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1. Introduction and statements of main results

Let $X$ be a smooth vector field on a smooth manifold $M$. In order to fix some notation, let us recall that the flow associated to $X$ is a pair $(V, \varphi)$, where $V \subset \mathbb{R} \times M$ is open and $\varphi : V \to M$, $(t, m) \mapsto \varphi_t(m)$, is a smooth map such that

(i) for any $m \in M$, $(0, m) \in V$, and $\varphi_0(m) = m$;
(ii) $\varphi_t(\varphi_s(m)) = \varphi_{t+s}(m)$ whenever $(s, m)$, $(t + s, m)$, $(t, \varphi_s(m)) \in V$;
(iii) for every $m \in M$

$$\frac{d}{dt}\varphi_t(m)\bigg|_{t=0} = X(m);$$

(iv) the pair $(V, \varphi)$ is maximal, i.e. if $(V', \varphi')$ is another pair satisfying (i), (ii) and (iii) then $V' \subset V$ and $\varphi' = \varphi_V$.

It is a standard result that any smooth vector field has a unique associated flow (and every flow comes from a vector field).

The vector field $X$ with associated flow $\varphi_t(\cdot)$ is complete (respectively right complete) if for every $m \in M$ $\varphi_t(m)$ is defined for every $t \in \mathbb{R}$ (respectively $t > 0$).

In the paper [W] Wu introduced the notion of taut complex manifold. Let us recall briefly such a definition. Let $M$ be a complex manifold and let

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

be the open unit disk. A sequence $f_n : \Delta \to M$ is said to be compactly divergent if for any choice of a compact set $H \subset \Delta$ and $K \subset M$, $f_n(H) \cap K = \emptyset$ for $n$ large enough. The complex manifold $M$ is taut if given any sequence $f_n : \Delta \to M$ of holomorphic maps admits a subsequence converging to a holomorphic map $f : \Delta \to M$ (uniformly on the compact subset of $\Delta$) or a subsequence compactly divergent.

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The purpose of this paper is to describe the behaviour as \( t \to +\infty \) of the flow \( \varphi_t(\cdot) \) associated to a right complete holomorphic vector field \( X \) on a taut complex manifold \( M \) and to investigate the link between the structure of the zeroes of the vector field \( X \) (i.e. the fixed point set for the flow \( \varphi_t(\cdot) \)) and the topology of the manifold \( M \).

If the manifold \( M \) is compact then the theory is quite trivial, since such a manifold does not admit any non trivial holomorphic vector field. So, throughout the paper \( M \) will be a non compact connected taut complex manifold, \( X \) a holomorphic vector field on \( M \) with associated flow \( \varphi_t(\cdot) \).

In order to explain our main results we need to introduce some notations. We denote by \( S^1 \) the circle group (the topological boundary of \( \Delta \)), by \( T^r = S^1 \times \cdots \times S^1 \) the standard \( r \)-dimensional torus Lie group and by \( S^n \) the unit sphere in \( \mathbb{R}^{n+1} \).

We denote by \( \text{Hol}(M, N) \) the space of all holomorphic maps between two complex manifold \( M \) and \( N \) endowed with the compact open topology, and by \( \text{Aut}(M) \) the group of all holomorphic automorphisms of \( M \).

Let \( X \) be right complete vector field on a complex manifold \( M \) with associated flow \( \varphi_t(\cdot) \). We set

\[
Z(X) = \{ m \in M \mid X(m) \} = 0.
\]

The flow \( \varphi_t(\cdot) \) on \( M \) is said
(i) **compact** if the family \( \{ \varphi_t(\cdot) \}_{t \geq 0} \) is relatively compact in \( \text{Hol}(M, M) \),
(ii) **compactly divergent** if for each pair of compact subsets \( H, K \subset M \) for some \( t_0 \) one has \( \varphi_t(H) \cap K = \emptyset \) if \( t \geq t_0 \).

