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Imbedding Vector Fields in Scalar Parabolic Dirichlet BVP_s

PETER POLÁČIK - KRZYSZTOF RYBAKOWSKI *

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. In this paper we study dynamical systems that are generated by semilinear parabolic problems of the form

$$(P_f)\qquad \begin{aligned} u_t - \Delta u &= f(x, u, \nabla u), && t > 0,\ x \in \Omega \\ u(x,t) &= 0, && t > 0,\ x \in \partial\Omega. \end{aligned}$$

where $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a C^1-function. Note that the operator $-\Delta$ on Ω with Dirichlet boundary condition on $\partial\Omega$ generates a sectorial operator on $L^p(\Omega)$, $p \geq 1$. If $p > N$ then a fractional power space X^α can be chosen such that it is continuously imbedded in $C^1(\bar{\Omega})$, and then (P_f) indeed defines a (local) dynamical system on X^α (see e.g. [He]).

It is known nowadays that the complexity of this class of dynamical systems depends very much on whether the space dimension N is one or higher. For $N = 1$, equation (P_f) has rather simple dynamics, no matter what function f we choose. For example, each bounded solution of such a one-dimensional problem is known to converge to an equilibrium (see [Ze, Ma, Ha-R]; see [Ha3] for a discussion of other results in one space dimension). The situation is quite different when $N > 1$. The solutions of (P_f) can exhibit very complicated behavior in this case.

Recently, an effort has been made by both authors to prove that the dynamics of higher-dimensional problems (P_f) can, in a sense, be arbitrary. A way to show this is by realization of ODEs in (P_f): to a given ODE, one tries

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to find a function f such that on some invariant manifold, (P_f) has the "same" dynamics as the given ODE. The present paper is a contribution to this effort.

To be more specific, consider an ODE

$$(O_h) \qquad \dot{\xi} = h(\xi), \qquad \xi \in B,$$

where B is an open set in $\mathbb{R}^n$ for some $n \geq 1$. We say that (O_h) *can be realized* in (P_f) if one can find a function f such that the corresponding problem (P_f) has the following property: there is a C^1 imbedding $\Lambda_f: B \to X^\alpha$ such that if $t \mapsto \xi(t)$ is a solution of (O_h) with $\xi = \xi_0$ then $t \mapsto u(t,\cdot) := \Lambda_f(\xi(t))$ is a solution of (P_f) with $u(0,\cdot) = \Lambda_f(\xi_0)$. In other words, the submanifold $\mathcal{M}_f = \{\Lambda_f(\xi) : \xi \in B\}$ is locally invariant for (P_f) and the flow of (P_f) on $\mathcal{M}_f$ is conjugate to the flow of (O_h), Λ_f being the conjugacy. If this is the case, we also say that (P_f) *realizes the vector field* h *on the invariant manifold* $\mathcal{M}_f$.

In [Po3], the first author has proved that if $N > 1$ is a given integer, Ω is an appropriate domain in $\mathbb{R}^N$, and n is an arbitrary positive integer, then the following realization results hold. If $B \subset \mathbb{R}^n$ is open and bounded, and $\tilde{h} \in C^1(\bar{B}, \mathbb{R}^n)$, then arbitrarily close to $\tilde{h}$ in the C^1 norm there is a function h such that (O_h), with appropriatly rescaled time, can be realized in (P_f). In addition, any linear equation (O_h) can be realized in (P_f).

An interesting feature in these results is that n, the dimension of the state space of (O_h), can be arbitrary and yet it is sufficient to consider the domain Ω in just two dimensions. As a consequence of these results, one can show that any persistent dynamical phenomena, such as transverse homoclinic orbits to hyperbolic periodic orbits, occur in (P_f), and that (P_f) can have trajectories dense in a high-dimensional invariant torus (for the latter, one uses the linear realization result). Of course, occurrence of very degenerate phenomena in (P_f) is not guaranteed by the density realization result, as it would be if all ODEs were realized.

