$$

Hence

$$\|D^\alpha v_0(t)\|_\infty < \frac{C}{t^{d/2}} \max_{r+|\beta|\le|\alpha|} \left\| |y|^r \left| D^\beta z_\phi\Big(y, -\frac{1}{t}\Big)\right| \right\|_\infty \tag{3.24}$$

and thus, by (3.19)

$$\|D^\alpha v_0(t)\|_\infty \le \frac{C}{t^{d/2}} \Big|z_\phi\Big(-\frac{1}{t}\Big)\Big\|_{X_{A_1}} < \frac{C}{t^{d/2}} \text{ for } t > \frac{1}{\delta} \text{ and } |\alpha| \le A_1 - 2. \tag{3.25}$$

Also, by (3.21), (3.23)

$$|e^{-c|x|} D^\alpha v_0(x, t)| < \frac{C}{t^{A_1/2}} \text{ for } t > \frac{1}{\delta} \text{ and } |\alpha| \le \frac{A_1}{2}. \tag{3.26}$$

We still make repeated use of estimation (3.25), (3.26) in what follows.

From (3.26), it follows that for $s < \frac{A_1}{2}$

$$\|(R v_0)(t)\|_{H^s} < \frac{C}{t^{A_1/2}} \tag{3.27}$$

and hence, by (3.12)

$$\|f_0(t)\|_{X_A} < \frac{C}{t^{A_1/2}} \tag{3.28}$$

from (3.13)

$$|a| = 0(R|v_0| + |v_0|^{4/d}) \tag{3.29}$$

and hence, by (3.25), (3.26), for $|\alpha| \leq \frac{A_1}{2}$

$$|D^\alpha a)(x, t)| < \frac{C}{t^{A_1/2}} + \frac{C}{t^2} < \frac{C}{t^2}. \tag{3.30}$$

and

$$|e^{-c|x|} D^\alpha a(x, t)| < \frac{C}{t^{A_1/2}}. \tag{3.31}$$

Similarly, by (3.14)

$$|D^\alpha b(x, t)| < \frac{C}{t^2} \tag{3.32}$$

and

$$|e^{-c|x|} D^\alpha b(x, t)| < \frac{C}{t^{A_1/2}}. \tag{3.33}$$

Coming back to (3.17), rewrite the right side as

$$-i \int_t^\infty e^{i(\tau - t)L} P_S[f_0 + aw + b\overline{w} + G(w)](\tau) d\tau \tag{3.34}$$

$$+$$

$$-i \int_t^\infty e^{i(\tau - t)L} P_M[f_0 + aw + b\overline{w} + G(w)](\tau) d\tau. \tag{3.35}$$

We first verify the estimate

$$\|w(t)\|_{H^s} < \frac{1}{t^{A_1/4}} \text{ for } s \leq A,\ t \geq \frac{1}{\delta} \tag{3.36}$$

deriving this inequality from (3.17), (3.35), (3.36).

It follows from (2.10) that

$$\begin{aligned} &\|(3.34)\|_{H^s} \\ &\leq \int_t^\infty [1 + (\tau - t)^3] \left\{ \int \left| [f_0 + aw + b\overline{w} + G(w)](x, \tau) \right| e^{-c|x|} dx \right\} d\tau \\ &\leq \int_t^\infty [1 + (\tau - t)^3] \left\{ \frac{C}{\tau^{A_1/2}} + \frac{C}{\tau^{A_1/2}} \|w(\tau)\|_{H^s} + C\|w(\tau)\|_{H^s}^2 \right\} d\tau \end{aligned} \tag{3.37}$$

by (3.28), (3.31), (3.33). Reintroducing (3.36) in (3.37) gives the bound

$$\|(3.34)\|_{H^s} \le \int_t^\infty [1+(\tau-t)^3]\left[\frac{C}{\tau^{A_1/2}}+\frac{C}{\tau^{A_1/2}}\right]d\tau < \frac{C}{t^{A_1/2-4}} < \frac{\delta}{t^{A_1/4}}. \tag{3.38}$$

Next, consider (3.35) and estimate from (2.9)

$$\begin{aligned}\|(3.35)\|_{H^s} &\le \int_t^\infty \|[f_0+aw+b\overline{w}+G(w)](\tau)\|_{H^s}d\tau\\ &\le \int_t^\infty \left\{\frac{C}{\tau^{A_1/2}}+\frac{C}{\tau^2}\|w(\tau)\|_{H^s}+C\|w(\tau)\|_{H^s}^2\right\}d\tau\\ &\le \int_t^\infty \left\{\frac{C}{\tau^{A_1/2}}+\frac{C}{\tau^{\frac14 A_1+2}}+\frac{C}{\tau^{A_1/2}}\right\}d\tau\\ &\le \frac{C}{t^{\frac14 A_1+1}} < \frac{C\delta}{t^{A_1/4}} \qquad (3.39)\end{aligned}$$

using (3.28), (3.30), (3.32), (3.36)..

