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On Veech's Conjecture for Harmonic Functions

W. HANSEN - N. NADIRASHVILI

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

0. - Introduction

Let λ denote the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$. For every $x \in \mathbb{R}^d$ and $r > 0$ let $B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}$ and $\lambda_{B(x,r)} = \lambda(B(x,r))^{-1} 1_{B(x,r)} \lambda$. A function $r > 0$ on a domain U in $\mathbb{R}^d$ is called *admissible* provided $B(x, r(x)) \subset U$ for every $x \in U$. Given an admissible function r on U, let us say that a Lebesgue measurable real function f on U is *r-median* if

$$f(x) = \lambda_{B(x,r(x))}(f)$$

for every $x \in U$. In [HN1, HN2, HN5] we proved the following converse to the mean value theorem for harmonic functions (for the case $U = \mathbb{R}^d$ see [HN4]):

THEOREM 0.1. *Let r be an admissible function on a proper subdomain U of $\mathbb{R}^d$. Let f be an r-median function on U which is bounded by some harmonic function on U and suppose that f is continuous or that r is locally bounded away from zero. Then f is harmonic.*

Simple counterexamples reveal that the boundedness condition for f cannot be completely dropped. However, under additional assumptions on r boundedness from one side is sufficient. Work of J.R. Baxter [Ba2], A. Cornea and J. Veselý [CV], W.A. Veech [Ve3], led to the following (for a detailed account of the history see [NV]):

THEOREM 0.2. *Let U be a Green domain in $\mathbb{R}^d$, let $\rho, r : U \to]0, \infty[$ and $\alpha > 0$ be such that, for all $x, y \in U$:*

$$\rho(x) \leq \operatorname{dist}(x, \partial U), \quad |\rho(x) - \rho(y)| \leq |x - y|, \quad \alpha\rho(x) < r(x) < (1-\alpha)\rho(x).$$

Then any r-median function $f \geq 0$ on U is harmonic.

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About twenty years ago W.A. Veech ([Ve3]) formulated the:

CONJECTURE 0.3. *Let U be a bounded domain in $\mathbb{R}^d$ and let r be an admissible function on U which is locally bounded away from zero, i.e., such that* $\inf r(K) > 0$ *for any compact subset K of U. Then every r-median function $f \geq 0$ on U is harmonic.*

Previous work by F. Huckemann [Hu] shows that this is true if $d = 1$.

In this paper we shall see, however, that the conjecture fails already for open balls in any $\mathbb{R}^d$, $d \geq 2$. In fact, even a weakened version of the conjecture where r and f are assumed to be continuous (or C^∞) is wrong. As in [HN3] our counterexample will be based on properties of the random walk given by the transition kernel $P : (x, A) \mapsto \lambda_{B(x,r(x))}(A)$.

1. - A measurable counterexample

Let U denote the open unit ball in $\mathbb{R}^d$, $d \geq 2$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\Omega = U^{\mathbb{N}_0}$, $X_i(\omega) = \omega_i$ and let $\mathcal{M}$ be the σ-algebra on Ω generated by X_i, $i \in \mathbb{N}_0$. As usual θ_j, $j \in \mathbb{N}_0$, will be the canonical shift $\theta_j : \Omega \to \Omega$ defined by $(\theta_j \omega)_i = \omega_{i+j}$, i.e., $X_i \circ \theta_j = X_{i+j}$. Given a Markov kernel P on U and $x \in U$, let P^x denote the probability measure on $(\Omega, \mathcal{M})$ such that $(\Omega, \mathcal{M}, P^x)$ is the random walk starting at x having transition kernel P, i.e., for all $n \in \mathbb{N}_0$ and Borel subsets $A_0, A_1, \ldots, A_n$ of U

$$
\begin{aligned}
&P^x[X_0 \in A_0, X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n] \\
(*)\qquad &= \varepsilon_x(A_0) \int_{A_1} P(x, dx_1) \int_{A_2} P(x_1, dx_2) \ldots \int_{A_n} P(x_{n-1}, dx_n).
\end{aligned}
$$

