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## Regularity of Minimizers for a Class of Membrane Energies

EMILIO ACERBI\* – IRENE FONSECA\*\* – NICOLA FUSCO\*

**Abstract.** Regularity properties for (local) minimizers of elastic energies have been challenging mathematical techniques for many years. Recently the interest has resurfaced due in part to the fact that existing partial regularity results do not suffice to ensure existence of (classical) solutions to problems involving free discontinuity sets. The analysis of such questions was started with the fundamental work of De Giorgi in the early 80's in connection with the Mumford-Shah model for image segmentation in computer vision, and later applied to some models for fracture mechanics, thin films, and membranes ([1], [18], [20]). In this paper it is shown that local minimizers in  $W^{1,2}(\Omega; \mathbb{R}^d)$  of the functional

$$\mathcal{F}_0(u, \Omega) := \int_{\Omega} \left[ \frac{1}{2} |Du|^2 + f(|v(u)|) \right] dx$$

are Hölder continuous of any exponent  $\gamma \in (0, 1)$ , where  $\Omega \subset \mathbb{R}^2$  is an open, bounded set,  $f$  is a (not necessarily convex) function growing linearly at infinity, and  $v(u)$  stands for the vector of all  $2 \times 2$  minors of  $Du$ . As a consequence, it is possible to obtain existence of “classical” minimizers in  $SBV(\Omega; \mathbb{R}^d)$

$$\mathcal{F}(u, \Omega) := \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + f(|v(u)|) \right] dx + \beta \int_{\Omega} |u - g|^q dx + \gamma H^{N-1}(S_u \cap \Omega)$$

where  $g \in L^\infty(\Omega; \mathbb{R}^d)$ ,  $q > 1$ ,  $\beta, \gamma > 0$ . These minimizers are “classical minimizers” in the sense that  $H^1(\overline{S_u} \setminus S_u \cap \Omega) = 0$  and  $u \in W^{1,2}(\Omega \setminus \overline{S_u})$ .

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### 1. – Introduction

De Giorgi's seminal work in the early 80's in the study of free discontinuity problems with relation to the model of Mumford Shah for image segmentation in computer vision ([9], [10], [11], [14], [16], [25]) has opened many doors

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into the study of mathematical questions relevant to the understanding of the behavior of thin films, membranes, and fractured elastic media (see [1], [7], [19], [20]). Although very different in physical nature and motivation, models for these problems have as a common feature the fact that, when searching for quasistatic stable or metastable solutions, one is led to minimizing among all pairs  $(K, u)$  an energy involving bulk and interfacial terms,

$$\mathcal{G}(K, u) := \int_{\Omega \setminus K} F(Du) dx + \beta \int_{\Omega \setminus K} |u - g|^q dx + \gamma H^{N-1}(K \cap \Omega)$$

where  $\Omega$  is an open, bounded subset of  $\mathbb{R}^N$ ,  $q > 1$ ,  $g \in L^\infty(\Omega; \mathbb{R}^d)$ ,  $\beta, \gamma > 0$ ,  $H^{N-1}$  stands for the  $(N-1)$ -dimensional Hausdorff measure,  $K$  is closed in  $\mathbb{R}^N$  and  $u : \Omega \setminus K \rightarrow \mathbb{R}^d$  is smooth.

In order to find minima for this energy, as it is usual one relaxes the spaces of admissible fields where lower semicontinuity and compactness results may be found more easily ([2], [3], [4]), and through regularity arguments one concludes that, indeed, these solutions live in the more restricted set of “classical” fields. De Giorgi and Ambrosio (see [2], [4], [15]) introduced the space *SBV* of functions of *special bounded variation*, i.e. *BV* functions  $u$  whose distributional derivative  $Du$ , which is a finite Radon measure, may be decomposed into an absolutely continuous part  $\nabla u \mathcal{L}^N$  with respect to the  $N$ -dimensional Lebesgue measure  $\mathcal{L}^N$ , and a singular part whose support is an  $(N-1)$ -dimensional rectifiable set  $S_u$ . This is the “jump set” of  $u$ , in the sense that  $u$  has traces  $u^+ \neq u^-$   $H^{N-1}$  a.e. on the two sides of  $S_u$ . They showed that

$$(1.1) \quad \mathcal{F}(u; \Omega) := \int_{\Omega} F(\nabla u) dx + \beta \int_{\Omega} |u - g|^q dx + \gamma H^{N-1}(S_u \cap \Omega)$$

admits a minimizer in *SBV* provided  $F$  is a convex function growing super-linearly at infinity and coercive. In the scalar-valued case, where  $u : \Omega \rightarrow \mathbb{R}$ , De Giorgi, Carriero and Leaci [16] proved that if  $F(\xi) = |\xi|^2$  then

$$H^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0$$

for any minimizer  $u \in SBV(\Omega; \mathbb{R})$  of  $\mathcal{F}$ , and so the pair  $(\overline{S_u}, u)$  is a “classical” minimizer of the original functional  $\mathcal{G}$ . Note that, in general, if  $u \in SBV$  then the set  $S_u$  is far from being closed, i.e.  $H^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) > 0$ , and it may be even dense in  $\Omega$ . The result of [16] has been extended to a quite broad class of convex functions  $F$  with growth  $p > 1$  at infinity, and further regularity on the set  $S_u$  has been obtained (see [1], [5], [6], [11], [13], [20]).