For every \( m \in M \), \( s \geq 0 \), set

\[
\Gamma_m(s) = \{ \varphi_t(m) \mid t \geq s \},
\]

\[
\Gamma_m = \bigcap_{s \geq 0} \Gamma_m(s);
\]

we also set

\[
E(X) = \{ m \in M \mid m \in \Gamma_m \},
\]

\[
T(X) = M \setminus E(X).
\]

(“\( E \)” and “\( T \)” stand respectively for “ergodic” and “transient”).

Let us recall that the Kuratowski limits of a family of subset \( A_t \subset M \), \( t \geq 0 \) are defined as

\[
K-\liminf_{t \to +\infty} A_t =
\]

\[
\{ m \in M \mid \forall U \text{ neighbourhood of } m \ \exists t_0 \geq 0 \text{ s.t. } \forall t \geq t_0 \ A_t \cap U \neq \emptyset \},
\]

\[
K-\limsup_{t \to +\infty} A_t =
\]

\[
\{ m \in M \mid \forall U \text{ neighbourhood of } m \ \forall t \geq 0 \ \exists t_0 \geq t \text{ s.t. } A_{t_0} \cap U \neq \emptyset \},
\]
If $u : [0, +\infty] \to M$ is a map then we will write $K - \liminf_{t \to +\infty} u(t)$ (respectively $K - \limsup_{t \to +\infty} u(t)$) instead of $K - \liminf_{t \to +\infty} \{u(t)\}$ (respectively $K - \limsup_{t \to +\infty} \{u(t)\}$).

Finally, as usual, $\pi_p(M)$, $H_p(M, G)$ and $H^p(M, G)$ will stand respectively for the $p$-th homotopy group (with respect to some base point in $M$), the $p$-th homology and cohomology groups of $M$ with coefficients in the abelian group $G$. The manifold $M$ is of finite topological type if for every $p \geq 0$

$$\dim \mathbb{Q} H^p(X, \mathbb{Q}) < +\infty$$

and for such a manifold the Euler characteristic is

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim \mathbb{Q} H^i(X, \mathbb{Q}).$$

Our main results on flows on taut manifolds are described by the followings theorems:

**Theorem 1.1.** Let $X$ be a right complete vector field on a (non compact) connected taut complex manifold $M$ with associated flow $\varphi_t(\cdot)$. Then $\varphi_t(\cdot)$ is either compact or compactly divergent. If it is compact then the following assertions hold:

(i) $E(X)$ is a not empty closed integral submanifold of $M$;
(ii) the restriction of $X$ to $E(X)$ is a complete vector field on $E(X)$;
(iii) if $L$ is an integral submanifold of $X$ and the restriction of $X$ to $L$ is a complete vector field on $L$ then $L \subset E(X)$;
(iv) the inclusion map $i : E(X) \to M$ is a homotopical equivalence between $E(X)$ and $M$;
(v) there exist a smooth toral action on $E(X)$

$$T^r \times E(X) \to E(X)$$

such that for all $m \in E(X)$ and any $s \geq 0$

$$\Gamma_m = \overline{\Gamma_m(s)} = T^r m.$$

In particular, $X(m) = 0$ (i.e. $\varphi_t(m) = m$) if, and only if, $m$ is a fixed point for such a toral action;

(vi) there exist a holomorphic retraction $\rho : M \to E(X)$ such that for all $m \in T(X)$

$$K - \limsup_{t \to +\infty} \varphi_t(m) = \Gamma_{\rho(m)}.$$

In particular $\varphi_t(m)$ converges to $\overline{m} \in M$ as $t \to +\infty$ if, and only if, $\overline{m} = \rho(m)$ and $X(\overline{m}) = 0$. 
THEOREM 1.2. Let $X$, $M$ and $\varphi_t(\cdot)$ be as in Theorem 1.1 and assume further that $M$ is of finite topological type and the flow $\varphi_t(\cdot)$ is compact. Then $Z(X)$ is a closed complex submanifold of finite topological type and
\[ \chi(Z(X)) = \chi(M). \]
In particular, if $\chi(M) \neq 0$ then $Z(X) \neq \emptyset$.

We recall that a topological space $S$ is acyclic (over $\mathbb{Q}$) if $\dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) = 0$ for every $i > 0$.