At the present time, results on realization of arbitrary ODEs in $\mathbb{R}^n$ are available only under restrictions on n. In [Po2], realizability of any ODE on $\mathbb{R}^n$ with $n \leq N = \dim \Omega$ has been shown for the PDE in (P_f) with Neumann boundary condition. The method of that paper, a rather elementary one, does not apply to Dirichlet boundary condition.

The Dirichlet problem (P_f) with Δ replaced by a general self-adjoint second-order differential operator L has been considered by the second author in [Ry1]. The main result of [Ry1] says that whenever the kernel $\ker L$ of the operator L on Ω (with Dirichlet boundary condition on $\partial\Omega$) satisfies a certain nondegeneracy condition, previously introduced in [Po1], then every sufficiently smooth (and sufficiently small) vector field h on $\mathbb{R}^n$, $n \leq \dim \ker L$, can be realized on a *center manifold* of (P_f), with an appropriate nonlinearity f.

The nondegeneracy condition can hold only if $\dim \ker L \leq N+1$; in [Po1] it was verified for a certain elliptic operator with $\dim \ker L = N+1$, and for a simpler operator, $\Delta + \text{const}$, under the weaker requirement $\dim \ker L = N$.

In this paper we first show that there is an analytic function $a: \mathbb{R}^N \to \mathbb{R}$

such that the differential operator $L := \Delta + a$ satisfies the above mentioned nondegeneracy condition on a ball in $\mathbb{R}^N$ with $\dim\ker L = N + 1$. (See Lemma 3). An application of Lemma 3 together with the main theorem of [Ry1] yields a center-manifold realization result for all sufficiently smooth vector fields h on $\mathbb{R}^n$, $n \leq N+1$. (See Theorem 1.) Here the vector field h has to be very smooth (of class C^{32} at least) and there is a loss of derivatives involved: if h is of class C^r then one obtains f of class C^{r-15}. All this is the consequence of the Nash-Moser inverse mapping theorem, used in the proof of the result in [Ry1].

On the other hand, if we do not place any restriction on the nature of the invariant manifold $\mathcal{M}_f$, in particular, if we do not insist that it be a center manifold, then Theorem 1 can drastically be improved. We thus arrive at Theorem 2, the main result of this paper, in which we prove realizability of arbitrary C^1-vector fields on $\mathbb{R}^n$, $n \leq N + 1$. There is no loss of derivatives in the latter theorem: if $m \geq 1$ and h is of class C^m then f can be chosen of class C^m. Moreover, the proof of Theorem 2 is much simpler than that of Theorem 1: it uses neither the Nash-Moser theorem nor the center manifold theory. The proof is based on Lemma 3 and some simple properties of semigroups generated by sectorial operators. It should be remarked that both Theorem 1 and Theorem 2 are also valid for other types of boundary conditions, like the Neumann or Robin problems.

The precise statements of the results and the proofs are given in the next section. For realization results in other classes of equations, we refer the reader to [Ha1, Ha2, Ry2, Fa-M] for the case of delay equations, to [Fi-P] for a nonlocal one-dimensional parabolic problem, and to [Da, Sa-F] for periodically forced parabolic problems.

2. - Statements of the results and their proofs

From now on we assume that $N \geq 2$ and Ω is the unit ball in $\mathbb{R}^N$. Fix $p > N$ and α with $(p+N)/2p < \alpha < 1$. It is well-known that the differential operator $-\Delta$ on Ω with Dirichlet boundary condition on $\partial\Omega$ defines a sectorial operator A on $X := L^p(\Omega)$ with domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The corresponding fractional power space X^α satisfies

$$X^\alpha \subset C^1(\bar{\Omega})$$

with continuous inclusion (see [He]).

For $m = 1, 2, \ldots$ let Y_m be the set of all functions

$$f\colon (x, s, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x, s, w) \in \mathbb{R}$$

such that for all $1 \leq j \leq m$ the Fréchet derivative $D^j_{(s,w)}f$ exists and is continuous and bounded on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. By $C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$, $m = 0, 1, \ldots$, we denote the space of C^m functions from $\mathbb{R}^{N+1}$ into itself that are continuous and bounded together with all derivatives up to the order m. Let $|\cdot|_{C_b^m}$ be the standard C^m supremum norm on $C_b^m(\mathbb{R}^{N+1},\mathbb{R}^{N+1})$.

