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Instability Phenomena for the Moment Problem

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Suppose K is a compact set in $\mathbb{C}^n$ or $\mathbb{R}^n$, and let μ be a finite complex Borel measure on K. In this paper we show, under appropriate conditions on K, that if the analytic or harmonic moments of μ decrease sufficiently rapidly (or grow sufficiently slowly) in a certain precise sense dependent on K, then these moments vanish identically. In the most favorable cases, it is then possible to conclude that $\mu = 0$. This phenomenon does not seem to have been noticed previously, even in the classical case of the power moment problem for a finite interval in $\mathbb{R}$.

In the sequel all measures are Borel.

1. - Holomorphic moments, $n = 1$

We begin with a discussion of the situation for $n = 1$.

THEOREM 1. *Let K be a compact set in the plane which does not contain the origin, and let μ be a finite complex measure on K with moments*

$$(1) \qquad a_j = \int_K \frac{d\mu(\varsigma)}{\varsigma^{j+1}} \quad j = 0, 1, 2, \cdots .$$

If

$$(2) \qquad \overline{\lim_{j\to\infty}} \sqrt[j]{|a_j|} < \frac{1}{\max_K |z|}$$

and K does not separate 0 from ∞ (i.e., 0 belongs to the unbounded component of $\mathbb{C}\backslash K$), then $a_j = 0$ for $j = 0, 1, 2, \cdots$.

If K does separate 0 from ∞, then for each sequence $\{a_j\}$ satisfying (2) *there is a measure μ on K having $\{a_j\}$ as its moment sequence, i.e., such that* (1) *holds.*

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PROOF. Suppose that K does not separate 0 and ∞. Let

$$F(z) = \int_K \frac{d\mu(\varsigma)}{\varsigma - z}.$$

Clearly, F is analytic off K. Expanding F in a power series about 0, we find

$$F(z) = \sum_{j=0}^{\infty} a_j z^j,$$

where the a_j are given by (1). If (2) holds, this series converges uniformly on an open disc containing K. Thus, the restriction of F to the unbounded component of $\mathbb{C}\backslash K$ extends to be analytic on the entire complex plane. Since $F(\infty) = 0$, F vanishes identically, so $a_j = 0$ for $j = 0, 1, 2, \cdots$.

Now suppose that K separates 0 from ∞. Denote by U the unbounded component of $\hat{\mathbb{C}}\backslash K$ and consider the space A of all continuous functions on $K \cup U$ which are analytic on U. By the maximum modulus principle, $\|f\|_A = \max_K |f(z)|$ defines a norm on A under which it is identified with a closed subspace of $C(K)$. Let $\{a_n\}$ be a sequence which satisfies (2), so that

$$R = \frac{1}{\overline{\lim_{j\to\infty}} \sqrt[j]{|a_j|}} > \max_K |z|$$

Then the function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on $\{z : |z| < R\}$, and we may choose ρ so that $\max_K |z| < \rho < R$. Now set

$$L(f) = \frac{1}{2\pi i} \int_\Gamma f(z)\varphi(z)dz \tag{3}$$

where Γ is the positively oriented circle of radius ρ about the origin. Clearly,

$$|L(f)| \leq \rho \max_\Gamma |\varphi(z)| \max_K |f(z)| \leq C(\rho, \varphi)\|f\|_A.$$

Thus L defines a continuous linear functional on A, which (by the Hahn-Banach Theorem) extends to all of $C(K)$. Thus there exists a finite complex measure μ on K such that

$$L(f) = \int_K f d\mu \quad f \in A. \tag{4}$$

On the other hand, we have

$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(z)}{z^{j+1}} dz.$$

Since the functions $f_j(z) = z^{-(j+1)}$, $j = 0, 1, 2, \cdots$, all belong to A, (3) yields $L(f_j) = a_j$. It then follows from (4) that the measure μ has moment sequence $\{a_j\}$, i.e., that (1) holds. □

Note, in particular, that one may choose $\{a_j\}$ such that (2) holds but $a_j \neq 0$ for all j.