For every Borel subset A of U let

$$T_A := \inf\{i \in \mathbb{N}_0 : X_i \in A\}$$

(where $\inf \emptyset = \infty$). A Borel function $v \geq 0$ on U is called P-supermedian if $Pv \leq v$. We recall that the function $x \mapsto P^x[T_A < \infty]$ is the smallest P-supermedian function v on U such that $v \geq 1$ on A ([DM], [Re]). Moreover, we shall use the (strong) Markov property for random walks (cf. [DM], [Re]) and we shall exploit the following simple fact which is intuitively clear and can easily be derived formally from (*): If P and Q are two Markov kernels on U such that $P(x, \cdot) = Q(x, \cdot)$ for every x in the complement of a Borel set B then

$$P^x[X_0 \in A_0, \ldots, X_n \in A_n, n \leq T_B] = Q^x[X_0 \in A_0, \ldots, X_n \in A_n, n \leq T_B]$$

for all $x \in U$, $n \in \mathbb{N}_0$, and Borel sets $A_0, \ldots, A_n$ in U. In particular, $P^x[T_B = n] = Q^x[T_B = n]$ for every $n \geq 0$.

Now let us choose points $x_n = (\xi_n, 0, \ldots, 0) \in \mathbb{R}^d$ and radii $0 < \rho_n < 1$, $n \in \mathbb{N}_0$, such that $\xi_0 = 0$, $\xi_n < \xi_{n+1}$, $\lim_{n\to\infty} \xi_n = 1$,

$$\{x_{n+1}\} = B(x_n, \rho_n) \cap \{x_n : j \neq n\}, \qquad x_{n+1} \notin B(x_n, \rho_n/2).$$

More precisely, fix $0 < \alpha < 1/4$, define

$$\alpha_n := (1-\alpha)^n \alpha, \qquad n = -1, 0, 1, 2, \ldots$$

and, for every $n \in \mathbb{N}_0$,

$$\xi_n := \sum_{j=0}^{n-1} \alpha_j = 1 - (1-\alpha)^n, \quad x_n := (\xi_n, 0, \ldots, 0) \in \mathbb{R}^d,$$

$$\rho_n := \frac{\alpha_{n-1} + \alpha_n}{2} = \left(1 - \frac{\alpha}{2}\right) \alpha_{n-1}.$$

Then, for every $n \in \mathbb{N}_0$,

$$\alpha_n - \frac{\rho_n}{2} = (1-a)\alpha_{n-1} - \frac{\rho_n}{2} > (1-\alpha)\,\rho_n - \frac{\rho_n}{2} > \alpha\,\rho_n,$$

$$\alpha_n + \alpha_{n+1} = (1 - \alpha + (1-\alpha)^2)\alpha_{n-1} > (2 - 3\alpha)\alpha_{n-1} > \alpha_{n-1},$$

$$\alpha_{n-1} - \rho_n = \left(\frac{1}{1 - \frac{\alpha}{2}} - 1 \right) \rho_n > \frac{\alpha}{2}\,\rho_n,$$

$$\rho_n - \alpha_n = \left(1 - \frac{1-\alpha}{1 - \frac{\alpha}{2}} \right) \rho_n > \frac{\alpha}{2}\,\rho_n\,.$$

Choosing real numbers $c_n > 0$ such that $c_n \leq \inf(\alpha/5, 2^{-n})\rho_n$ and defining

$$C_n := B(x_n, c_n), \qquad n = 0, 1, 2, \ldots,$$

we hence obtain that, for every $x \in C_0$,

$$B(x, \rho_0/2) \cap C_1 = B(x, \rho_0) \cap C_2 = \emptyset, \qquad C_1 \subset B(x, \rho_0)$$

and, for every $x \in C_n$, $n \geq 1$,

$$B(x, \rho_n/2) \cap C_{n+1} = B(x, \rho_n) \cap (C_{n-1} \cup C_{n+2}) = \emptyset, \qquad C_{n+1} \subset B(x, \rho_n).$$

Moreover, for every $n \in \mathbb{N}_0$ and every $x \in C_n$,

$$\rho_n \leq 2\alpha(1 - |x|)$$

since $2\alpha(1 - |x|) \geq 2\alpha(1 - \xi_n - c_n) = 2\alpha_n - 2\alpha c_n \geq \alpha_{n-1} - 2\alpha c_n \geq \rho_n + \frac{\alpha}{2}\rho_n - 2\alpha c_n \geq \rho_n$.