THEOREM 1.3. Let $X$, $M$ and $\varphi_t(\cdot)$ be as in Theorem 1.2 and assume further $M$ is a connected acyclic manifold. Then $Z(X)$ also is a connected acyclic connected (closed) submanifold of $M$.

THEOREM 1.4. Let $X$, $M$ and $\varphi_t(\cdot)$ be as in Theorem 1.2. If the even rational homotopy groups of $M$ vanish then $Z(X)$ is either empty or a connected complex submanifold of $M$.

I thank Prof. Angelo Vistoli for some helpful conversation on the subject.

2. – Proofs

Let $X$, $M$ and $\varphi_t(\cdot)$ be as in Theorem 1.1. Since the law of composition is continuous in $\text{Hol}(M, M)$ then the family
\[ \mathcal{F} = \{ \varphi_t(\cdot) \mid t \in [0, 1] \}. \]
is compact (with respect to the compact open topology on $\text{Hol}(M, M)$). Set also $f = \varphi_1 : M \to M$; then the sequence $f^n = f \circ \cdots \circ f = \varphi_n : M \to M$ of the iterates of $f$ is either relatively compact in $\text{Hol}(M, M)$ or compactly divergent ([A1]).

Assume the sequence $f^n$ relatively compact. Since every $\varphi_t(\cdot)$ is of the form $f^n \circ \varphi_s$ with $\varphi_s \in \mathcal{F}$, then $\{\varphi_t\}$ is a relatively compact family in $\text{Hol}(M, M)$. Assume now that the sequence $f^n$ is compactly divergent. Let $H$ and $K$ be any two compact subset of $M$. Then, being $\mathcal{F}$ a compact family,
\[ H' = \bigcup_{s \in [0, 1]} \varphi_s(H) \]
is a compact subset of $M$. Since, by assumption, the sequence $f^n$ is compactly divergent then there exists $n_0$ such that $f^n(H') \cap K = \emptyset$ for all $n \geq n_0$. Then, for every $t \geq n_0$, writing $t = n + s$ with $n \in \mathbb{N}$ and $s \in [0, 1]$, \[ \varphi_t(H) \cap K = f^n(\varphi_s(H)) \cap K \subset f^n(H') \cap K = \emptyset, \]
which shows that the flow \( \varphi_t(\cdot) \) is compactly divergent. The first statement of Theorem 1.1 is thus proved.

Assume now that the flow \( \varphi_t(\cdot) \) is compact. Let \( S \) be the closure of the family \( \{ \varphi_t(\cdot) \}_{t \geq 0} \) in \( \text{Hol}(M, M) \). Then \( S \) is a compact topological abelian semigroup with respect to the law of composition of (holomorphic) maps. It is a standard fact of the theory of compact (semi-)topological semi-groups that \( S \) contains an idempotent element \( \rho \) (i.e. \( \rho \) satisfies \( \rho^2 = \rho \)) such that the semigroup \( G = \rho S \) is a compact topological abelian group with identity \( \rho \) ([Ru]).

The map \( \rho : M \to M \) is therefore a retraction of \( M \) onto its image \( N = \rho(M) \). It follows from a result of Rossi ([R]) that \( N \) is a closed complex submanifold of \( M \).

Let \( m \in N \). Then, since \( S \) is abelian,

\[
\varphi_t(m) = \varphi_t(\rho(m)) = \rho(\varphi_t(m)) \quad \Rightarrow \quad \varphi_t(m) \in N,
\]

that is, \( N \) is invariant under each \( \varphi_t \), or, equivalently, \( N \) is an integral submanifold of \( X \).

Moreover, since \( G \) is a group, it easily follows that the restriction of each \( \varphi_t \) to \( N \) is an automorphism of \( N \), and setting

\[
\psi_t(m) = \begin{cases} 
  \varphi_t(m) & \text{if } t \geq 0 \\
  \varphi^{-1}_t(m) & \text{if } t < 0,
\end{cases}
\]

then \( \psi_t(\cdot) \) is a smooth one parameter group of automorphisms of \( N \) such that

\[
\left. \frac{d}{dt} \psi_t(m) \right|_{t=0} = X(m).
\]

It follows that the restriction of \( X \) to \( N \) is a complete vector field on \( N \).