An analogous reasoning, based on expanding the function $F(z)$ in a Laurent series about the point $z = \infty$, yields:

THEOREM 1′. *Let K be a compact set in the plane which does not contain the origin, and let μ be a finite complex measure on K with moments*

$$b_j = \int_K \varsigma^j \, d\mu(\varsigma) \quad j = 0, 1, 2, \cdots. \tag{1′}$$

If

$$\overline{\lim_{j\to\infty}} \sqrt[j]{|b_j|} < \min_K |z| \tag{2′}$$

and K does not separate 0 from ∞, then $b_j = 0$ for all $j = 0, 1, 2, \cdots$.

If K does separate 0 from ∞, then for each sequence $\{b_j\}$ satisfying (2′) there is a measure μ on K having $\{b_j\}$ as its moment sequence, i.e., such that (1′) holds.

COROLLARY 1. *Suppose K has empty interior, $\mathbb{C}\backslash K$ is connected, and $0 \notin K$. Then (2) and (2′) each imply that $\mu = 0$.*

PROOF. Suppose that (2′) holds. Then, by Theorem 1′, all analytic moments (1′) vanish. Taking linear combinations shows that μ is orthogonal to all analytic polynomials. By Mergelyan's Theorem, any function in $C(K)$ can be uniformly approximated on K by such polynomials. Thus μ annihilates all elements of $C(K)$ and hence vanishes identically. If (2) holds, the proof of Theorem 1 shows that $F(z)$ vanishes identically. Thus the coefficients of its Laurent expansion about ∞ (given by (1′)) are identically zero, so again $\mu = 0$. □

When K is an interval that does not contain the origin, it is possible to strengthen Corollary 1 under the assumption that μ does not place any mass at the (relevant) endpoint of $[a, b]$. Specifically, we have

COROLLARY 2. *Let μ be a finite complex measure on $[a, b] \subset \mathbb{R}$, where $0 < a < b$. If either $\mu(\{b\}) = 0$ and*

$$\overline{\lim_{j\to\infty}} \left(\left| \int_a^b \frac{d\mu(t)}{t^{j+1}} \right| \right)^{1/j} \leq \frac{1}{b}$$

or $\mu(\{a\}) = 0$ *and*

$$\overline{\lim_{j\to\infty}}\left(\left|\int_a^b t^j d\mu(t)\right|\right)^{1/j} \leq a,$$

then $\mu = 0$.

PROOF. Suppose

$$\overline{\lim_{j\to\infty}}\left(\left|\int_a^b \frac{d\mu(t)}{t^{j+1}}\right|\right)^{1/j} \leq \frac{1}{b}.$$

Let

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{d\mu(t)}{t-z},$$

so that $F(z)$ is analytic on $\mathbb{C}\backslash\{b\}$. For $y > 0$ we have

$$F(x+iy) - F(x-iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y d\mu(t)}{(x-t)^2+y^2} = (P_y * \mu)(x),$$

where P_y is the Poisson kernel for the upper half plane. Since $(P_y * \mu)(x)dx$ converges weak$*$ to μ as $y \to 0$, we have

$$\lim_{y\to 0}\int_{-\infty}^{\infty} f(x)[F(x+iy) - F(x-iy)]dx = \int_a^b f(x)d\mu(x)$$

for any function $f \in C_0(\mathbb{R})$.

Now fix $c \in (a, b)$ and let $\varphi \in C_0(\mathbb{R})$ be supported in $[0, c]$. We have

$$\int_a^c \varphi(x)d\mu(x) = \lim_{y\to 0}\int_0^c \varphi(x)[F(x+iy) - F(x-iy)]dx = 0$$

since F is analytic on $[0, c]$. Taking the sup over all such φ satisfying $|\varphi(x)| \leq 1$, we obtain $|\mu|([a, c)) = 0$. It follows that $|\mu|([a, b)) = 0$; and, since $\mu(\{b\}) = 0$, $\mu = 0$. □

This result applies in particular to absolutely continuous measures, i.e., functions in $L^1([a, b])$.