In the vectorial case  $u : \Omega \rightarrow \mathbb{R}^d$ ,  $d \geq 2$ , Fonseca and Francfort (see [18]) showed that functionals of the type (1.1) appear naturally in the study of effective energies for fractured elastic materials.

In this paper we show that local minimizers  $u \in W^{1,2}(\Omega; \mathbb{R}^d)$  of the functional

$$\mathcal{F}_0 : v \in W^{1,2}(\Omega; \mathbb{R}^d) \mapsto \int_{\Omega} \left[ \frac{1}{2} |Du|^2 + f(|v(u)|) \right] dx$$

are in  $C_{\text{loc}}^{0,\gamma}(\Omega; \mathbb{R}^d)$  for all  $\gamma \in (0, 1)$ , where  $\Omega \subset \mathbb{R}^2$  is open and bounded,  $f$  grows linearly at infinity, and  $\nu(u)$  stands for the vector of all  $2 \times 2$  minors of  $Du$ . In turn, this regularity will entail that minimizers  $u \in SBV(\Omega; \mathbb{R}^d)$  of

$$\mathcal{F}(u, \Omega) := \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + f(|\nu(u)|) \right] dx + \beta \int_{\Omega} |u - g|^q dx + \gamma H^1(S_u \cap \Omega)$$

are ‘‘classical’’ mimimizers in that  $u \in W^{1,2}(\Omega \setminus \overline{S_u}; \mathbb{R}^2)$  and  $H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0$ . In the case where  $d = 3$  we may consider these energies as associated to thin films or membranes (see [7], [12], [19], [23]). We remark that  $f$  does not need to be convex. This regularity result was already obtained for  $d = 2$ ,  $\nu(u) = \det \nabla u$ , and when  $f$  is convex (see [1]). As in that paper, here we use an argument similar to the one introduced by Bauman, Phillips and Owen [8], and used by Dougherty [17]; precisely, we obtain regularity of local minimizers by means of the higher integrability of two auxiliary combinations of the derivatives of  $u$ ,  $A := (|D_1 u|^2 - |D_2 u|^2)/2$ ,  $B := D_1 u \wedge D_2 u$ , which turn out to be harmonic functions in the case where  $f(|\nu(u)|) = |\nu(u)|$  (see Proposition 3.2).

## 2. – Statements and Preliminary Results

If  $\Omega \subset \mathbb{R}^N$  is open, we say that a function of *bounded variation*  $u \in BV(\Omega; \mathbb{R}^d)$  is a function of *special bounded variation* (see [2], [3], [4], [14], [15]),  $u \in SBV(\Omega; \mathbb{R}^d)$ , if, denoting by  $S_u$  the complement of the set of Lebesgue points of  $u$ , the distributional derivative  $Du$  is represented by

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu H^{N-1} \llcorner S_u,$$

where  $\nabla u$  is the Radon-Nikodým derivative of the finite Radon measure  $Du$  with respect to the  $N$ -dimensional Lebesgue measure  $\mathcal{L}^N$ ,  $\nu$  is the normal to the rectifiable set  $S_u$ ,  $u^+$  and  $u^-$  are the traces of  $u$  on  $S_u$ , and  $H^{N-1}$  denotes the  $(N - 1)$ -dimensional Hausdorff measure.

In the sequel we consider  $\Omega$  to be a bounded, open subset of  $\mathbb{R}^2$ , and we let  $f : [0, +\infty) \rightarrow [0, \infty)$  to be a  $C^1$  function such that

(H1)  $f(t) \leq C(1 + t)$  for some  $C > 0$ ;

(H2) there exist  $M \in [0, +\infty)$  such that

$$\lim_{t \rightarrow +\infty} f'(t) = M;$$

(H3) there exist  $\alpha, C > 0$  such that for all  $t \geq 1$

$$\left| f'(t) - \frac{f(t)}{t} \right| \leq \frac{C}{t^\alpha}.$$

It is not restrictive to assume that

$$0 < \alpha < 1$$

and in what follows we will work under this assumption. Also, in order to simplify the notation the value of the constant  $C$  may change from one line to the next, and  $B_R$ ,  $R > 0$ , will denote a generic open ball of radius  $R$ , centered at  $x \in \Omega$ , and such that  $B_R \subset \Omega$ .

Given  $u \in SBV(\Omega; \mathbb{R}^d)$  we define

$$v(u) := \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2},$$

the 2-covector whose components are the  $2 \times 2$  subdeterminants of  $\nabla u$ .