Take $0 < b_0 < c_0/2$, $a_0 = b_0/2$, and define

$$A_0 = \overline{B(0, a_0)}, \qquad B_0 = B(0, b_0).$$

In order to understand the idea of our counterexample let us assume for a moment that we have chosen real numbers $0 < a_n < c_n$, $n \in \mathbb{N}$. Then we take $A_n = \overline{B(x_n, a_n)}$ and we may define a measurable function $f \geq 0$ on U by

$$f = \begin{cases} \prod_{j=1}^{n} \dfrac{\rho_{j-1}^d - a_{j-1}^d}{a_j^d} & \text{on } A_n, n \geq 0, \\ 0 & \text{on } U \setminus \bigcup_{n=0}^{\infty} A_n. \end{cases}$$

Obviously, f is not harmonic, but it is r-median if we define

$$r(x) = \begin{cases} \rho_n, & x \in A_n, n \geq 0, \\ \inf\left(\alpha(1 - |x|),\ \operatorname{dist}\left(x, \bigcup_{n=0}^{\infty} A_n\right)\right), & x \in U \setminus \bigcup_{n=0}^{\infty} A_n. \end{cases}$$

Of course, r is not locally bounded away from zero. We may, however, modify this construction and obtain an r-median function for which r is locally bounded away from zero. To that end we shall arrange that the random walk given by the kernel

$$P : (x, A) \mapsto \lambda_{B(x, r(x))}(A)$$

has almost no chance to get to A_{n+1} except to go to A_n first and then to hit A_{n+1} at the next step.

Let us see how this can be achieved. We choose a continuous function $0 < \rho \leq \rho_0$ on $U \setminus \{x_n : n \in \mathbb{N}\}$ such that $\rho = \rho_0$ on A_0 and

$$\rho(x) \leq \inf\left(\alpha(1 - |x|),\ \frac{1}{3} \inf_{n \in \mathbb{N}} |x - x_n|\right)$$

for every $x \in U \setminus (B_0 \cup \{x_n : n \in \mathbb{N}\})$, take

$$C_n' = B(x_n, c_n/2), \qquad C' = \bigcup_{n=1}^{\infty} C_n',$$

and consider the Markov kernel Q given by

$$Q(x,\cdot) = \begin{cases} \lambda_{B(x,\rho(x))}, & x \in U \setminus C', \\ \varepsilon_x, & x \in C'. \end{cases}$$

Using the associated random walk we define a function g on U by

$$g(x) = Q^x[T_{A_0} < \infty], \quad x \in U.$$

Obviously,

$$\liminf_{|x| \downarrow a_0} g(x) \geq \lim_{|x| \mapsto a_0} Q^x[X_1 \in A_0] = \left(\frac{a_0}{\rho_0}\right)^d > 0.$$

Since ρ is continuous and $Qg = g$ on $U \setminus A_0$, we know that g is continuous on $U \setminus (A_0 \cup C')$ and that $\{y \in U \setminus (A_0 \cup C') : g(y) = 0\}$ is an open set in $U \setminus (A_0 \cup C')$. Therefore $g > 0$ on $U \setminus C'$ and, for every $n \in \mathbb{N}$,

$$\beta_n := \inf\left\{g(y) : \frac{c_n}{2} \leq |y - x_n| \leq c_n\right\} > 0.$$

