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Continuous dynamical systems on Taut complex manifolds

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1. – Introduction and statements of main results

Let X be a smooth vector field on a smooth manifold M. In order to fix some notation, let us recall that the flow associated to X is a pair (V, φ), where $V \subset \mathbb{R} \times M$ is open and $\varphi : V \to M$, $(t, m) \mapsto \varphi_t(m)$, is a smooth map such that

(i) for any $m \in M$, $(0, m) \in V$, and $\varphi_0(m) = m$;
(ii) $\varphi_t(\varphi_s(m)) = \varphi_{t+s}(m)$ whenever (s, m), $(t + s, m)$, $(t, \varphi_s(m)) \in V$;
(iii) for every $m \in M$

$$\frac{d}{dt}\varphi_t(m)\bigg|_{t=0} = X(m);$$

(iv) the pair (V, φ) is maximal, i.e. if (V', φ') is an other pair satisfying (i), (ii) and (ii) then $V' \subset V$ and $\varphi' = \varphi_{|V}$.

It is a standard result that any smooth vector field has a unique associated flow (and every flow comes from a vector field).

The vector field X with associated flow $\varphi_t(\cdot)$ is *complete* (respectively *right complete*) if for every $m \in M$ $\varphi_t(m)$ is defined for every $t \in \mathbb{R}$ (respectively $t > 0$).

In the paper [W] Wu introduced the notion of *taut* complex manifold. Let us recall briefly such a definition. Let M be a complex manifold and let

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

be the open unit disk. A sequence $f_n : \Delta \to M$ is said to be *compactly divergent* if for any choice of a compact set $H \subset \Delta$ and $K \subset M$, $f_n(H) \cap K = \emptyset$ for n large enough. The complex manifold M is *taut* if given any sequence $f_n : \Delta \to M$ of holomorphic maps admits a subsequence converging to a holomorphic map $f : \Delta \to M$ (uniformely on the compact subset of Δ) or a subsequence compactly divergent.

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The purpose of this paper is to describe the behaviour as $t \to +\infty$ of the flow $\varphi_t(\cdot)$ associated to a right complete holomorphic vector field X on a taut complex manifold M and to investigate the link between the structure of the zeroes of the vector field X (i.e. the fixed point set for the flow $\varphi_t(\cdot)$) and the topology of the manifold M.

If the manifold M is compact then the theory is quite trivial, since such a manifold does not admit any non trivial holomorphic vector field. So, throughout the paper M will be a non compact connected taut complex manifold, X a holomorphic vector field on M with associated flow $\varphi_t(\cdot)$.

In order to explain our main results we need to introduce some notations.

We denote by S^1 the circle group (the topological boundary of Δ), by $T^r = S^1 \times \cdots \times S^1$ the standard r-dimensional torus Lie group and by S^n the unit sphere in $\mathbb{R}^{n+1}$.

We denote by $\mathrm{Hol}(M, N)$ the space of all holomorphic maps between two complex manifold M and N endowed with the compact open topology, and by $\mathrm{Aut}(M)$ the group of all holomorphic automorphisms of M.

Let X be right complete vector field on a complex manifold M with associated flow $\varphi_t(\cdot)$. We set

$$Z(X) = \{m \in M \mid X(m)\} = 0 .$$

The flow $\varphi_t(\cdot)$ on M is said

(i) *compact* if the family $\{\varphi_t(\cdot)\}_{t \geq 0}$ is relatively compact in $\mathrm{Hol}(M, M)$,

(ii) *compactly divergent* if for each pair of compact subsets H, $K \subset M$ for some t_0 one has $\varphi_t(H) \cap K = \emptyset$ if $t \geq t_0$.

For every $m \in M$, $s \geq 0$, set

$$\Gamma_m(s) = \{\varphi_t(m) \mid t \geq s\} ,$$
$$\Gamma_m = \bigcap_{s \geq 0} \overline{\Gamma_m(s)} ;$$

we also set

$$E(X) = \{m \in M \mid m \in \Gamma_m\} ,$$
$$T(X) = M \setminus E(X) .$$

("E" and "T" stand respectively for "ergodic" and "transient").

Let us recall that the Kuratowsky limits of a family of subset $A_t \subset M$, $t \geq 0$ are defined as

$$K - \liminf_{t \to +\infty} A_t =$$
$$\{m \in M \mid \forall U \text{ neighbourhood of } m \; \exists t_0 \geq 0 \text{ s.t. } \forall t \geq t_0 \; A_t \cap U \neq \emptyset\} ,$$

$$K - \limsup_{t \to +\infty} A_t =$$
$$\{m \in M \mid \forall U \text{ neighbourhood of } m \; \forall t \geq 0 \; \exists t_0 \geq t \text{ s.t. } A_{t_0} \cap U \neq \emptyset\} ,$$

If $u : [0, +\infty[\to M$ is a map then we will write $K - \liminf_{t\to+\infty} u(t)$ (respectively $K - \limsup_{t\to+\infty} u(t)$) instead of $K - \liminf_{t\to+\infty}\{u(t)\}$ (respectively $K - \limsup_{t\to+\infty}\{u(t)\}$).