Consider the energies

$$\mathcal{G}(K, u) := \int_{\Omega \setminus K} \left[ \frac{1}{2} |\nabla u|^2 + f(|v(u)|) \right] dx + \beta \int_{\Omega \setminus K} |u - g|^q dx + \gamma H^1(\Omega \cap K),$$

$$\mathcal{F}(u, \Omega) := \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 + f(|v(u)|) \right] dx + \beta \int_{\Omega} |u - g|^q dx + \gamma H^1(S_u \cap \Omega),$$

and

$$\mathcal{F}_0(u, \Omega) := \int_{\Omega} \left[ \frac{1}{2} |Du|^2 + f(|v(u)|) \right] dx.$$

DEFINITION 2.1. We say that  $u \in W^{1,2}(\Omega; \mathbb{R}^d)$  is a  $W^{1,2}$ -local minimizer of

$$I(v, \Omega) := \int_{\Omega} F(\nabla v) dx, \quad v \in W^{1,2}(\Omega; \mathbb{R}^d)$$

if

$$I(u, B_R(x_0)) = \min \left\{ I(v, B_R(x_0)) : v \in u + W_0^{1,2}(B_R(x_0); \mathbb{R}^d) \right\}$$

for all balls  $B_R(x_0) \subset \Omega$ .

The main result of this paper is the following theorem.

THEOREM 2.2. *If  $u \in W^{1,2}(\Omega, \mathbb{R}^d)$  is a  $W^{1,2}$ -local minimizer of  $\mathcal{F}_0$  then  $u \in C_{\text{loc}}^{0,\gamma}$  for all  $\gamma \in (0, 1)$ .*

In the proof of Theorem 2.2 we will use classical arguments of regularity theory within the framework of the *Morrey spaces*  $L^{p,\lambda}$ ; for a detailed study of these methods we refer the reader to [21], [24].

DEFINITION 2.3. Given  $\lambda \geq 0$  we say that  $f \in L^{p,\lambda}(\Omega; \mathbb{R})$  if there exists a constant  $C > 0$  such that

$$\int_{B_\rho(x) \cap \Omega} |f|^p dx \leq C \rho^\lambda$$

for all  $x \in \Omega$  and  $0 < \rho < \text{diam } \Omega$ . The function  $f$  is said to be in  $L_{\text{loc}}^{p,\lambda}(\Omega)$  if  $f \in L^{p,\lambda}(\Omega')$  for all  $\Omega' \subset\subset \Omega$ .

It can be shown that, with  $\Omega \subset \mathbb{R}^2$ ,

$$L^{p,0}(\Omega) = L^p(\Omega), \quad L^{p,2}(\Omega) = L^\infty(\Omega), \quad L^{p,\lambda}(\Omega) = \{0\} \quad \text{if } \lambda > 2,$$

and that  $L^{p,\lambda}(\Omega)$  is a Banach space endowed with the norm

$$\|f\|_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{B_\rho(x) \cap \Omega} |f|^p dx \right\}^{\frac{1}{p}}.$$

Morrey proved that (see Theorem 3.5.2, [24])

LEMMA 2.4. *If  $u \in W_{\text{loc}}^{1,2}(\Omega)$  and  $Du \in L_{\text{loc}}^{2,\lambda}(\Omega)$  for some  $0 < \lambda < 2$  then  $u \in C_{\text{loc}}^{0,\lambda/2}(\Omega)$ .*

In light of Lemma 2.4, we will prove Theorem 2.2 by showing that if  $u$  is a  $W^{1,2}$ -local minimizer of  $\mathcal{F}_0$  then for all  $0 \leq \lambda < 2$

$$(2.1) \quad \int_{B_\rho} |Du|^2 dx \leq C \left( \frac{\rho}{R} \right)^\lambda \int_{B_R} |Du|^2 dx + C\rho^\lambda$$

for all  $0 < \rho < R$  with  $B_R \subset\subset \Omega$ .

As a corollary we obtain,

COROLLARY 2.5. *Let  $u \in SBV(\Omega; \mathbb{R}^d)$  be a minimizer for  $\mathcal{F}$ . Then  $(\overline{S_u}, u)$  is a minimizer for  $\mathcal{G}$  among all pairs  $(K, v)$  with  $K \subset \Omega$  closed and  $v \in W^{1,2}(\Omega \setminus K; \mathbb{R}^d)$ . Moreover,*

$$H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

Following the argument introduced by De Giorgi, Carriero and Leaci [16], and outlined in [1], the corollary holds provided we can show that  $W^{1,2}$ -local minimizers of

$$v \in W^{1,2}(B_1; \mathbb{R}^d) \mapsto \int_{B_1} \left[ \frac{1}{2} |Dv|^2 + M|v(v)| \right] dx$$

satisfy an estimate of the type

$$\int_{B_\rho} \left[ \frac{1}{2} |Du|^2 + M|v(u)| \right] dx \leq C\rho^\lambda \int_{B_1} \left[ \frac{1}{2} |Du|^2 + M|v(u)| \right] dx + C\rho^\lambda,$$

for some  $0 < \lambda < 2$  and  $0 < \rho \leq 1$  or, equivalently,

$$\int_{B_\rho} |Du|^2 dx \leq C\rho^\lambda \int_{B_1} |Du|^2 dx + C\rho^\lambda.$$

We conclude that the assertion of the corollary holds true provided we prove (2.1).