Now let $n \in \mathbb{N}_0$ and suppose that real numbers $a_j \in \left]0, c_j/4\right[$, $j = 0, \ldots, n$, have already been chosen. Define

$$\gamma_n := \prod_{j=1}^{n} \left(\frac{\rho_{j-1}}{a_j}\right)^d$$

(in particular, $\gamma_0 = 1$). Then there exists $b_{n+1} \in \left]0, c_{n+1}/2\right[$ such that $B_{n+1} := B(x_{n+1}, b_{n+1})$ satisfies

$$\begin{aligned} R_1^{B_{n+1}} &:= \inf\{s : s \geq 0 \text{ superharmonic on } U, s \geq 1 \text{ on } B_{n+1}\} \\ &\leq 2^{-(n+2d)} \beta_{n+1}/\gamma_n \quad \text{on } U \setminus C'_{n+1}. \end{aligned}$$

Take

$$a_{n+1} = \frac{b_{n+1}}{2}, \qquad A_{n+1} = \overline{B(x_{n+1}, a_{n+1})}.$$

By Harnack's inequality there exists $c \in \mathbb{R}^+$ such that for every harmonic function $h \geq 0$ on U

$$h(x_2) \leq ch(x_1)$$

and we clearly may assume that b_2 and hence a_2 are chosen so small that

$$\gamma_2 > (e^4\gamma_1 + 1)c.$$

Having obtained the sequences of balls (A_n) and (B_n) we define $r : U \to]0,1[$ by

$$r(x) := \begin{cases} \rho(x), & x \in U \setminus \bigcup_{n=1}^{\infty} B_n, \\ \rho_n, & x \in A_n, n \in \mathbb{N}, \\ \rho_n/2, & x \in B_n \setminus A_n, n \in \mathbb{N}. \end{cases}$$

Then $r(x) \leq 2\alpha(1-|x|)$ for every $x \in U$ and $\inf r(K) > 0$ for every compact K in U.

Let P denote the corresponding Markov kernel, i.e.,

$$P(x, \cdot) = \lambda_{B(x,r(x))}, \qquad x \in U.$$

Using the associated random walk we define measurable functions f_n on U, $n \in \mathbb{N}_0$, by

$$f_n(x) := \gamma_n P^x[T_{A_n} < \infty], \qquad x \in U.$$

Obviously, $0 \leq f_n \leq \gamma_n$. The sequence (f_n) is increasing since

$$\begin{aligned} f_{n+1}(x) &\geq \gamma_{n+1} P^x[T_{A_n} < \infty, X_{T_{A_n}+1} \in A_{n+1}] \\ &= \gamma_{n+1} \int\limits_{[T_{A_n} < \infty]} P^{X_{T_{A_n}}}[X_1 \in A_{n+1}]\, dP^x \\ &= \gamma_{n+1} \left(\frac{a_{n+1}}{\rho_n}\right)^d P^x[T_{A_n} < \infty] = \gamma_n P^x[T_{A_n} < \infty] = f_n(x) \end{aligned}$$

for every $x \in U$. Let

$$f := \lim_{n \to \infty} f_n.$$

Since obviously $Pf_n(x) = f_n(x)$ for every $n \in \mathbb{N}_0$ and every $x \in U \setminus A_n$, we conclude that $Pf = f$.

We intend to show that, for every $n \in \mathbb{N}$,

$$(*) \qquad f_{n+1} \leq (1 + 2^{2-n}) f_n \qquad \text{on } U \setminus C_{n+1}.$$

Since $\prod\limits_{n=1}^{\infty}(1 + 2^{2-n}) \leq \exp\left(\sum\limits_{n=1}^{\infty} 2^{2-n}\right) = e^4$, we then obtain that, for every $n \in \mathbb{N}$,

$$f \leq e^4 f_n \qquad \text{on } U \setminus \bigcup_{j=n+1}^{\infty} C_j.$$

In particular, f turns out to be locally bounded on U and

$$f(x_2) \geq f_2(x_2) = \gamma_2 > ce^4\gamma_1 = ce^4 f_1(x_1) \geq cf(x_1),$$

so f is not harmonic. Thus (*) will yield that f is a counterexample to Veech's conjecture.