Finally, as usual, $\pi_p(M)$, $H_p(M, G)$ and $H^p(M, G)$ will stand respectively for the p-th homotopy group (with respect to some base point in M), the p-th homology and cohomology groups of M with coefficients in the abelian group G. The manifold M is of *finite topological type* if for every $p \geq 0$

$$\dim_{\mathbb{Q}} H^p(X, \mathbb{Q}) < +\infty$$

and for such a manifold the *Euler characteristic* is

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) .$$

Our main results on flows on taut manifolds are described by the followings theorems:

THEOREM 1.1. *Let X be a right complete vector field on a (non compact) connected taut complex manifold M with associated flow $\varphi_t(\cdot)$. Then $\varphi_t(\cdot)$ is either compact or compactly divergent. If it is compact then the following assertions hold:*

(i) *$E(X)$ is a not empty closed integral submanifold of M;*

(ii) *the restriction of X to $E(X)$ is a complete vector field on $E(X)$;*

(iii) *if L is an integral submanifold of X and the restriction of X to L is a complete vector field on L then $L \subset E(X)$;*

(iv) *the inclusion map $i : E(X) \to M$ is a homotopical equivalence between $E(X)$ and M;*

(v) *there exist a smooth toral action on $E(X)$*

$$T^r \times E(X) \to E(X)$$

such that for all $m \in E(X)$ and any $s \geq 0$

$$\Gamma_m = \overline{\Gamma_m(s)} = T^r m .$$

In particular, $X(m) = 0$ (i.e. $\varphi_t(m) = m$) if, and only if, m is a fixed point for such a toral action;

(vi) *there exist a holomorphic retraction $\rho : M \to E(X)$ such that for all $m \in T(X)$*

$$K - \limsup_{t\to+\infty} \varphi_t(m) = \Gamma_{\rho(m)} .$$

In particular $\varphi_t(m)$ converges to $\overline{m} \in M$ as $t \to +\infty$ if, and only if, $\overline{m} = \rho(m)$ and $X(\overline{m}) = 0$.

THEOREM 1.2. *Let X, M and $\varphi_t(\cdot)$ be as in Theorem 1.1 and assume further that M is of finite topological type and the flow $\varphi_t(\cdot)$ is compact. Then $Z(X)$ is a closed complex submanifold of finite topological type and*

$$\chi(Z(X)) = \chi(M) .$$

In particular, if $\chi(M) \neq 0$ then $Z(X) \neq \emptyset$.

We recall that a topological space S is *acyclic* (over $\mathbb{Q}$) if $\dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) = 0$ for every $i > 0$.

THEOREM 1.3. *Let X, M and $\varphi_t(\cdot)$ be as in Theorem 1.2 and assume further M is a connected acyclic manifold. Then $Z(X)$ also is a connected acyclic connected (closed) submanifold of M.*

THEOREM 1.4. *Let X, M and $\varphi_t(\cdot)$ be as in Theorem 1.2. If the even rational homotopy groups of M vanish then $Z(X)$ is either empty or a connected complex submanifold of M.*

I thank Prof. Angelo Vistoli for some helpful conversation on the subject.

2. – Proofs

Let X, M and $\varphi_t(\cdot)$ be as in Theorem 1.1. Since the law of composition is continuous in $\mathrm{Hol}(M, M)$ then the family

$$\mathcal{F} = \{\varphi_t(\cdot) \mid t \in [0, 1]\} .$$

is compact (with respect to the compact open topology on $\mathrm{Hol}(M, M)$). Set also $f = \varphi_1 : M \to M$; then the sequence $f^n = f \circ \cdots \circ f = \varphi_n : M \to M$ of the iterates of f is either relatively compact in $\mathrm{Hol}(M, M)$ or compactly divergent ([A1]).

Assume the sequence f^n relatively compact. Since every $\varphi_t(\cdot)$ is of the form $f^n \circ \varphi_s$ with $\varphi_s \in \mathcal{F}$, then $\{\varphi_t\}$ is a relatively compact family in $\mathrm{Hol}(M, M)$. Assume now that the sequence f^n is compactly divergent. Let H and K be any two compact subset of M. Then, being $\mathcal{F}$ a compact family,

$$H' = \bigcup_{s \in [0,1]} \varphi_s(H)$$

is a compact subset of M. Since, by assumption, the sequence f^n is compactly divergent then there exists n_0 such that $f^n(H') \cap K = \emptyset$ for all $n \geq n_0$. Then, for every $t \geq n_0$, writing $t = n + s$ with $n \in \mathbb{N}$ and $s \in [0, 1[$,

$$\varphi_t(H) \cap K = f^n\big(\varphi_s(H)\big) \cap K \subset f^n(H') \cap K = \emptyset ,$$

which shows that the flow $\varphi_t(\cdot)$ is compactly divergent. The first statement of Theorem 1.1 is thus proved.