The following two lemmas may be found in [21] (see Chapter 3, Theorem 3.1, page 87, and Lemma 2.1, respectively).

LEMMA 2.6. *Let  $\lambda < 2$ , let  $f \in L^{2,\lambda}(B_R; \mathbb{R}^2)$ , and let  $v \in W^{1,2}(B_R; \mathbb{R})$  satisfy*

$$\Delta v = \operatorname{div} f \quad \text{in } B_R.$$

*Then  $Dv \in L_{\text{loc}}^{2,\lambda}(B_R; \mathbb{R}^2)$ , and for every  $\rho \leq R$*

$$\int_{B_\rho} |Dv|^2 dx \leq C \left( \frac{\rho}{R} \right)^\lambda \int_{B_R} |Dv|^2 dx + C\rho^\lambda \|f\|_{L^{2,\lambda}(B_R)}^2.$$

LEMMA 2.7. *Let  $\phi : [0, +\infty) \rightarrow [0, +\infty)$  be a nonnegative, nondecreasing function, such that*

$$\phi(\rho) \leq H \left[ \left( \frac{\rho}{R} \right)^\gamma + \varepsilon \right] \phi(R) + KR^\beta$$

*for all  $0 < \rho < R \leq R_0$  and for some constants  $H, K \geq 0$  and  $0 < \beta < \gamma$ . Then there exist constants  $\varepsilon_0 = \varepsilon_0(H, \gamma, \beta)$ ,  $C = C(H, \gamma, \beta)$  such that*

$$\phi(\rho) \leq C \left[ \left( \frac{\rho}{R} \right)^\beta \phi(R) + K\rho^\beta \right]$$

*for all  $0 < \rho < R \leq R_0$ .*

LEMMA 2.8. *Let  $p > 1$  and  $0 \leq \lambda < 2$ . If  $f_{ij} \in L_{\text{loc}}^{p,\lambda}(\Omega)$  for  $i, j \in \{1, 2\}$  and  $u \in L_{\text{loc}}^1(\Omega)$  is a distributional solution of*

$$\Delta u = \sum D_{ij} f_{ij}$$

*then  $u \in L_{\text{loc}}^{p,\lambda}(\Omega)$ .*

PROOF. Let  $B_R \subset\subset \Omega$  and for every  $i, j$  let  $v_{ij}$  be the solution of (see Theorem 9.15 and Lemma 9.17, [22])

$$\begin{cases} \Delta v_{ij} = f_{ij} \\ v_{ij} \in W_0^{1,p}(B_R) \cap W^{2,p}(B_R), \end{cases}$$

and we set

$$w := \sum D_{ij} v_{ij}.$$

Then  $w \in L^p(B_R)$  and  $\|w\|_{L^p(B_R)} \leq C \sum \|f_{ij}\|_{L^p(B_R)}$ . In addition,  $\Delta w = \sum D_{ij} f_{ij}$  in  $\mathcal{D}'$ , so that the function

$$v := u - w$$

is harmonic, i.e.  $\Delta v = 0$ . Hence

$$\sup_{B_{R/2}} |v| \leq C(p) \left( \frac{1}{|B_R|} \int_{B_R} |v|^p dx \right)^{1/p},$$

from which we deduce that for every  $\rho \leq R/2$  (thus, for all  $0 < \rho \leq R$ )

$$\int_{B_\rho} |v|^p dx \leq C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |v|^p dx.$$

We have

$$\begin{aligned} \int_{B_\rho} |u|^p dx &\leq C \int_{B_\rho} (|v|^p + |w|^p) dx \\ &\leq C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |v|^p dx + C \int_{B_R} |w|^p dx \\ &\leq C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |u|^p dx + C \int_{B_R} |w|^p dx \\ &\leq C \left(\frac{\rho}{R}\right)^2 \int_{B_R} |u|^p dx + CR^\lambda. \end{aligned}$$

By Lemma 2.7 we deduce that for all  $0 < \rho \leq R$

$$\begin{aligned} \int_{B_\rho} |u|^p dx &\leq C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |u|^p dx + C\rho^\lambda \\ &\leq \rho^\lambda \left[ \frac{C}{R^\lambda} \int_{B_R} |u|^p dx + C \right], \end{aligned}$$

and so  $u \in L_{\text{loc}}^{p,\lambda}(\Omega)$ . □

We end this section with a list of algebraic inequalities, following an argument introduced in [8] (see also [17]).