REMARK. The first part of Theorems 1 and 1′ hold not only for measures but for distributions of compact support and for analytic functionals as well; the proofs remain the same. As a consequence of this, we have:

COROLLARY 3. *Let f be an entire function such that*

$$|f(z)| \leq \gamma(1+|z|)^N e^{r|\operatorname{Im} z|}, \quad z \in \mathbb{C}. \tag{5}$$

Suppose that for some $\rho > 0$ one has

$$\overline{\lim_{j\to\infty}} \sqrt[j]{\left|\sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \frac{f^{(k)}(0)}{[-i(r+\rho)]^k}\right|} < \frac{\rho}{r+\rho}. \tag{6}$$

Then $f \equiv 0$.

If, in addition, $f \in L^2(\mathbb{R})$ (i.e., f is in the Wiener class [A1, p. 166]*), we may relax the inequality in* (6) *to allow equality.*

PROOF. If (5) holds, then $f(z) = u(e^{-ixz})$, where u is a distribution supported on $[-r, r]$, cf. [Ru, p. 183]. Now

$$f(z)e^{-i(r+\rho)z} = u(e^{-i(x+r+\rho)z}) = u_1(e^{-ixz}),$$

where u_1 is a distribution whose support lies in $[\rho, \rho+2r]$. It follows that

$$\begin{aligned} u_1(x^j) &= \frac{1}{(-i)^j} \frac{d^j}{dz^j} [f(z)e^{-i(r+\rho)z}]\bigg|_{z=0} \\ &= i^j \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} f^{(k)}(0)[-i(r+\rho)]^{j-k}. \end{aligned}$$

By Corollary 1 (and the previous Remark), it follows that $u_1 = 0$ and hence $f \equiv 0$. When $f \in L^2(\mathbb{R})$, the distribution u_1 is a function in $L^2([\rho, \rho+2r])$, so the result follows from Corollary 2. □

2. - Holomorphic moments, $n > 1$

We shall follow the conventional notations for multi-indices. Thus, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ an n-tuple of non-negative integers and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ we shall write $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. For K a compact set in $\mathbb{C}^n$, set $d_\alpha(K) = \max_K |z^\alpha|$ and $\tilde{K} = \bigcup_{j=1}^{n} \{w : \Gamma_{z_j} \cap K \neq \emptyset\}$, where $\Gamma_{z_j} = \{w \in \mathbb{C}^n : w_j = z_j\}$. Put $\hat{\mathbb{C}}^n = \hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \cdots \times \hat{\mathbb{C}}$ (n times).

THEOREM 2. *Let K be a compact set in $\mathbb{C}^n$ such that $\varsigma_j \neq 0$, $j = 1, \ldots, n$, for $\varsigma \in K$ and the points 0 and $(\infty, \ldots, \infty)$ belong to the same connected component of $\hat{\mathbb{C}}^n \backslash \tilde{K}$. Let μ be a finite complex measure on K with moments*

$$a_\alpha = \int_K \frac{d\mu}{\varsigma^{\alpha+I}}, \tag{7}$$

where $I = (1, \ldots, 1)$.

If

$$\overline{\lim_{|\alpha|\to\infty}} \sqrt[|\alpha|]{|\alpha_\alpha| d_\alpha(K)} < 1, \tag{8}$$

then $a_\alpha = 0$ *for all* $\alpha \in (\mathbb{Z}^+)^n$.

PROOF. Let

$$F(z) = \int\limits_K \frac{d\mu(\varsigma)}{(\varsigma_1 - z_1)\dots(\varsigma_n - z_n)}.$$

Clearly, F is analytic on $\hat{\mathbb{C}}^n \backslash \tilde{K}$. Expanding F in a power series about 0, we find

$$F(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha,$$

where the a_α are given by (7). Since (8) holds, the series converges uniformly on a complete Reinhardt domain $\mathcal{D}$ containing K ([AM] cf. [F, pp. 49-51]). Denote by U the component of $\hat{\mathbb{C}}^n \backslash \tilde{K}$ containing 0 and $(\infty, \dots, \infty)$. Clearly, F is analytic on $\mathcal{D} \cup U$. In particular, it is analytic on the union of n polydiscs about the origin, the j-th of which has j-radius equal ∞. Since the envelope of holomorphy of this union is clearly all of $\mathbb{C}^n$, the series for F converges everywhere and thus defines an entire function. But

$$\lim_{|z|\to\infty} F(z) = 0.$$

Thus F vanishes identically, so $a_\alpha = 0$ for all α such that $\alpha_j \geq 0$, $j = 1, 2, \dots, n$. □

COROLLARY 4. *Let* K *and* μ *be as in Theorem 2. If every function in* $C(K)$ *is uniformly approximable on* K *by polynomials in* $1/z_1, \dots, 1/z_n$, *then* (8) *implies* $\mu = 0$.