Assume now that the flow $\varphi_t(\cdot)$ is compact. Let S be the closure of the familty $\{\varphi_t(\cdot)\}_{t\geq 0}$ in $\mathrm{Hol}(M, M)$. Then S is a compact topological abelian semigroup with respect to the law of composition of (holomorphic) maps. It is a standard fact of the theory of compact (semi-)topological semi-groups that S contain an idempotent element ρ (i.e. ρ satisfies $\rho^2 = \rho$) such that the semigroup $G = \rho S$ is a compact topological abelian group with identity ρ ([Ru]).

The map $\rho : M \to M$ is therefore a retraction of M onto its image $N = \rho(M)$. It follows from a result of Rossi ([R]) that N is a closed complex submanifold of M.

Let $m \in N$. Then, since S is abelian,

$$\varphi_t(m) = \varphi_t\big(\rho(m)\big) = \rho\big(\varphi_t(m)\big) \quad \Rightarrow \quad \varphi_t(m) \in N\,,$$

that is, N is invariant under each φ_t, or, equivalently, N is an integral submanifold of X.

Moreover, since G is a group, it easily follows that the restriction of each φ_t to N is an automorphism of N, and setting

$$\psi_t(m) = \begin{cases} \varphi_t(m) & \text{if } \ t \geq 0 \\ \varphi_{-t}^{-1}(m) & \text{if } \ t < 0\,, \end{cases}$$

then $\psi_t(\cdot)$ is a smooth one parameter group of automorphisms of N such that

$$\frac{d}{dt}\psi_t(m)\bigg|_{t=0} = X(m)\,.$$

It follows that the restriction of X to N is a complete vector field on N.

Before going further we need to recall some basic fact on the "Kobayashi distance". Such a pseudodistance

$$k_M : M \times M \to [0, +\infty[$$

is defined for every (connected) complex manifold M, and can be characterized as the biggest one among all the pseudodistances $\delta : M \times M \to [0, +\infty[$ such that for all $f \in \mathrm{Hol}(\Delta, M)$ and all $z,\ w \in \Delta$

$$\delta\big(f(z), f(w)\big) \leq \omega(z, w)\,,$$

where $\omega(z, w)$ is the Poincaré distance on Δ, i.e. the integrated form of the Poincaré metric

$$\frac{dz\,\overline{dz}}{(1 - z\overline{z})^2}\,.$$

If it happens that k_M is a distance, that is $k_M(m, m') > 0$ if $m \neq m'$ then the complex manifold M is said *hyperbolic*.

We summarize here the results on hyperbolic manifold that we need in the sequel. For references see e.g. [K], [A2].

(i) if $f \in \mathrm{Hol}(M, N)$ then, for each pair of points m and $m' \in M$,

$$k_N\big(f(m),\ f(m')\big) \leq k_M(m, m')\,;$$

(ii) if N is a complex submanifold of M and M is hyperbolic then N is hyperbolic;

(iii) the group $\mathrm{Aut}(M)$ of an hyperbolic complex manifold M is a Lie group acting smoothly on M;

(iv) every taut manifold is hyperbolic.

Caming back to the proof of Theorem 1.1, set

$$K = \{u_{|N} \mid u \in S\}$$

Then K, being a compact connected abelian subgroup of $\mathrm{Aut}(N)$, which by the assertions above is a Lie group, it is a Lie group. But a compact connected abelian Lie group is isomorphic (as Lie group) to a standard real torus T^r for some $r \geq 0$. Thus we have a toral action

$$T^r \times E(X) \to E(X)\,.$$

Since every $u \in K$ is limit of a sequence of maps in the flow φ_t, it easily follows that for every $m \in N$ and $s \geq 0$

$$\overline{\Gamma_m(s)} = \Gamma_m = T^r m\,,$$

which clearly implies that $N \subset E(X)$.

Now let $m \in M \backslash N$. Choose a sequence $f_n = \varphi_{t_n}$ converging in $\mathrm{Hol}(M, M)$ to ρ. Given $\varepsilon > 0$, pick n large enough in such a way that $k_M\big(f_n(m), \rho(m)\big) < \varepsilon$. Then, for every $t \geq t_n$,

$$\begin{aligned} &k_M\big(\varphi_t(m), \varphi_{t-t_n}(\rho(m))\big) \\ &\quad = k_M\big(\varphi_{t-t_n}(f_n(m)), \varphi_{t-t_n}(\rho(m))\big) \leq k_M\big(f_n(m), \rho(m)\big) < \varepsilon\,, \end{aligned}$$

whence

$$k_M\big(\varphi_t(m), T^r\rho(m)\big) \leq k_M\big(\varphi_t(m), \varphi_{t-t_n}(\rho(m))\big) < \varepsilon\,.$$

(Where, by definition, for a subset $A \subset M$ we set $k_M(m, A) = \inf_{m'}\{k_M(m, m') \mid m' \in A\}$).