Let  $P, Q \in \mathbb{R}^d$  and set

$$A := \frac{|P|^2 - |Q|^2}{2}, \quad B := P \cdot Q, \quad v := P \wedge Q.$$

LEMMA 2.9. *We have*

- i)  $2\sqrt{A^2 + B^2} \leq |P|^2 + |Q|^2$ ;
- ii)  $0 \leq |P|^2 + |Q|^2 - 2|v| \leq 2\sqrt{A^2 + B^2}$ ;
- iii) if  $v = 0$  then  $|P|^2 + |Q|^2 = 2\sqrt{A^2 + B^2}$ ;
- iv) if  $\alpha, \beta \in \mathbb{R}^d$  and  $v \neq 0$  then

$$\left| \frac{1}{|v|} v \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \right|^2 \leq 4\sqrt{A^2 + B^2} (|\alpha|^2 + |\beta|^2).$$

PROOF. Since

$$|v|^2 = \sum_{i < j} |P_i Q_j - P_j Q_i|^2 = \frac{1}{2} \sum_{i,j} |P_i Q_j - P_j Q_i|^2 = |P|^2 |Q|^2 - (P \cdot Q)^2,$$



we have

$$|P|^2|Q|^2 = B^2 + |v|^2,$$

and so

$$4A^2 = (|P|^2 + |Q|^2)^2 - 4|P|^2|Q|^2 = (|P|^2 + |Q|^2)^2 - 4(B^2 + |v|^2),$$

and

$$4(A^2 + B^2) = (|P|^2 + |Q|^2)^2 - 4|v|^2.$$

Clearly i) and iii) follow. In addition, we have that

$$(|P|^2 + |Q|^2)^2 - 4|v|^2 \geq 0$$

hence

$$0 \leq |P|^2 + |Q|^2 - 2|v| \leq \sqrt{(|P|^2 + |Q|^2)^2 - 4|v|^2} = 2\sqrt{A^2 + B^2},$$

which yields assertion ii).

Now remark that if  $v \neq 0$  then  $P \neq 0$  and, setting

$$Q' := Q - \frac{P \cdot Q}{|P|^2} P,$$

then also  $Q' \neq 0$ . Define the orthonormal vectors

$$P_1 := \frac{P}{|P|}, \quad Q_1 := \frac{Q'}{|Q'|}.$$

We write

$$P = p P_1, \quad Q = s P_1 + q Q_1$$

with

$$p := |P|, \quad q := |Q'|, \quad s := \frac{P \cdot Q}{|P|}.$$

Note that

$$v = pq P_1 \wedge Q_1, \quad |v| = pq,$$

and that if  $v \in \mathbb{R}^d$  then

$$(P_1 \wedge Q_1) \cdot (P_1 \wedge v) = v \cdot Q_1, \quad (P_1 \wedge Q_1) \cdot (v \wedge Q_1) = v \cdot P_1.$$

We have

$$\begin{aligned} & \frac{1}{|v|} v \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \\ &= (P_1 \wedge Q_1) \cdot (p P_1 \wedge \beta - s P_1 \wedge \alpha + q \alpha \wedge Q_1) \\ & \quad - (p P_1 \cdot \alpha + s P_1 \cdot \beta + q Q_1 \cdot \beta) \\ &= [(q - p) P_1 - s Q_1] \cdot \alpha + [-s P_1 + (p - q) Q_1] \cdot \beta \\ &= v_1 \cdot \alpha + v_2 \cdot \beta, \end{aligned}$$

with

$$v_1 := (q - p) P_1 - s Q_1 \quad \text{and} \quad v_2 := -s P_1 + (p - q) Q_1.$$

We have

$$|v_1 \cdot \alpha + v_2 \cdot \beta|^2 \leq (|v_1|^2 + |v_2|^2)(|\alpha|^2 + |\beta|^2) = 2(|P|^2 + |Q|^2 - 2|v|)(|\alpha|^2 + |\beta|^2),$$

which, together with ii), concludes the proof of iv).  $\square$

### 3. – Proof of the regularity theorem

In this section we assume that  $u \in W^{1,2}(\Omega; \mathbb{R}^d)$  is a local minimizer of  $\mathcal{F}_0$ .

PROPOSITION 3.1. *If  $Du \in L_{\text{loc}}^{2,\lambda}(\Omega; \mathbb{R}^d)$  for some  $0 \leq \lambda < 2$  then  $Du \in L_{\text{loc}}^{2,q_0(\lambda)}(\Omega; \mathbb{R}^d)$ , where  $q_0(\lambda) := \alpha + \lambda(1 - \alpha/2)$ .*

Before proceeding with the proof of this result, we remark that using an iterative scheme where

$$\lambda_0 := 0, \quad \lambda_{k+1} := q_0(\lambda_k),$$

then

$$\lim_{k \rightarrow +\infty} \lambda_k = \lim_{k \rightarrow +\infty} \alpha \sum_{i=0}^{k-1} \left(1 - \frac{\alpha}{2}\right)^i = 2,$$

hence (2.1) will follow for all  $0 \leq \lambda < 2$  and, as justified in Section 2, this suffices to assert Theorem 2.2.