PROOF. According to Theorem 2, all $a_\alpha = 0$. Consequently, for every polynomial P one has

$$\int\limits_K P\left(\frac{1}{z_1}, \dots, \frac{1}{z_n}\right) d\mu = 0.$$

Thus, for every $\varphi \in C(K)$,

$$\int\limits_k \varphi d\mu = 0,$$

so $\mu = 0$. □

When K is a subset of the real subspace $\mathbb{R}^n$ of $\mathbb{C}^n$ which does not intersect the coordinate planes, the approximation condition of Corollary 4 holds (by the Stone Weierstrass Theorem). Thus we have:

COROLLARY 5. *Let K be a compact set in $\mathbb{R}^n \subset \mathbb{C}^n$ which does not intersect the coordinate planes. If* (8) *holds, then* $\mu = 0$.

PROOF. In this case, the projection of K onto each complex coordinate plane does not separate 0 and ∞ there. Thus, the points 0 and $(\infty, \dots, \infty)$ belong to the same connected component of $\hat{\mathbb{C}}^n \backslash \tilde{K}$, so the hypotheses of Theorem 2 and Corollary 4 hold, and hence $\mu = 0$. □

In analogy with Corollary 3, we have also:

COROLLARY 6. *Let f be an entire function in $\mathbb{C}^n$ such that*

$$|f(z)| \leq \gamma(1 + |z|)^N e^{r|\operatorname{Im} z|}, \quad z \in \mathbb{C}^n.$$

Suppose that for some $\rho > 0$ we have

$$\overline{\lim_{|\alpha|\to\infty}} \left(d_\alpha(B) \left| \sum_{\substack{\beta_k = 0 \\ k=1,\dots,n}}^{\alpha_k} \frac{\alpha!}{\beta!(\alpha-\beta)!} \frac{\partial^{|\beta|} f(0)}{\partial z_n^{\beta_n} \cdots \partial z_1^{\beta_1}} [-i(r+\rho)]^{|\alpha-\beta|} \right| \right)^{1/|\alpha|} < 1, \tag{9}$$

where

$$B = \left\{ x \in \mathbb{R}^n \subset \mathbb{C}^n : \left| \frac{1}{x_1} - r - \rho \right|^2 + \dots + \left| \frac{1}{x_n} - r - \rho \right|^2 \leq r^2 \right\}$$

and $\alpha! = \alpha_1! \dots \alpha_n!$. Then $f \equiv 0$.

If, in addition, $f \in L^2(\mathbb{R}^n)$, we may relax the inequality in (9) *to allow equality.*

Denote by J the collection of 2^n vectors of the form $p = (p_1, p_2, \dots, p_n)$ where $p_j = \pm 1$ for each j. A vector $p \in J$ operates on a point $z = (z_1, \dots, z_n)$ with nonzero coordinates via $p(z) = (z_1^{p_1}, z_2^{p_2}, \dots, z_n^{p_n})$. Let pK be the image of K under this mapping and set

$$a_\alpha^p = \int_{pK} \frac{d(\mu \circ p^{-1})}{\varsigma^{\alpha+I}}.$$

The following extension of Theorem 2 obtains.

THEOREM 3. *Let K and μ be as in Theorem 2 with $\mathbb{C}^n \backslash \tilde{K}$ connected. If*

$$\min_{p\in J} \overline{\lim_{|\alpha|\to\infty}} \sqrt[|\alpha|]{|a_\alpha^p| d_\alpha(pK)} < 1 \tag{10}$$

holds in place of (8), *then all the moments $a_\alpha^p = 0$ (so that $a_\beta = 0$ for all $\beta \in \mathbb{Z}^n$).*

COROLLARY 7. *If, in addition, every function in $C(K)$ is uniformly approximable on K by polynomials in $z_1, \dots, z_n$ and $1/z_1, \dots, 1/z_n$, then* (10) *implies* $\mu = 0$.