From (3.38), (3.39), the apriori bound (3.36) follows taking δ small enough (or, alternatively, for $\|\phi\|_{X_{A_1}}$ sufficiently small).

Next estimate the $\|w(t)\|_{L^2(|x|^A dx)}$. We verify the estimate

$$\|w(t)\|_{L^2(|x|^A dx)} \le \frac{C}{t^{A_1/5}} \text{ for } t > \frac{1}{\delta}. \tag{3.40}$$

From (3.38), we also get

$$\|(3.34)\|_{L^2(|x|^A dx)} \le C\|(34)\|_2 < \frac{\delta}{t^{A_1/4}}. \tag{3.41}$$

To estimate $\|(3.35)\|_{L^2(|x|^A dx)}$, apply (2.11). We get thus

$$\|(3.35)\|_{L^2(|x|^A dx)} \le C\int_t^\infty \|[f_0+aw+b\overline{w}+G(w)](\tau)\|_{L^2(|x|^A dx)}d\tau \tag{3.42}$$

$$+\,C\int_t^\infty (1+|t-\tau|^A)\|[f_0+aw+b\overline{w}+G(w)(\tau)\|_{H^A}d\tau. \tag{3.43}$$

From (3.28), (3.30), (3.32), (3.36) and (3.40), it follows

$$(3.48) \le \int_t^\infty \left\{\frac{C}{\tau^{A_1/2}}+\frac{C}{\tau^2}\|w(\tau)\|_{L^2(|x|^A dx)}+C\|w(\tau)\|_\infty\|w(\tau)\|_{L^2(|x|^A dx)}\right\}d\tau$$

$$< \int_t^\infty \left\{\frac{C}{\tau^{A_1/2}}+\frac{C}{\tau^{2+A_1/5}}+\frac{C}{\tau^{A_1/4+A_1/5}}\right\}d\tau < \frac{1}{t^{1+A_1/5}} < \frac{C\delta}{t^{t^{A_1/5}}}. \tag{3.44}$$

For (3.43), we get from (3.36), (3.30), (3.32)

$$(3.43) \leq C \int_t^\infty [1 + (\tau - t)^A] \left\{ \frac{C}{\tau^{A_1/2}} + C \| w(\tau) \|_{H^A} \right\}$$

$$\leq C \int_t^\infty \tau^A \left\{ \frac{C}{\tau^{A_1/2}} + \frac{C}{\tau^{A_1/4}} \right\} d\tau < \frac{C}{t^{A_1/4 - A - 1}} < \frac{\delta}{t^{A_1/5}} \tag{3.45}$$

taking A_1 sufficiently large. Hence (3.40) holds and from (3.36), (3.40)

$$\| w(t) \|_{X_A} < \frac{C}{t^{A_1/5}} < \frac{1}{t^{2A}} \text{ for } t > \frac{1}{\delta} \tag{3.46}$$

establishing (3.7).

This completes the proof of Theorem 1′ and hence Theorem 1.

REMARK. The following observation will be useful in the proof to Theorem 2. Given $\phi \in \mathcal{P}_{A_1}$, A_1 sufficiently large depending on A, the function $w = w_\phi \in \mathcal{C}([-\delta, 0), X_A)$ in (1.23), which we constructed above, has clearly a Lipschitz dependence on ϕ, i.e. for $t \in [-\delta, 0]$

$$\| w_\phi(t) - w_\psi(t) \|_{X_A} \leq C |t|^A \| \phi - \psi \|_{X_{A_1}} \text{ for } \phi, \psi \in \mathcal{P}_{A_1}. \tag{3.47}$$

This fact is easily verified from the preceding argument.

4. – Proof of Theorem 2

We will use the following

LEMMA 4.1. *Let $x_1, \dots, x_J \in \mathbb{R}^d$ and $t_1, \dots, t_J \in \mathbb{R}$ be such that*

$$|x_j - x_k| \geq 1 \textit{ for } 1 \leq j \neq k \leq J. \tag{4.2}$$

Fix $j = 1, \dots, J$, an integer A and an index $\alpha \in \mathbb{Z}^d$, $|\alpha| \leq A$. There is a Schwartz function $\eta_{j\alpha}$ satisfying

$$D^\beta (e^{it_k \Delta} \eta_{j\alpha})(x_k) = \delta_{jk} \delta_{\alpha\beta} \textit{ for } 1 \leq k \leq J, \ |\beta| \leq A. \tag{4.3}$$

Moreover, if we restrict the system $\{(x_j, t_j)\}$, subject to (4.2), to a bounded set, the functions $\eta_{j\alpha}$ are subject to uniform estimates.