The proof of Proposition 3.1 uses higher integrability properties of the functions

$$A := \frac{|D_1 u|^2 - |D_2 u|^2}{2}, \quad B := (D_1 u) \cdot (D_2 u),$$

where  $D_1 u$  and  $D_2 u$  stand for the column vectors in  $\mathbb{R}^d$  of the derivatives of  $u$  with respect to  $x_1$  and to  $x_2$ , respectively.

PROPOSITION 3.2. *The functions  $A$  and  $B$  solve the system*

$$\begin{cases} \Delta A = D_{11}^2 g - D_{22}^2 g \\ \Delta B = 2D_{12}^2 g, \end{cases}$$

where

$$g := f(|v(u)|) - |v(u)| f'(|v(u)|).$$

In addition, if  $Du \in L_{\text{loc}}^{2,\lambda}(\Omega; \mathbb{R}^{2d})$  for some  $0 \leq \lambda < 2$  then  $\sqrt{|A| + |B|} \in L_{\text{loc}}^{2,2\alpha+\lambda(1-\alpha)}(\Omega; \mathbb{R})$ .

PROOF. Consider  $\Phi := (\varphi, \psi) \in C_0^1(\Omega; \mathbb{R}^2)$ , and let  $\varepsilon > 0$  be small enough so that with  $\Phi_\varepsilon(x) := x + \varepsilon\Phi(x)$ , then  $\Phi_\varepsilon : \Omega \rightarrow \Omega$  is a smooth diffeomorphism satisfying

$$\det D\Phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div} \Phi(x) + \omega_1(x, \varepsilon),$$

$$\det D\Phi_\varepsilon^{-1}(y) = 1 - \varepsilon \operatorname{div} \Phi(\Phi_\varepsilon^{-1}(y)) + \omega_2(y, \varepsilon),$$

where  $\omega_i(\cdot, \varepsilon)/\varepsilon \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ , uniformly in  $\Omega$ . Set

$$u_\varepsilon(y) := u\left(\Phi_\varepsilon^{-1}(y)\right), \quad y \in \Omega.$$

We have

$$\begin{aligned}
\int_{\Omega} |Du_{\varepsilon}(y)|^2 dy &= \int_{\Omega} |Du(\mathbb{I} - \varepsilon D\Phi)|^2(\Phi_{\varepsilon}^{-1}(y)) dy + o(\varepsilon) \\
&= \int_{\Omega} |Du(\mathbb{I} - \varepsilon D\Phi)|^2(1 + \varepsilon \operatorname{div} \Phi) dx + o(\varepsilon) \\
&= \int_{\Omega} |Du|^2 dx + \varepsilon \int_{\Omega} [|Du|^2 \operatorname{div} \Phi - 2Du D\Phi \cdot Du] dx + o(\varepsilon),
\end{aligned}$$

where the inner product of two  $d \times 2$  matrices  $\xi$  and  $\eta$  is defined as  $\xi \cdot \eta := \operatorname{trace}(\xi^T \eta)$ .

On the other hand, since

$$v(u_{\varepsilon}(y)) = [\det D\Phi_{\varepsilon}^{-1}(y)] v(u)(\Phi_{\varepsilon}^{-1}(y)),$$

we also have that, setting  $\Omega_{\varepsilon} := \{x \in \Omega : |\varepsilon \operatorname{div} \Phi - \omega_2| |v(u)| \neq 0\}$ ,

$$\begin{aligned}
\int_{\Omega} f(|v(u_{\varepsilon}(y))|) dy &= \int_{\Omega} f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|v(u)|) \det D\Phi_{\varepsilon} dx \\
&= \int_{\Omega_{\varepsilon}} [f(|v(u)|) + (-\varepsilon \operatorname{div} \Phi + \omega_2)|v(u)| f'(|v(u)|)] \det D\Phi_{\varepsilon} dx \\
&\quad + \int_{\Omega_{\varepsilon}} \left[ \frac{f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|v(u)|) - f(|v(u)|)}{(-\varepsilon \operatorname{div} \Phi + \omega_2)|v(u)|} - f'(|v(u)|) \right] \\
&\quad \quad (-\varepsilon \operatorname{div} \Phi + \omega_2)|v(u)| \det D\Phi_{\varepsilon} dx \\
&\quad + \int_{\Omega \setminus \Omega_{\varepsilon}} f(|v(u)|) \det D\Phi_{\varepsilon} dx \\
&= \int_{\Omega} f(|v(u)|) \det D\Phi_{\varepsilon} dx \\
&\quad + \int_{\Omega} (-\varepsilon \operatorname{div} \Phi + \omega_2)|v(u)| f'(|v(u)|) \det D\Phi_{\varepsilon} dx + o(\varepsilon), \\
&= \int_{\Omega} f(|v(u)|) dx + \varepsilon \int_{\Omega} [f(|v(u)|) - |v(u)| f'(|v(u)|)] \operatorname{div} \Phi dx + o(\varepsilon),
\end{aligned}$$