Denoting by $R(K)$ the uniform closure on K of the rational functions which are holomorphic on K, we see that when K satisfies the conditions of Corollary 7, one has $C(K) = R(K)$. It does not seem easy to find general conditions which insure this. For instance, one may have $R(K) \neq C(K)$ even when K is polynomially convex and contains no ordinary analytic structure.

EXAMPLE. Let X be a Swiss Cheese [Z1, pp. 69-70], i.e., a compact set in $\mathbb{C}$ without interior such that $R(X) \neq C(X)$. It is well-known that there exists $\varphi \in R(X)$ such that z and φ generate $R(X)$, i.e., polynomials in z and φ are uniformly dense in $R(X)$. (This is the Bishop-Hoffman Theorem; for the proof, cf. [Ro, Theorem 3.6].) We may clearly choose X and φ so that 0 belongs to the unbounded component of X and $\varphi(z) \neq 0$ on X. Now set $\Phi(z) = (z, \varphi(z))$ and put $K = \Phi(X)$. Evidently, K contains no analytic discs. Let $P(K)$ be the uniform algebra of functions uniformly approximable on K by polynomials. Now $\Phi^* F = F \circ \Phi$ defines an algebra isomorphism of $C(K)$ onto $C(X)$ which maps $P(K)$ onto $R(X)$. Since $R(X) \neq C(X)$ we have $P(K) \neq C(K)$. On the other hand, since the spectrum of the Banach algebra $R(X)$ is X, the spectrum of $P(K)$ is K, i.e., K is polynomially convex. Hence, by the Oka-Weil Theorem, $R(K) = P(K)$, so that $R(K) \neq C(K)$.

3. - Harmonic moments

In discussing harmonic moments, it will be convenient to consider the cases $n = 2$ and $n \geq 3$ separately. We begin with $n \geq 3$. Denote by B_1 the open unit ball in $\mathbb{R}^n$ and by ∂B_1 its boundary, the unit sphere. Let $\{P_{j,s}\}$ be an orthonormal basis of homogeneous harmonic polynomials in $L^2(\partial B_1)$, where j is the degree of $P_{j,s}$ and $s = 1, \dots, \sigma(j, n)$.

THEOREM 4. *Let K be a compact set in $\mathbb{R}^n$ ($n \geq 3$) which does not contain the origin, and let μ be a finite complex measure on K with moments*

$$a_{j,s} = \int_K \frac{\overline{P_{j,s}(x)} d\mu(x)}{|x|^{n+2j-2}}. \tag{11}$$

If

$$\overline{\lim_{j \to \infty}} \max_s \sqrt[j]{|a_{j,s}|} < \frac{1}{\max\limits_K |x|}, \tag{12}$$

and K does not separate 0 from ∞, then $a_{j,s} = 0$ for all j, s.

If K does separate 0 *from* ∞, *then for each collection of numbers* $\{a_{j,s}\}$ *satisfying* (12) *there is a measure on* K *such that* (11) *holds.*

PROOF. Consider the Newtonian potential

$$F(y) = \int_K \frac{d\mu(x)}{|x-y|^{n-2}}.$$

Clearly, F is harmonic on $\mathbb{R}^n \backslash K$. For the fundamental solution of the Laplace equation in $\mathbb{R}^n$ ($n \geq 3$), one has the expansion ([D], cf. [A1, Lemmas 36.3 and 38.5])

$$\frac{1}{|x-y|^{n-2}} = \frac{1}{|x|^{n-2}} + \Omega_n(n-2) \sum_{j=1}^{\infty} \sum_{s=1}^{\sigma(j,n)} \frac{P_{j,s}(y)\overline{P_{j,s}(x)}}{(n+2j-2)|x|^{n+2j-2}}, \tag{13}$$

where Ω_n is the area of the unit sphere ∂B_1, and the series on the right-hand side of (13) converges uniformly together with all derivatives on compact subsets of the cone $\{(x,y) \in \mathbb{R}^{2n} : |y| < |x|\}$. Expanding F in a series of homogeneous polynomials in a neighborhood of 0, we find