Before going further we need to recall some basic fact on the “Kobayashi distance”. Such a pseudodistance

\[
k_M : M \times M \to [0, +\infty[.
\]

is defined for every (connected) complex manifold \( M \), and can be characterized as the biggest one among all the pseudodistances \( \delta : M \times M \to [0, +\infty[ \) such that for all \( f \in \text{Hol}(\Delta, M) \) and all \( z, w \in \Delta \)

\[
\delta(f(z), f(w)) \leq \omega(z, w),
\]

where \( \omega(z, w) \) is the Poincaré distance on \( \Delta \), i.e. the integrated form of the Poincaré metric

\[
\frac{dz \overline{dz}}{(1 - \overline{z}z)^2}.
\]

If it happens that \( k_M \) is a distance, that is \( k_M(m, m') > 0 \) if \( m \neq m' \) then the complex manifold \( M \) is said \textit{hyperbolic}. 
We summarize here the results on hyperbolic manifold that we need in the sequel. For references see e.g. [K], [A2].

(i) if \( f \in \text{Hol}(M, N) \) then, for each pair of points \( m \) and \( m' \) in \( M \),

\[
k_N(f(m), f(m')) \leq k_M(m, m');
\]

(ii) if \( N \) is a complex submanifold of \( M \) and \( M \) is hyperbolic then \( N \) is hyperbolic;

(iii) the group \( \text{Aut}(M) \) of an hyperbolic complex manifold \( M \) is a Lie group acting smoothly on \( M \);

(iv) every taut manifold is hyperbolic.

Coming back to the proof of Theorem 1.1, set

\[
K = \{ u_{|N} \mid u \in S \}
\]

Then \( K \), being a compact connected abelian subgroup of \( \text{Aut}(N) \), which by the assertions above is a Lie group, it is a Lie group. But a compact connected abelian Lie group is isomorphic (as Lie group) to a standard real torus \( T^r \) for some \( r \geq 0 \). Thus we have a toral action

\[
T^r \times E(X) \to E(X).
\]

Since every \( u \in K \) is limit of a sequence of maps in the flow \( \varphi_t \), it easily follows that for every \( m \in N \) and \( s \geq 0 \)

\[
\overline{\Gamma_m(s)} = \Gamma_m = T^r m,
\]

which clearly implies that \( N \subset E(X) \).

Now let \( m \in M \setminus N \). Choose a sequence \( f_n = \varphi_{t_n} \) converging in \( \text{Hol}(M, M) \) to \( \rho \). Given \( \varepsilon > 0 \), pick \( n \) large enough in such a way that \( k_M(f_n(m), \rho(m)) < \varepsilon \). Then, for every \( t \geq t_n \),

\[
k_M(\varphi_t(m), \varphi_{t-t_n}(\rho(m))) = k_M(\varphi_{t-t_n}(f_n(m)), \varphi_{t-t_n}(\rho(m))) \leq k_M(f_n(m), \rho(m)) < \varepsilon,
\]

whence

\[
k_M(\varphi_t(m), T^r \rho(m)) \leq k_M(\varphi_{t-t_n}(\rho(m))) < \varepsilon.
\]

(Where, by definition, for a subset \( A \subset M \) we set \( k_M(m, A) = \inf_{m'} \{ k_M(m, m') \mid m' \in A \} \).)

The estimates above, together with the fact that \( \{ \varphi_t(\rho(m)) \}_{t \geq 0} \) is dense in \( T^r \rho(m) \), implie that for every \( s \geq 0 \)

\[
\overline{\Gamma_{\rho(m)}(s)} = \Gamma_m = T^r \rho(m) \subset N,
\]

and also \( m \in T(X) \), being \( m \in M \setminus N \).
Since \( m \in M \setminus N \) is arbitrary, it follows then that \( M \setminus N \subset T(X) \), which combined with the previously proved inclusion \( N \subset E(X) \) yields the equalities \( N = E(X) \) and \( M \setminus N = T(X) \).

All that proves statements (i), (ii), (v) and (vi) of Theorem 1.1.