Our first theorem is a center-manifold realization result:

THEOREM 1. *There is an analytic function $a:\mathbb{R}^N \to \mathbb{R}$ with the following property: for every $m \geq 17$ there is an $\epsilon_m > 0$ such that for every vector field $h \in C_b^{m+15}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})$ with $|h|_{C_b^{m+15}} < \epsilon_m$ there is a nonlinearity $f \in Y_m$ with the property that equation (P_f) realizes the vector field h on the* global center manifold $\mathcal{M}_c$ of (P_f), *relative to the operator $L = \Delta + a$ on Ω, with Dirichlet boundary condition on $\partial\Omega$. The manifold $\mathcal{M}_c$ is given by an imbedding $\Lambda:\mathbb{R}^{N+1} \to X^\alpha$ of class C^m.*

We recall that if P denotes the $L^2(\Omega)$-orthogonal projection of X^α onto $\ker L$ then $\mathcal{M}_c$ is defined as the set of all $u_0 \in X^\alpha$ for which there is a solution $u:\mathbb{R} \to X^\alpha$ with $u(0) = u_0$ and $t \mapsto (I-P)u(t)$ bounded in X^α.

The following result is the main contribution of this paper:

THEOREM 2. *There is a $\delta_1 > 0$ such that for every $h \in C_b^1(\mathbb{R}^{N+1},\mathbb{R}^{N+1})$ with $|h|_{C_b^1} < \delta_1$ there is a nonlinearity $f \in Y_1$ and an invariant manifold $\mathcal{M}_f$ of (P_f) with the property that equation (P_f) realizes the vector field h on $\mathcal{M}_f$. If in addition $h \in C_b^m(\mathbb{R}^{N+1},\mathbb{R}^{N+1})$ then f can be chosen such that $f \in Y_m$ and the manifold $\mathcal{M}_f$ is given by an imbedding $\Lambda_f:\mathbb{R}^{N+1} \to X^\alpha$ of class C^m.*

Note that rescaling time in (O_h) has the effect of multiplying h by a constant. Thus the assumption that h be small in the C_b^1 norm is hardly a significant restriction.

We prepare the proofs of Theorem 1 and Theorem 2 by a few lemmas. The first one establishes the nondegeneracy property mentioned in the Introduction.

LEMMA 3. *There exists a real analytic function $a:\mathbb{R}^N \to \mathbb{R}$ such that the kernel of the operator $\Delta + a$ on Ω, under Dirichlet boundary condition on $\partial\Omega$, is spanned by $N+1$ linearly independent eigenfunctions $\phi_1,\ldots,\phi_{N+1}$ with the following property: If*

$$R(x) := \det \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega$$

then $R(x) \neq 0$ for some $x \in \Omega$.

PROOF. Let $a(x)$ be a real analytic function on $\mathbb{R}^N$ that is radially symmetric: $a = a(r) = a(|x|)$. Following [Po1, Po4], we consider the eigenvalue problems

$$\begin{aligned} v_{rr} + \frac{N-1}{r} v_r + a(r)v - \mu v &= 0, \\ v_r(0) = v(1) &= 0, \end{aligned} \tag{1}$$

and

$$w_{rr} + \frac{N-1}{r}\, w_r + \left(a(r) - \frac{N-1}{r^2}\right) w - \nu w = 0, \tag{2}$$
$$w(1) = 0, \quad w \quad \text{regular at} \quad r = 0.$$