$$F(y) = \int_K \frac{d\mu(x)}{|x|^{n-2}} + \Omega_n(n-2) \sum_{j=1}^{\infty} \sum_{s=1}^{\sigma(j,n)} \frac{a_{j,s}}{n+2j-2} P_{j,s}(y),$$

where the $a_{j,s}$ are given by (11). If (12) holds, then the series converges uniformly on some ball B_R with radius $R > \max_K |x|$. Then the restriction of F to the unbounded component of $\mathbb{R}^n \backslash K$ extends to be harmonic on all of $\mathbb{R}^n$. Since $F(\infty) = 0$, it follows that $F \equiv 0$, so $a_{j,s} = 0$ for all j, s.

Now suppose that K separates 0 from ∞. Denote by U the unbounded component of $(\mathbb{R}^n \cup \{\infty\}) \backslash K$ and consider the space H of all continuous functions on $K \cup U$ which are harmonic on U. By the maximum principle, $\|f\|_H = \max_K |f(x)|$ defines a norm on H under which H is (identified with) a closed subspace of $C(K)$. Now suppose that the $a_{j,s}$ satisfy (12), so that

$$R = \frac{1}{\overline{\lim_{j\to\infty}} \max_s \sqrt[j]{|a_{j,s}|}} > \max_K |x|.$$

Choose ρ such that $\max_K |x| < \rho < R$ and set

$$\varphi(x) = \sum_{j,s} a_{j,s} \rho^{-1} P_{j,s}(x). \tag{14}$$

This series converges uniformly on each ball $\overline{B}_r$, $r < R$, and defines a harmonic function on B_R. Put

$$L(f) = \int_{\partial B_\rho} f(x)\varphi(x)d\sigma. \tag{15}$$

Clearly,

$$L(f) \leq \Omega_n \rho^{n-1} \max_{\partial B_\rho} |\varphi(x)| \max_K |f(x)| \leq C(\rho, \varphi)\|f\|.$$

Thus L defines a continuous linear functional on H, which (by the Hahn-Banach Theorem) extends to all of $C(K)$. Thus there exists a finite complex measure μ on K such that

$$L(f) = \int_K f d\mu \quad f \in H. \tag{16}$$

Now the functions

$$f_{j,s}(x) = \frac{\overline{P_{j,s}(x)}}{|x|^{n+2j-2}} \quad s = 1, 2, \cdots, \sigma(j, n) \quad j = 0, 1, 2, \cdots,$$

obtained by applying the Kelvin transformation with respect to ∂B_1 to $P_{j,s}(x)$, all belong to H. Thus by (14) and (15) we have

$$\begin{aligned} L(f_{j,s}) &= \int_{\partial B_\rho} \frac{\overline{P_{j,s}(x)}}{|x|^{n+2j-2}} \varphi(x)d\sigma \\ &= \int_{\partial B_\rho} \sum_{k,t} \frac{a_{k,t}}{\rho} \frac{1}{|x|^{n+2j-2}} \overline{P_{j,s}(x)} P_{k,t}(x)d\sigma \\ &= \int_{\partial B_1} \sum_{k,t} \frac{a_{k,t}}{\rho} \frac{1}{\rho^{n+2j-2}} \rho^j \overline{P_{j,s}(y)} \rho^k P_{k,t}(y)\rho^{n-1} d\sigma \\ &= \sum_{k,t} a_{k,t}\rho^{k-j} \int_{\partial B_1} \overline{P_{j,s}(y)} P_{k,t}(y)d\sigma \\ &= a_{j,s} \end{aligned}$$

by the orthonormality of $\{P_{j,s}\}$ on ∂B_1. It now follows from (16) that μ has the $a_{j,s}$ as its harmonic moments, i.e., that (11) holds. □

In analogy with Theorem 1′ we have the following analogue of Theorem 4.