Let us prove statement (iii).

Let \( L \subset M \) be an integral submanifold of the vector field \( X \), and assume that the restriction of \( X \) to \( L \) is complete. Let \( \psi_t(.) \) be the flow on \( L \) associated to \( X \). It is an one parameter group of automorphisms of \( L \), which is a hyperbolic complex manifold, relatively compact in \( \text{Aut}(L) \). Its closure \( G \) in \( \text{Aut}(L) \) is therefore a compact connected abelian Lie group, whence it is isomorphic to a \( r \)-dimensional torus \( T^r \). As before, given \( m \in L \), we obtain \( \Gamma_m = T^m \), whence \( m \in \Gamma_m \), that is \( m \in E(X) \). Since \( m \in L \) is arbitrary the inclusion \( L \subset E(X) \) follows.

It remains to prove statement (iv) of Theorem 1.1.

Because the spaces \( M \) and \( E(X) = \rho(M) \) are connected (complex) manifold, they have some \( CW \)-complex structure, and hence it suffices to prove that the induced group homomorphisms

\[
i_\ast : \pi_k(E(X)) \to \pi_k(M)
\]

are isomorphisms in each dimension \( k \).

The identity \( \rho \circ i = \text{id}_N \) yields \( \rho_* \circ i_* = \text{id}_{N_*} \), and hence \( i_* \) is a monomorphism. We will prove that \( i_* \) is an epimorphism showing that \( \rho_* \) is a monomorphism.

Let \([u] \in \pi_k(E(X))\), represented by the continuous map \( u : S^k \to E(X) \), be in the kernel of \( \rho_* \). Let \( f^n = \varphi_n \) (here the “\( n \)” in “\( f^n \)” is an index written as superscript for “typographical” reason) be a sequence converging to \( \rho \) in \( \text{Hol}(M, M) \). Then, since \( M \) is a manifold, \( f^n([u]) = [u \circ f^n] = [u \circ \rho] = \rho_*([u]) = 0 \) for \( n \) large enough. Obviously \( f^n = \varphi_n \) is homotopic to \( \varphi_0 \), which is the identity map on \( M \), and hence \( [u] = [u \circ \varphi_0] = [u \circ f^n] = \rho_*([u]) = 0 \).

The proof of Theorem 1.1 is complete.

Let now \( X, M, \varphi_t(.) \) and \( F \). By Theorem 1.1, after replacing \( M \) with a closed submanifold homotopically equivalent to \( M \) if necessary, we may assume that \( X \) is a complete vector field, that is \( E(X) = M \). Again by Theorem 1.1 there exists a toral action

\[
T^r \times M \to M
\]

such that \( \Gamma_m = T^m \) for each \( m \in M \). Obviously the set \( F \) is then the fixed point set of such an action. Theorem 1.2 then follows immediatly from the following

**Theorem 2.1.** Let \( T^r \times V \to V \) be a smooth action on the (not necessarily connected) smooth manifold \( V \), anf let \( F \) be the fixed point set for such an action. If \( V \) is a manifold of finite topological type then also \( F \) it is and

\[
\chi(F) = \chi(V).
\]
PROOF. Let us write $T^{r} = S^{1} \times T^{r-1}$, and let $F_{1}$ be the fixed point set for $S^{1}$. Since a smooth action of a compact Lie group is linearizable in a neighbourhood of each fixed point, it follows that $F_{1}$ is a smooth (obviously closed) submanifold of $V$. Moreover $T^{r-1}$ acts on $F_{1}$. After $r$ steps we so obtain a chain

$$F = F_{r} \subset \cdots \subset F_{1} \subset V$$

of closed submanifold with $F_{i+1}$ the fixed point set of a circle action on $F_{i}$.

Then, clearly it suffices to prove the theorem for $r = 1$, that is for a circle action, which is done, e.g., in [Br] Theorem 10.9 (and remark below). \qed

Theorems 1.3 and 1.4 follow immediately from the corresponding statements on toral actions (see, e.g., respectively [B, Theorem 5.3 (a)] and [H, Theorems (IV.5)]).

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