Let μ_2 denote the second eigenvalue of (1) and ν_1 the first eigenvalue of (2). It is well-known that both μ_2 and ν_1 are eigenvalues of the operator $\Delta + a(r)$ on Ω under Dirichlet bondary condition; the eigenfunction ϕ_1 of (1) corresponding to μ_2 is at the same time a radially symmetric eigenfunction of $\Delta + a(r)$, while to ν_1 there correspond N nonsymmetric eigenfunctions of $\Delta + a(r)$ given by

$$\phi_{i+1}(x) = w(r)\, \frac{x_i}{r}, \quad r = |x|, \quad i = 1, \ldots, N, \tag{3}$$

where w is the eigenfunction of (2) corresponding to ν_1 (cf. [Po1, Sect. 3]). (We remark that ϕ_{i+1} is the unique, up to scalar multiples, eigenfunction of $\Delta + a(r)$ that is odd in x_i and positive in $\{x \in \Omega : x_i > 0\}$.) If $\nu_1 = \mu_2$ then $\phi_1, \ldots, \phi_{N+1}$ are linearly independent eigenfunctions of $\Delta + a(r)$ that span the kernel of $\Delta + a(r) - \nu_1$. Therefore the lemma will be proved if we show that $\nu_1 = \mu_2$ implies $R(x) \not\equiv 0$ and that $\nu_1 = \mu_2$ actually holds for some real analytic radially symmetric function. For functions $\phi_1 = \phi(r)$ and $\phi_2, \ldots, \phi_{N+1}$ of the form (3), the determinant $R(x)$ has been calculated in [Po1, Sect. 3]. It is shown there that $R(x) \not\equiv 0$ provided the following relations are satisfied

$$\phi_{1r}(0) = 0, \quad \phi_1(0) \neq 0, \quad w(0) = 0, \quad w_r(0) \neq 0. \tag{4}$$

To see that these relations hold, first note that, as the eigenfunctions of $\Delta + a(r)$ are real analytic in Ω, $\phi_1(r)$ and $w(r)$ are real analytic near $r = 0$ (more precisely, they are restrictions of real analytic functions). By (1), we have $\phi_{1r}(0) = 0$. If $\phi_1(0) = 0$, then these two equalities together with (1) imply that all derivatives of ϕ_1 at 0 vanish, hence $\phi_1 \equiv 0$. But this is impossible for an eigenfunction. Thus $\phi_1(0) \neq 0$. Next, multiplying the equation in (2) by r^2 and letting $r \to 0$, we obtain $w(0) = 0$. Again, $w_r(0) = 0$ leads to the contradiction $w \equiv 0$. We have thus proved all the relations in (4).

In order to find a function $a = a(r)$ such that $\nu_1 = \nu_1(a)$ and $\mu_2 = \mu_2(a)$ coincide, we argue as in [Po4]. If a_1 and a_2 are such that $\nu_1(a_1) > \mu_2(a_1)$ and $\nu_1(a_2) < \mu_2(a_2)$ then $\nu_1(a) = \mu_2(a)$ for some a of the form $a = sa_1 + (1-s)a_2$. Smooth functions a_1, a_2 that satisfy the above relations and in addition are constant near $r = 0$ were found in [Po4] (see the proof of Proposition 3.2 and Remark A.2 in [Po4]). We can clearly approximate a_1, a_2 by real analytic radially symmetric functions such that the inequalities remain unchanged. The resulting function a is then real analytic as desired. The lemma is proved. □

PROOF OF THEOREM 1. Lemma 3 implies that the operator $L = \Delta + a$ on Ω, with Dirichlet boundary condition on $\partial\Omega$, satisfies the nondegeneracy condition from [Po1] formulated in Definition 2.1 of [Ry1] (with $\dim\ker L = N+1$). Now use Theorem 2.3 of [Ry1]. For $m \geq 17$ let $\epsilon_m := \epsilon$ where ϵ is as in that theorem and for $|h|_{C_b^{m+15}} < \epsilon_m$ let the nonlinearity σ_0 and the invariant manifold $\mathcal{M}_{\sigma_0}$ be as in that theorem. Set

$$f(x,s,w) := a(x)s + \sigma(x,s,w), \qquad (x,s,w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Then the theorem implies that $f \in Y_m$ and the manifold $\mathcal{M}_{\sigma_0}$ is given by an imbedding $\Lambda: \mathbb{R}^{N+1} \to X^\alpha$ of class C^m. (Note that the space Y_m defined above is different from the space Y_m defined in [Ry1].) The fact that $\mathcal{M}_{\sigma_0} = \mathcal{M}_{\mathrm{c}}$ follows from the center manifold theorem (cf. Proposition 2.2 in [Ry1] and its proof contained, in part, e.g. in [Ry3]). Theorem 1 is proved. □

We shall need a few more preliminary results before we can prove Theorem 2.