The estimates above, together with the fact that $\{\varphi_t(\rho(m))\}_{t \geq 0}$ is dense in $T^r\rho(m)$, implie that for every $s \geq 0$

$$\overline{\Gamma_{\rho(m)}(s)} = \Gamma_m = T^r\rho(m) \subset N\,,$$

and also $m \in T(X)$, being $m \in M \setminus N$.

Since $m \in M \setminus N$ is arbitrary, it follows then that $M \setminus N \subset T(X)$, which combined with the previously proved inclusion $N \subset E(X)$ yields the equalities $N = E(X)$ and $M \setminus N = T(X)$.

All that proves statements (i), (ii), (v) and (vi) of Theorem 1.1.

Let us prove statement (iii).

Let $L \subset M$ be an integral submanifold of the vector field X, and assume that the restriction of X to L is complete. Let $\psi_t(\cdot)$be the flow on L associated to X. It is an one parameter group of automorphisms of L, which is a hyperbolic complex manifold, relatively compact in Aut(L). Its closure G in Aut(L) is therefore a compact connected abelian Lie group, whence it is isomorphic to a r-dimensional torus T^r. As before, given $m \in L$, we obtain $\Gamma_m = T^r m$, whence $m \in \Gamma_m$, that is $m \in E(X)$. Since $m \in L$ is arbitrary the inclusion $L \subset E(X)$ follows.

It remains to prove statement (iv) of Theorem 1.1.

Because the spaces M and $E(X) = \rho(M)$ are connected (complex) manifold, they have some CW-complex structure, and hence it suffices to prove that the induced group homomorphisms

$$i_* : \pi_k(E(X)) \to \pi_k(M)$$

are isomorphisms in each dimension k.

The identity $\rho \circ i = id_N$ yields $\rho_* \circ i_* = id_{N*}$, and hence i_* is a monomorphism. We will prove that i_* is an epimorphism showing that ρ_* is a monomorphism.

Let $[u] \in \pi_k(E(X))$, represented by the continuous map $u : S^k \to E(X)$, be in the kernel of ρ_*. Let $f^n = \varphi_{t_n}$ (here the "n" in "f^n" is an index written as superscript for "typographical" reason) be a sequence converging to ρ in Hol(M, M). Then, since M is a manifold, $f^n_*([u]) = [u \circ f^n] = [u \circ \rho] = \rho_*([u]) = 0$ for n large enough. Obviously $f^n = \varphi_{t_n}$ is homotopic to φ_0, which is the identity map on M, and hence $[u] = [u \circ \varphi_0] = [u \circ f^n] = \rho_*([u]) = 0$.

The proof of Theorem 1.1 is complete.

Let now X, M, $\varphi_t(\cdot)$ and F. By Theorem 1.1, after replacing M with a closed submanifold homotopically equivalent to M if necessary, we may assume that X is a complete vector field, that is $E(X) = M$. Again by Theorem 1.1 there exists a toral action

$$T^r \times M \to M$$

such that $\Gamma_m = T^r m$ for each $m \in M$. Obvoiusly the set F is then the fixed point set of such an action. Theorem 1.2 then follows immediatly from the following

THEOREM 2.1. *Let $T^r \times V \to V$ be a smooth action on the (not necessarily connected) smooth manifold V, anf let F be the fixed point set for such an action. If V is a manifold of finite topological type then also F it is and*

$$\chi(F) = \chi(V).$$

PROOF. Let us write $T^r = S^1 \times T^{r-1}$, and let F_1 be the fixed point set for S^1. Since a smooth action of a compact Lie group is linearizable in a neighbourhood of each fixed point, it follows that F_1 is a smooth (obviously closed) submanifold of V. Moreover T^{r-1} acts on F_1. After r steps we so obtain a chain

$$F = F_r \subset \cdots \subset F_1 \subset V$$

of closed submanifold with F_{i+1} the fixed point set of a circle action on F_i.

Then, clearly it suffices to prove the theorem for $r = 1$, that is for a circle action, which is done, e.g., in [Br] Theorem 10.9 (and remark below). □

Theorems 1.3 and 1.4 follow immediatly from the corresponding statements on toral actions (see, e.g., respectively [B, Theorem 5.3 (a)] and [H, Teorem (IV.5)]).

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