PROOF. Since

$$e^{it\Delta}\phi = \int e^{i(|\lambda|^2 t + \lambda x)}\, \widehat{\phi}(\lambda) d\lambda \tag{4.4}$$

one has

$$D^\beta (e^{it_k\Delta}\phi)(x_k) = \int (i\lambda)^\beta\, e^{i(|\lambda|^2 t_k + \lambda x_k)}\, \widehat{\phi}(\lambda) d\lambda. \tag{4.5}$$

Denote by $\psi_{k\beta}(1 \leq k \leq J,\ |\beta| \leq A)$ the function

$$\psi_{k\beta}(\lambda) = (i\lambda)^\beta\, e^{i(|\lambda|^2 t_k + \lambda x_k)}\, \gamma(\lambda) \tag{4.6}$$

where γ is a smooth, compactly supported function, such that

$$\gamma(\lambda) = 1 \text{ for } |\lambda| \leq 1. \tag{4.7}$$

The lemma will clearly follow from the fact that the system $\{\psi_{k\beta} | 1 \leq k \leq J, |\beta| \leq A\}$ consists of linearly independent functions, or equivalently

$$\det(\langle \psi_{k\beta},\ \psi_{k',\beta'}\rangle) \neq 0. \tag{4.8}$$

Assume this were not the case, then there would be coefficients $\overline{a} = \{a_{k\beta}\} \subset \mathbb{C}$, $\overline{a} \neq 0$ s.t.

$$\sum a_{k\beta}\psi_{k\beta}(\lambda) = 0 \text{ for all } \lambda. \tag{4.9}$$

Hence, by (4.7)

$$\sum_k \Big[\sum_\beta a_{k\beta}(i\lambda)^\beta\Big] e^{i(|\lambda|^2 t_k + \lambda . x_k)} = 0 \text{ for } |\lambda| < 1. \tag{4.10}$$

Observe that the left side of (4.10) extends to an entire function on $\mathbb{C}^d$, which consequently vanishes identically. Using induction on the number of summands, taking (4.2) into account, and derivative considerations, one easily reaches a contradiction. The uniformity statement results from a standard compactness consideration.

COROLLARY 4.11. *Let the system $\{(x_j,\ t_j)\}$ be as in Lemma 4.1 and A, A_1 integers. Then, given complex numbers $\overline{a} = (a_{j,\alpha})_{\substack{1\leq j\leq J\\ |\alpha|\leq A}}$, there is a Schwartz function η such that*

$$D^\alpha(e^{it_j\Delta}\eta)(x_j) = a_{j\alpha} \textit{ for all } 1 \leq j \leq J,\ |\alpha| \leq A \tag{4.12}$$

and

$$\|\eta\|_{X_{A_1}} \leq C|\overline{a}|. \tag{4.13}$$

In fact $\eta = \eta_{\overline{a}}$ depends linearly on $\overline{a}$.

The constant $C = C(J,\ A,\ A_1)$ in (4.13) is again uniform if we restrict the system $\{(x_j,\ t_j)\}$ to a bounded set, subject to condition (4.2).

We let $d = 1$ or $d = 2$ again

LEMMA 4.14. *Fix* $\varepsilon > 0$, *a large integer* A, $x_1 \in \mathbb{R}^d$, $|x_1| > 1$ *and* $0 > t_1 > -\delta$, δ *small enough.*

Then (1.1) *has a blowup solution* u *on* $[-\delta, 0[$ *of the form*

$$u = \tilde{R} + u_1 \tag{4.15}$$

where $\tilde{R}$ *is given by* (1.8) *and such that* u_1 *is smooth on* $[-\delta, \delta]$, *solving* (1.1) *on* $[0, \delta]$

$$\|u_1(t)\|_{X_A} < \varepsilon \tag{4.16}$$

and

$$D^\alpha u(x_1, t_1) = 0 \text{ for all } |\alpha| \leq A. \tag{4.17}$$

Thus u *is a perturbation of* $\tilde{R}$ *such that* (4.17) *holds.*

PROOF. We will construct u by an iterative process, based on Theorem 1′ and (3.47). Recall first (1.8)

$$\tilde{R}(x, t) = \frac{1}{t^{d/2}} e^{\frac{|x|^2-4}{4it}} R\left(\frac{x}{t}\right) \tag{4.18}$$

implying that

$$|D^\alpha \tilde{R}(x_1, t_1)| < |t_1|^{-\frac{d}{2}-|\alpha|} e^{-c\frac{|x_1|}{|t_1|}} < e^{-c\frac{1}{\delta}} \overset{\delta\to 0}{\to} 0. \tag{4.19}$$