LEMMA 4. *Let G be the set of all $x \in \Omega$ such that $R(x) \neq 0$. Then G is open and $\bar{\Omega} \setminus G$ has N-dimensional Lebesgue measure zero.*

PROOF. Since the eigenfunctions ϕ_i, and hence also R are real analytic on Ω (e.g., by pp. 207-210 in [BJS]), the result follows from Lemma 3 and the well-known general result (easily proved by induction on N, using Fubini-Tonelli theorem) that the zero set of a nontrivial real analytic function defined on an open subset of $\mathbb{R}^N$ has measure zero. □

LEMMA 5. *Let G be as in Lemma 4. For every $k \in \mathbb{N}$ there is a function $b \in C^\infty(\bar{\Omega})$ with* $\operatorname{supp} b \subset G$ *such that*

$$\lambda < -k$$

for every eigenvalue λ of the operator $\Delta + a + b$ on Ω with Dirichlet boundary condition on $\partial\Omega$.

PROOF. Let

$$c := \max_{x \in \bar{\Omega}} |a(x)|.$$

For $\epsilon > 0$ let G_ϵ be the set of all x with $\operatorname{dist}(x, \bar{\Omega} \setminus G) \geq \epsilon$. Choose a function $b_\epsilon \in C^\infty(\bar{\Omega})$ with $\operatorname{supp} b_\epsilon \subset G$ and such that

$$b_\epsilon(x) \equiv -c - k - 1, \qquad x \in G_\epsilon$$

$$-c - k - 1 \leq b_\epsilon(x) \leq 0, \qquad x \in \bar{\Omega}.$$

We shall show that the lemma holds with b replaced by b_ϵ for $\epsilon > 0$ sufficiently small. The reason for this is quite obvious: for $\bar{b} = -c - k - 1$ the eigenvalues are less than $-k$ and, since $\bar{\Omega} \setminus G$ has measure 0, b_ϵ is close to $\bar{b}$. We give the details. Suppose the claim is not true. Then there are sequences (λ_n), (u_n), (ϵ_n) and (b_n) such that $\epsilon_n \to 0$, $b_n := b_{\epsilon_n}$,

$$\Delta u_n = -(a + b_n)u_n + \lambda_n u_n \qquad \text{on } \Omega \tag{5}$$

$$u_n = 0 \qquad \text{on } \partial\Omega$$

$$\lambda_n \geq -k \tag{6}$$

for all $n \in \mathbb{N}$. We may assume that

$$(u_n | u_n) \equiv 1$$

where $(\cdot|\cdot)$ denotes the scalar products on both $L^2(\Omega, \mathbb{R})$ and $L^2(\Omega, \mathbb{R}^N)$. It follows that

$$-(\nabla u_n | \nabla u_n) + ((a + b_n)u_n | u_n) = \lambda_n \tag{7}$$

Since $a + b_n \leq c$ on $\bar{\Omega}$, this implies that

$$\lambda_n \leq c$$

so by (6)

$$-k \leq \lambda_n \leq c$$

for all $n \in \mathbb{N}$. Thus the right hand side of (5) is bounded in $L^2(\Omega)$ so (u_n) is bounded in $H^2(\Omega)$. Passing to a subsequence if necessary, we may therefore assume that there is a $\bar{u} \in H^1(\Omega)$ such that

$$u_n \to \bar{u} \quad \text{in } H^1(\Omega).$$

In particular,

$$(\bar{u} | \bar{u}) = 1. \tag{8}$$

Moreover, by Sobolev imbedding theorems, there is a $q > 2$ such that

$$u_n \to \bar{u} \quad \text{in } L^q(\Omega).$$

Now set

$$\bar{b}(x) \equiv -c - k - 1, \qquad x \in \bar{\Omega}.$$

For every $x \in G$, $b_n(x) \to \bar{b}(x)$. Thus, by Lemma 4, $b_n \to \bar{b}$ a.e. on Ω. Since (b_n) is bounded in $L^\infty(\Omega)$, it follows from the dominated convergence theorem that