because by Lebesgue's dominated convergence, by (H1), and due to the boundedness of  $f'$ ,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega_{\varepsilon}} \left| \frac{f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|v(u)|) - f(|v(u)|)}{(-\varepsilon \operatorname{div} \Phi + \omega_2)|v(u)|} - f'(|v(u)|) \right| \\
|v(u)| \left| \operatorname{div} \Phi - \frac{\omega_2}{\varepsilon} \right| |1 + \varepsilon \operatorname{div} \Phi + \omega_1| dx = 0.
\end{aligned}$$

By the local minimality of  $u$  we have  $\mathcal{F}_0(u_{\varepsilon}) - \mathcal{F}_0(u) \geq 0$ , from which the Euler-Lagrange equation can be easily obtained,

$$\int_{\Omega} \left[ \frac{1}{2} |Du|^2 \operatorname{div} \Phi - Du D\Phi \cdot Du \right] dx = \int_{\Omega} [|v(u)| f'(|v(u)|) - f(|v(u)|)] \operatorname{div} \Phi dx$$

for every  $\Phi = (\varphi, \psi) \in C_0^1(\Omega; \mathbb{R}^2)$ . This equation may be rewritten as

$$\int_{\Omega} [A(D_2\psi - D_1\varphi) - B(D_1\psi + D_2\varphi)] dx = \int_{\Omega} -g(D_1\varphi + D_2\psi) dx,$$

that is,

$$\begin{cases} D_1A + D_2B = D_1g \\ D_2A - D_1B = -D_2g, \end{cases}$$

and the first assertion follows. By (H3)

$$|g| \leq C(1 + |v(u)|^{1-\alpha})$$

and so, assuming that  $Du \in L_{\text{loc}}^{2,\lambda}(\Omega; \mathbb{R}^{2d})$  we have that  $|v(u)| \in L_{\text{loc}}^{1,\lambda}(\Omega; \mathbb{R})$  and

$$g \in L_{\text{loc}}^{\frac{1}{1-\alpha},\lambda}(\Omega).$$

We may now use Lemma 2.8 to obtain that

$$A, B \in L_{\text{loc}}^{\frac{1}{1-\alpha},\lambda}(\Omega),$$

and by Hölder inequality we conclude that

$$\sqrt{|A| + |B|} \in L_{\text{loc}}^{2,2\alpha+\lambda(1-\alpha)}(\Omega). \quad \square$$

Finally, in order to prove Proposition 3.1 we introduce the following notation:

$$\begin{aligned} q(\lambda) &:= 2\alpha + \lambda(1 - \alpha), \\ \Omega_0 &:= \{x \in \Omega : |v(u)| = 0\}, \\ \Omega'_0 &:= \{x \in \Omega : |v(u)| > 0\}, \\ \Omega_K &:= \{x \in \Omega : 0 < |v(u)| \leq K\}, \\ \Omega'_K &:= \{x \in \Omega : |v(u)| > K\}. \end{aligned}$$

**PROOF OF PROPOSITION 3.1.** Fix  $\phi \in W_0^{1,2}(\Omega; \mathbb{R}^d)$  and assume that  $Du \in L_{\text{loc}}^{2,\lambda}(\Omega; \mathbb{R}^{2d})$  for some  $0 \leq \lambda < 2$ . For  $\varepsilon \in \mathbb{R}$  set  $u_\varepsilon(x) := u(x) + \varepsilon\phi(x)$ . Define

$$P := D_1u, \quad Q := D_2u, \quad \alpha := D_1\phi, \quad \beta = D_2\phi, \quad v := v(u).$$

Since

$$v(u_\varepsilon) = v(u) + \varepsilon P \wedge \beta + \varepsilon \alpha \wedge Q + \varepsilon^2 \alpha \wedge \beta,$$

we have

$$\begin{aligned} \int_{\Omega} f(|v(u_\varepsilon)|) dx - \int_{\Omega} f(|v|) dx &= \varepsilon \int_{\Omega'_0} f'(|v|) \frac{v}{|v|} \cdot (P \wedge \beta + \alpha \wedge Q) dx \\ &\quad + |\varepsilon| \int_{\Omega_0} f'(0) |P \wedge \beta + \alpha \wedge Q| dx + o(\varepsilon). \end{aligned}$$