Fix A_1 sufficiently large. Using (4.11), take $\eta_1 \in \mathcal{S}$ satisfying

$$D^\alpha \eta_1(0, 0) = 0 \text{ for } |\alpha| \leq A_1 \tag{4.20}$$

$$D^\alpha(e^{it_1\Delta}\eta_1)(x_1) = -D^\alpha \tilde{R}(x_1, t_1) \text{ for } |\alpha| \leq A \tag{4.21}$$

$$\|\eta_1\|_{X_{A_1}} = \kappa_1 < C(A_1)e^{-c\,1/\delta} \tag{4.22}$$

where (4.22) follows from (4.13), (4.19).

Applying Theorem 1′, cf. (3.47), with $\eta_1 \in \mathcal{P}_{A_1}$, we get $w_{\eta_1} \in \mathcal{C}([-\delta, 0]; X_A)$ such that

$$U_1 = \tilde{R} + z_{\eta_1} + w_{\eta_1} \tag{4.23}$$

solves (1.1) on $[-\delta, 0[$ and

$$\|w_{\eta_1}(t)\|_{X_{A+2}} < C|t|^A \|\eta_1\|_{X_{A_1}} \tag{4.24}$$

thus from (4.23), (4.21) and the integral equation, it follows that

$$D^\alpha U_1(x_1, t_1) = i D^\alpha \left\{ \int_{t_1}^0 e^{i(t_1-\tau)\Delta}(z_{\eta_1}|z_{\eta_1}|^{4/d})(\tau)d\tau \right\}(x_1) + D^\alpha w_{\eta_1}(x_1, t_1)$$
$$= (4.25) + (4.26).$$

From (4.22), we estimate for $|\alpha| \le A$

$$|(4.25)| < \left\| \int_{t_1}^0 e^{i(t_1-\tau)\Delta}(z_{\eta_1}|z_{\eta_1}|^{4/d})(\tau)d\tau \right\|_{H^{|\alpha|+2}} < C|t_1|.\|z_{\eta_1}\|^{4/d+1}_{H^{|\alpha|+2}}$$

$$< C\delta.\|\eta_1\|^{4/d+1}_{X_{A+2}} < C\delta\kappa_1^{4/d+1} \tag{4.27}$$

and from (4.22), (4.24)

$$|(4.26)| < \|w_{\eta_1}(t_1)\|_{H^{|\alpha|+2}} < \|w_{\eta_1}(t_1)\|_{X_{A+2}} < C\delta^A\kappa_1. \tag{4.28}$$

Replacing $U_0 = \tilde{R}$ by U_1, (4.27), (4.28) give thus

$$|D^\alpha U_1(x_1, t_1)| < \delta\kappa_1 \text{ for } |\alpha| \le A. \tag{4.29}$$

Choose next $\eta_2 \in \mathcal{S}$ such that

$$D^\alpha \eta_2(0, 0) = 0 \text{ for } |\alpha| \le A_1 \tag{4.30}$$
$$D^\alpha(e^{it_1\Delta}\eta_2)(x_1) = -D^\alpha U_1(x_1, t_1) \text{ for } |\alpha| \le A \tag{4.31}$$
$$\|\eta_2\|_{X_{A_1}} = \kappa_2 < C(A_1)\delta\kappa_1 < \delta^{1/2}\kappa_1 \tag{4.32}$$

applying again (4.11). Considering $\eta_1 + \eta_2 \in \mathcal{P}_{A_1}$, we get $w_{\eta_1+\eta_2}$ s.t.

$$U_2 = \tilde{R} + z_{\eta_1+\eta_2} + w_{\eta_1+\eta_2} \tag{4.33}$$

solves (1.1) on $[-\delta, 0[$ and, by (3.47),

$$\|w_{\eta_1+\eta_2}(t) - w_{\eta_1}(t)\|_{X_{A+2}} < C|t|^A\|\eta_2\|_{X_{A_1}}. \tag{4.34}$$