$$a + b_n \to a + \bar{b} \quad \text{in } L^r(\Omega)$$

for every r with $1 \leq r < \infty$. Define r such that

$$(2/q) + (1/r) = 1.$$

It follows from Hölder's inequality that

$$((a + b_n)u_n | u_n) \to ((a + \bar{b})\bar{u} | \bar{u})$$

$$(\nabla u_n | \nabla u_n) \to (\nabla \bar{u} | \nabla \bar{u})$$

Thus, from (7) and (8)

$$\lambda_n \to -(\nabla \bar{u} | \nabla \bar{u}) + ((a + \bar{b})\bar{u} | \bar{u}) \leq -k - 1,$$

contradicting (6). The lemma is proved. □

For every globally Lipschitzian map $h: \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ let $\pi_h: \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be the global flow generated by h. In other words, $\pi_h(t, \xi_0) := \xi(t)$ where $t \mapsto \xi(t)$, $t \in \mathbb{R}$, is the unique solution of the initial value problem

$$\dot{\xi}(t) = h(\xi(t)), \quad t \in \mathbb{R},$$

$$\xi(0) = \xi_0,$$

By differentiating this equation and using standard arguments we easily derive the following essentially well-known fact:

LEMMA 6. *For every $m \in \mathbb{N}$ there is a constant $\tilde{c}_m$ such that for every vector field $h \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ the flow π_h is of class C^m and for every $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$*

$$|D_\xi^m \pi_h(t, \xi_0)|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \leq \tilde{c}_m \exp(mL|t|)$$

where $L := |h|_{C_b^m}$.

Applying the higher-order chain rule to the composite map $h \circ \pi_h$ and using Lemma 6 we obtain:

LEMMA 7. *For every $m \in \mathbb{N}$ there is a constant c_m such that for every vector field $h \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and for every $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$*

$$|D_\xi^m (h \circ \pi_h)(t, \xi_0)|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \leq c_m L \exp(mL|t|)$$

where $L := |h|_{C_b^m}$.

PROOF OF THEOREM 2. We begin with some preliminary remarks. Given an ODE (O_h), with $|h|_{C_b^1}$ sufficiently small, we have to find a nonlinearity f and an imbedding $\Lambda: \mathbb{R}^{N+1} \to X^\alpha$ such that for each solution $\xi(t)$ of (O_h) the function $u(t,x) = \Lambda(\xi(t))(x)$ is a solution of (P_f). The latter means (dropping the argument t)

$$D_\xi \Lambda(\xi)(x)h(\xi) - \Delta_x \Lambda(\xi)(x) = f(x, \Lambda(\xi)(x), \nabla_x \Lambda(\xi)(x)). \tag{9}$$

We look for f and Λ in the form

$$\begin{aligned} f(x,s,w) &= a(x)s + g(x,s,w), \\ \Lambda(\xi)(x) &= \Phi(x) \cdot \xi + \Gamma(\xi)(x), \end{aligned} \tag{10}$$

where $a(x)$ is as in Lemma 3, $\Phi(x) = (\phi_1(x), \dots, \phi_{N+1}(x))$ with $\phi_1(x), \dots, \phi_{N+1}(x)$ as in Lemma 3, and $g \in Y_1$ and $\Gamma: \mathbb{R}^{N+1} \to X^\alpha$ are functions to be found. The construction of g and Γ is based on the following idea. If $\Gamma(\xi)$ is "sufficiently small" then for each $x \in G$, with G as in Lemma 4, the mapping $\xi \mapsto (\Lambda(\xi)(x), \nabla_x \Lambda(\xi)(x))$ is a diffeomorphism of $\mathbb{R}^{N+1}$. Thus for $x \in G$ we can choose g such that $g(x, \Lambda(\xi)(x), \nabla_x \Lambda(\xi)(x))$ equals any given function of ξ; we shall require this function to equal to $b(x)\Gamma(\xi)(x)$, where $b(x)$ is as in Lemma 5 and Γ is still to be found. (For $x \notin G$, we set $g(x,s,w) = 0$.) Substituting this expression for g and (10) into (9), we obtain that $\Gamma(\xi)(x)$ must satisfy

$$D_\xi \Gamma(\xi)(x)h(\xi) - \Delta_x \Gamma(\xi)(x) - (a(x) + b(x))\Gamma(\xi)(x) = -\Phi(\xi) \cdot h(\xi)$$