Local minimality of  $u$  entails

$$\limsup_{\varepsilon \rightarrow 0^-} \frac{\mathcal{F}_0(u_\varepsilon, \Omega) - \mathcal{F}_0(u, \Omega)}{\varepsilon} \leq 0,$$

and so

$$\int_{\Omega} Du \cdot D\phi \, dx + \int_{\Omega'_0} f'(|v|) \frac{v}{|v|} \cdot (P \wedge \beta + \alpha \wedge Q) \, dx \leq \int_{\Omega_0} f'(0) |P \wedge \beta + \alpha \wedge Q| \, dx.$$

We have

$$\begin{aligned} & (M+1) \int_{\Omega} Du \cdot D\phi \, dx + M \int_{\Omega'_0} \left[ \frac{v}{|v|} \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \right] \, dx \\ & + \int_{\Omega'_0 \cap \Omega_K} (f'(|v|) - M) \frac{v}{|v|} \cdot (P \wedge \beta + \alpha \wedge Q) \, dx \\ & \leq C \int_{\Omega_0} |Du| |D\phi| \, dx + \omega_K \int_{\Omega'_K} |Du| |D\phi| \, dx, \end{aligned}$$

where

$$\omega_K := \sup_{t \geq K} |M - f'(t)|.$$

We recall that by (H2)

$$\omega_K \rightarrow 0 \quad \text{as } K \rightarrow +\infty.$$

By Lemma 2.9 iii), iv), we deduce that

$$(3.1) \quad \begin{aligned} & (M+1) \int_{\Omega} Du \cdot D\phi \, dx + \int_{\Omega} G \cdot D\phi \, dx \\ & \leq C \int_{\Omega} \sqrt{|A| + |B|} |D\phi| \, dx + \omega_K \int_{\Omega} |Du| |D\phi| \, dx \end{aligned}$$

with  $G = (G_1, G_2)$  and

$$\begin{aligned} G_1 & := \chi_{\Omega'_0 \cap \Omega_K} (M - f'(|v|)) \frac{v}{|v|} \wedge Q \\ G_2 & := \chi_{\Omega'_0 \cap \Omega_K} (f'(|v|) - M) \frac{v}{|v|} \wedge P, \end{aligned}$$

and where  $\chi_A$  stands for the characteristic function of the set  $A$ . By Lemma 2.9 ii), iii), and recalling that on  $\Omega_K$  we have  $|v| \leq K$ , we have

$$|G| \leq C(K)(1 + \sqrt{|A| + |B|}), \quad \text{a.e. in } \Omega,$$

and by Proposition 3.2 we deduce that  $G \in L^{2,q(\lambda)}(\Omega; \mathbb{R}^d)$ . Next, for a fixed ball  $B_R \subset\subset \Omega$  we compare  $u$  with the solution of the Dirichlet problem

$$(3.2) \quad \begin{cases} (M+1)\Delta v = \operatorname{div} G & \text{in } B_R \\ v - u \in W_0^{1,2}(B_R; \mathbb{R}). \end{cases}$$

By Lemma 2.6  $Dv \in L_{\text{loc}}^{2,q(\lambda)}(B_R; \mathbb{R}^2)$  and for all  $0 < \rho \leq R$

$$(3.3) \quad \int_{B_\rho} |Dv|^2 dx \leq C \left(\frac{\rho}{R}\right)^{q(\lambda)} \int_{B_R} |Dv|^2 dx + C(K)\rho^{q(\lambda)}.$$

From (3.1) and (3.2) we have for all  $\phi \in W_0^{1,2}(B_R; \mathbb{R}^d)$

$$(M+1) \int_{B_R} (Du - Dv) \cdot D\phi dx \leq C \int_{\Omega \cap B_R} \sqrt{|A|+|B|} |D\phi| dx + \omega_K \int_{B_R} |Du| |D\phi| dx.$$

Therefore, taking  $\phi := u - v$ , and using the fact that by the definition of  $G$  and by (3.2)

$$|G| \leq C|Du|, \quad \int_{B_R} |Dv|^2 \leq C \int_{B_R} |Du|^2,$$

we have

$$\int_{B_R} |Du - Dv|^2 dx \leq C \int_{B_R} (|A| + |B|) dx + C\omega_K \int_{B_R} |Du|^2 dx.$$

Using (3.3) we now obtain

$$\int_{B_\rho} |Du|^2 dx \leq C \left[ \left(\frac{\rho}{R}\right)^{q(\lambda)} + \omega_K \right] \int_{B_R} |Du|^2 dx + C(K)R^{q(\lambda)},$$

and if  $K$  is large enough, so that  $\omega_K$  is small, from Lemma 2.7 we conclude that for all  $0 < \lambda' < q(\lambda)$

$$(3.4) \quad \int_{B_\rho} |Du|^2 dx \leq C \left(\frac{\rho}{R}\right)^{\lambda'} \int_{B_R} |Du|^2 dx + C\rho^{\lambda'},$$

and thus (3.4) holds true for  $\lambda' = q_0(\lambda)$ .  $\square$

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