We have

$$U_2 = U_1 + z_{\eta_1+\eta_2} - z_{\eta_1} + w_{\eta_1+\eta_2} - w_{\eta_1}$$
$$= U_1 + e^{it\Delta}\eta_2 + i\int_{t_1}^0 e^{i(t_1-\tau)\Delta}(z_{\eta_1+\eta_2}|z_{\eta_1+\eta_2}|^{4/d} - z_{\eta_1}|z_{\eta_1}|^{4/d})(\tau)d\tau$$
$$+ w_{\eta_1+\eta_2} - w_{\eta_1}. \tag{4.35}$$

Thus (4.35) gives for $|\alpha| \leq A$, by (4.31), (4.34), (4.32)

$$\begin{aligned} |D^\alpha U_2(x_1, t_1)| &\leq \int_{t_1}^0 \|(z_{\eta_1+\eta_2}|z_{\eta_1+\eta_2}|^{4/d} - z_{\eta_1}|z_{\eta_1}|^{4/d})(\tau)\|_{H^{|\alpha|+2}} d\tau + C|t_1|^A \kappa_2 \\ &< C\delta\kappa_1^{4/d}.\|z_{\eta_1+\eta_2} - z_{\eta_1}\|_{H^{|\alpha|+2}} + C\delta^A\kappa_2 \\ &< C\delta\kappa_1^{4/d}\|\eta_2\|_{H^{|\alpha|+2}} + C\delta^A\kappa_2 \\ &< \delta\kappa_2 \end{aligned} \tag{4.36}$$

The continuation of the process is clear. In general

$$U_s = \tilde{R} + z_{\eta_1+\cdots+\eta_s} + w_{\eta_1+\cdots+\eta_s} \tag{4.37}$$

solves (1.1) on $[-\delta, 0[$, where

$$\eta = \lim_{s\to\infty} (\eta_1 + \cdots + \eta_s) \text{ in } X_{A_1} \tag{4.38}$$

$$z_\eta = \lim z_{\eta_1+\cdots+\eta_s} \text{ in } \mathcal{C}([-\delta, \delta] : X_{A_1}) \tag{4.39}$$

solves (1.1) on $[0, \delta]$ and

$$w_\eta = \lim w_{\eta_1+\cdots+\eta_s} \text{ in } \mathcal{C}([-\delta, 0]; X_{A+2}) \tag{4.40}$$

satisfies

$$\|w_\eta(t)\|_{X_{A+2}} \leq C\kappa_1|t|^A. \tag{4.41}$$

Thus, by construction

$$u = \lim U_s = \tilde{R} + z_\eta + w_\eta = \tilde{R} + u_1 \tag{4.12}$$

satisfies (4.17) and

$$\|u_1(t)\|_{X_A} \leq \|\eta\|_{X_{A_1}} + \|w_\eta(t)\|_{X_{A+2}} < \kappa_1 \to 0 \text{ for } \delta \to 0 \tag{4.43}$$

so that the conditions of Lemma 4.14 hold.

Proof of Theorem 2. We let $(x_0, t_0) = (0, 0)$, $0 < |t_1| < \delta$ and apply Lemma 4.14 to get a blowup solution U on $[-2\delta, 0[$ of the form

$$U = \tilde{R} + U_1 \tag{4.44}$$

such that U_1 is smooth on $[-2\delta, 2\delta]$, solving (1.1) on $[0, 2\delta]$ and

$$\|U_1(t)\|_{X_{A_1}} \leq 1,\ |t| < 2\delta \tag{4.45}$$

$$D^\alpha U(x_1, t_1) = 0 \text{ for all } |\alpha| \leq A_1. \tag{4.46}$$

Thus, by (4.45), (4.46)

$$\phi = U(\cdot - x_1, t_1) = \tilde{R}(\cdot - x_1, t_1) + U_1(\cdot - x_1, t_1) \tag{4.47}$$

is in $\mathcal{P}_{A_1}$; the bounds on $\|\phi\|_{A_1}$ depending obviously on $|t_1|$. Application of Theorem 1′ yields for

$$t_1 > t_2,\ t_1 - t_2 < \delta(|t_1|) < \delta \tag{4.48}$$

a blowup solution U' on $[t_2, t_1[$ of the form

$$U' = \tilde{R}_1 + U + w \tag{4.49}$$

where

$$w \in \mathcal{C}([t_2, t_1]; X_A),\ \ \|w(t)\|_{X_A} \leq |t - t_1|^A. \tag{4.50}$$

Extend w to $[t_2, \infty[$ by letting $w(t) = 0$ for $t > t_1$, cf. (4.50). Define

$$u(t) = \begin{cases} U(t) & \text{for } t \in]t_1, 0[\\ U'(t) & \text{for } t \in]t_2, t_1[\end{cases} \tag{4.51}$$

clearly satisfying the conditions of Theorem 2.

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