(we have used the fact that the ϕ_i are in the kernel of $\Delta + a(x)$). Equivalently, we need to find $\Gamma(\xi)$ such that for each solution $\xi(t)$ of (O_h) the function $v(t,x) = \Gamma(\xi(t))(x)$ satisfies

$$v_t - \Delta v - (a(x) + b(x))v = -\Phi(\xi) \cdot h(\xi(t)).$$

As we also require that v be defined for each t and bounded, the variation of constants easily leads to a formula for v, hence for Γ. This formula is a starting point in the detailed construction that follows next. We verify that it yields Γ and f of class C^m if $h \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$.

Let $a(x)$ and G be as in Lemmas 3 and 4. Let $m \geq 1$ and let $h \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ be such that $|h|_{C_b^1} < \delta_1$, where δ_1 is a constant specified below. Denote $L_j = |h|_{C_b^j}$, $j = 0, \dots, m$. Choose a $k \in \mathbb{N}$ such that $k - mL_m > 0$ and $k > L_1 + 1$, and let b be the corresponding function from Lemma 5. Let $T(t)$, $t \geq 0$, be the analytic semigroup on X generated the operator $-(\Delta + a + b)$ on Ω with Dirichlet boundary condition on $\partial\Omega$. It is well-known that the corresponding fractional power spaces of this operator are identical as sets and isomorphic as normed spaces to the fractional power spaces of the operator A defined above. In particular, by our choice of b, we have the estimate

$$|T(t)u|_\alpha \leq C_\alpha t^{-\alpha} e^{-kt} |u|_\alpha, \quad t > 0, \ u \in X^\alpha \tag{11}$$

for some constant C_α (cf. [He, Sect. 1.5]). Here, $|\cdot|_\alpha$ is the norm in X^α. Let $|\Phi| = |(\phi_1, \dots, \phi_{N+1})| := \sum_{l=1}^{N+1} |\phi_l|_\alpha$. For $\xi \in \mathbb{R}^{N+1}$ define

$$\Gamma(\xi) = \Gamma_h(\xi) := -\int_0^\infty T(s)\Phi \cdot h(\pi(-s,\xi))\, ds \tag{12}$$

where $\pi := \pi_h$. The integrand in (12) is continuous into X^α and its X^α-norm is bounded by the function

$$g(s) := C_\alpha |\Phi| L_0 s^{-\alpha} e^{-ks}.$$

This latter function is integrable, so the integral in (12) converges in X^α. Thus

$$\Gamma\colon \mathbb{R}^{N+1} \to X^\alpha$$

is defined and bounded globally. Moreover, in view of Lemma 7, for every j with $1 \le j \le m$ and every $\xi \in \mathbb{R}^{N+1}$ the j-th order Fréchet derivative at ξ of the integrand in (12) is bounded in the $\mathcal{L}^j((\mathbb{R}^{N+1})^j, X^\alpha)$-norm by the function

$$g_j(s) := C_\alpha c_j |\Phi| L_j s^{-\alpha} e^{-(k-jL_j)s}$$

Since this function is integrable, it follows that $\Gamma \in C_b^m(\mathbb{R}^{N+1}, X^\alpha)$. Moreover,

$$\int_0^\infty g_j(s) ds \le C_\alpha c_j |\Phi| L_j (1/(1-\alpha) + (1/(k - jL_j))). \tag{13}$$