THEOREM 4′. *Let K be a compact set in $\mathbb{R}^n$ $(n \geq 3)$ which does not contain the origin and let μ be a finite complex measure on K with moments*

$$a_{j,s} = \int_K P_{j,s}(x) d\mu(x). \tag{11′}$$

If

$$\overline{\lim_{j\to\infty}} \max_s \sqrt[j]{|a_{j,s}|} < \min_K |x|, \tag{12′}$$

and K does not separate 0 from ∞, then all the moments $a_{j,s} = 0$.

If K does separate 0 from ∞, then for each collection $\{a_{j,s}\}$ of numbers satisfying (12′) *there is a measure on K such that* (11′) *holds.*

The two-dimensional analogues of Theorems 4 and 4′ are also valid. Here one has $\sigma(j,2) = 2$ for all $j \geq 1$. One may take $P_{j,1}(x_1, x_2) = \text{Re}(x_1 + ix_2)^j$ and $P_{j,2}(x_1, x_2) = \text{Im}(x_1 + ix_2)^j$, up to a normalizing constant. Writing $x_1 + ix_2 = z$, one considers, in place of the Newtonian potential, the logarithmic potential

$$F(z) = \int_K \log|\varsigma - z| d\mu(\varsigma),$$

which is again harmonic on $\mathbb{R}^2 \backslash K$. If (12) or (12′) holds, the restriction of F to the unbounded component of $\mathbb{R}^2 \backslash K$ extends to be harmonic on all of $\mathbb{R}^2$. Evidently, $|F(z)| \leq C \log|z|$ for large $|z|$. It follows from a version of Liouville's Theorem [Bu, Corollary 6.33] that F is constant. To see that $F = 0$, observe that

$$\begin{aligned} F(\infty) = \lim_{z\to\infty} F(z) &= \lim_{z\to\infty} \int_K \log|\varsigma - z| d\mu(\varsigma) \\ &= \lim_{z\to\infty} \int_K \log|z| d\mu(\varsigma) + \lim_{z\to\infty} \int_K \log\left|1 - \frac{\varsigma}{z}\right| d\mu(\varsigma) \\ &= \mu(K) \lim_{z\to\infty} \log|z|, \end{aligned}$$

since $\log\left|1 - \frac{\varsigma}{z}\right|$ tends uniformly to 0 on K as $z \to \infty$. But the left hand side is finite, so we must have $\mu(K) = 0$. Thus $F(\infty) = 0$ and F vanishes identically.

Denote by $h(K)$ the hull of K, i.e., the union of K with all the bounded components of its complement. Recall ([L, p. 307]) that a set E is *thin* at $x_0 \in \mathbb{R}^n$ if either E does not have x_0 as a limit point or there exists a function v superharmonic on $\mathbb{R}^n$ such that

$$v(x_0) < \underline{\lim_{x\to x_0}} v(x) \quad x \in E \backslash \{x_0\}. \tag{17}$$

COROLLARY 8. *Suppose that* $\operatorname{int} K = \emptyset$ *and* $\mathbb{R}^n \backslash h(K)$ *is not thin at any point of* K. *Then if* $0 \notin K$ *and* K *does not separate* 0 *from* ∞, (12) *and* (12′) *each imply that* $\mu \equiv 0$.

This follows from Theorem 4 and 4′ together with the fact ([Br], cf. [D]) that in this case every function in $C(K)$ is the uniform limit on K of harmonic polynomials.

As an immediate consequence we have:

COROLLARY 9. *Suppose that* K *has zero Lebesgue measure in* $\mathbb{R}^n$, $\mathbb{R}^n \backslash K$ *is connected, and* $0 \notin K$. *Then* (12) *and* (12′) *each imply that* $\mu \equiv 0$.

Indeed, since $\mathbb{R}^n \backslash K$ is connected, $h(K) = K$. But $\mathbb{R}^N \backslash K$ is nowhere thin if K has Lebesgue measure 0, since otherwise (17) would be inconsistent with the super-mean-value property of superharmonic functions for balls around x_0.

4. - Final comments

This work is in large measure a continuation of [A2], where problems of Morera type were considered for non-closed curves and pieces of hypersurfaces. For further information on such problems see [Z2], [Z3], and [BCPZ]. Additional motivation for the questions considered here is in [AR] and the papers listed there.

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