Define the map

$$\Lambda = \Lambda_h\colon \mathbb{R}^{N+1} \to X^\alpha$$

by

$$\xi \mapsto \Phi \cdot \xi + \Gamma_h(\xi).$$

Now let U be an open set with $\operatorname{supp} b \subset U \subset \bar{U} \subset G$. For every $x \in G$ the map

$$M(x)\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$$

$$\xi \mapsto (\Phi(x) \cdot \xi, \nabla\Phi(x) \cdot \xi)$$

is a linear isomorphism, by Lemma 3. Since $\bar{U}$ is compact

$$M := \sup_{x \in \bar{U}} |(M(x))^{-1}| < \infty.$$

Using (13) with $j = 1$ and the relation $k > L_1 + 1$, we see that there is a constant $\delta_1 > 0$ (independent of h and k) such that whenever $|h|_{C_b^1} < \delta_1$

$$\sup_{x \in \bar{U}, \xi \in \mathbb{R}^{N+1}} (|D_\xi \Gamma(\xi)(x)| + |\nabla_x D_\xi \Gamma(\xi)(x)|) < 1/M.$$

(Here we have also used the fact that X^α is continuously imbedded in $C^1(\bar{\Omega})$.) For such an h, the contraction mapping principle and the implicit function theorem implies that for $x \in \bar{U}$ the map

$$\psi_x : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1},$$
$$\xi \mapsto (\Lambda(\xi)(x), \nabla_x \Lambda(\xi)(x))$$

is a diffeomorphism of class C^m and for all j with $0 \leq j \leq m$ the map

$$(x, z) \in \bar{U} \times \mathbb{R}^{N+1} \mapsto D^j(\psi_x)^{-1}(z) \in \mathcal{L}^j((\mathbb{R}^{N+1})^j, \mathbb{R}^{N+1})$$

is continuous and bounded. In particular we obtain that $\xi \mapsto \Lambda(\xi)$ is an imbedding of $\mathbb{R}^{N+1}$ into X^α. Define for $z = (s, w) \in \mathbb{R}^{N+1}$ and $x \in \bar{\Omega}$

$$f(x, z) = \begin{cases} a(x)s, & \text{if } x \notin \operatorname{supp} b; \\ a(x)s + b(x)\Gamma((\psi_x)^{-1}(z))(x), & \text{if } x \in U; \end{cases}$$

Since $\operatorname{supp} b \subset U$, the definition of f is unambiguous and the smoothness properties proved so far imply that $f \in Y_m$. We shall show that f satisfies the assertions of Theorem 2. To this end, first note that

$$\Gamma(\xi) = \Gamma_h(\xi) = -\int_{-\infty}^{0} T(-s)\Phi \cdot h(\pi(s, \xi))ds$$

for all ξ. Hence for all t_0, $t \in \mathbb{R}$ with $t_0 < t$

$$\begin{aligned}
\Gamma(\pi(t,\xi)) &= -\int_{-\infty}^{0} T(-s)\Phi \cdot h(\pi(t+s,\xi))ds \\
&= -\int_{-\infty}^{t} T(t-s)\Phi \cdot h(\pi(s,\xi))ds \\
&= -T(t-t_0)\int_{-\infty}^{t_0} T(t_0-s)\Phi \cdot h(\pi(s,\xi))ds \\
&\quad -\int_{t_0}^{t} T(t-s)\Phi \cdot h(\pi(s,\xi))ds \\
&= T(t-t_0)\Gamma(\pi(t_0,\xi)) - \int_{t_0}^{t} T(t-s)\Phi \cdot h(\pi(s,\xi))ds.
\end{aligned}$$

Since the function $s \mapsto \Phi \cdot h(\pi(s,\xi))$ is locally Hölderian into X it follows that the function

$$v\colon t \in \mathbb{R} \mapsto \Gamma(\pi(t,\xi)) \in X^\alpha$$

is differentiable and

$$\dot{v}(t) = (\Delta + a + b)v(t) - \Phi \cdot h(\pi(t,\xi)), \quad t \in \mathbb{R}.$$

Therefore the definition of f and Lemma 3 obviously imply that the function

$$u\colon t \in \mathbb{R} \mapsto \Lambda(\pi(t,\xi)) \in X^\alpha$$

solves the equation

$$\dot{u}(t) = \Delta u(t) + f(\cdot, u(t)(\cdot), \nabla u(t)(\cdot)), \quad t \in \mathbb{R}.$$

The theorem is proved. □

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