In order to prove (*) let us first establish a general lemma (for the purpose of this section it will be sufficient to take $E = \emptyset$, i.e., $T_E = \infty$):

LEMMA. 1.1. *Let P be a Markov kernel on U, let A, B, C, A_0, C_0, E be Borel subsets of U such that $A \subset B \subset C \subset U \setminus C_0$, $A_0 \subset C_0$ and let*

$0 < \delta \le \varepsilon < 1/6$ such that $P(y, A) = 0$ for every $y \in U \backslash (A_0 \cup B \cup E)$, $P(y, C_0) \le \varepsilon$ for every $y \in A_0$, $P^y[T_{A_0} < \infty] \le \varepsilon$ for every $y \in U \backslash C_0$, $P(y, C) \le \varepsilon$ for every $y \in B$, and $P^y[T_B < T_E] \le \delta P^y[T_{A_0} < \infty]$ for every $y \in U \backslash C$. Then for every $x \in U \backslash C$

$$P^x[T_A < T_E] \le (1 + 3\varepsilon)\left(\sup_{y \in A_0} P(y, A) + \delta \sup_{y \in B \backslash E} P(y, A)\right) P^x[T_{A_0} < \infty].$$

PROOF. Fix $x \in U \backslash C$ and let

$$S = \inf\{j \in \mathbb{N} : X_j \in A_0\}.$$

For every $y \in A_0$,

$$\begin{aligned} P^y[X_1 \notin C_0, S < \infty] &= P^y[X_1 \notin C_0, T_{A_0} \circ \theta_1 < \infty] \\ &= \int_{[X_1 \notin C_0]} P^{X_1}[T_{A_0} < \infty]\, dP^y \le \varepsilon, \end{aligned}$$

hence

$$P^y[S < \infty] \le P^y[X_1 \in C_0] + P^y[X_1 \notin C_0, S < \infty] \le 2\varepsilon.$$

We define an increasing sequence (S_m) of stopping times by

$$S_1 := T_{A_0}, \quad S_{m+1} := S_m + S \circ \theta_{S_m}.$$

Clearly, for every $\omega \in \Omega$,

$$\{k \in \mathbb{N}_0 : X_k(\omega) \in A_0\} = \{S_m(\omega) : m \in \mathbb{N}, S_m(\omega) < \infty\}.$$

For every $m \in \mathbb{N}$,

$$\begin{aligned} P^x[S_m < \infty, X_{S_m+1} \in A] &= \int_{[S_m < \infty]} P^{X_{S_m}}[X_1 \in A]\, dP^x \\ &\le \sup_{y \in A_0} P(y, A) P^x[S_m < \infty]. \end{aligned}$$

Moreover,

$$\begin{aligned} P^x[S_{m+1} < \infty] &= P^x[S_m < \infty, S \circ \theta_{S_m} < \infty] \\ &= \int_{[S_m < \infty]} P^{X_{S_m}}[S < \infty]\, dP^x \le 2\varepsilon P^x[S_m < \infty], \end{aligned}$$

hence by induction

$$P^x[S_{m+1} < \infty] \le (2\varepsilon)^m P^x[S_1 < \infty].$$

Therefore

$$\sum_{m=1}^{\infty} P^x[S_m < \infty, X_{S_m+1} \in A] \leq P^x[T_{A_0} < \infty] \sup_{y \in A_0} P(y, A) \sum_{k=0}^{\infty} (2\varepsilon)^k$$

where $\sum_{k=0}^{\infty} (2\varepsilon)^k = (1-2\varepsilon)^{-1} \leq 1 + 3\varepsilon$ since $\varepsilon \leq \frac{1}{6}$.

Define similarly a sequence (T_m) by

$$T := \inf\{j \in \mathbb{N} : X_j \in B\}, \quad T_1 = T_B, \quad T_{m+1} = T_m + T \circ \theta_{T_m}.$$

We know that $P(y, C) \leq \varepsilon$ for every $y \in B$ and $P^y[T_B < T_E] \leq \delta P^x[T_{A_0} < \infty] \leq \delta \leq \varepsilon$ for every $y \in U \backslash C$. Arguing in a similar way as for the sequence (S_m) we hence obtain that

$$\sum_{m=1}^{\infty} P^x[T_m < T_E, X_{T_m+1} \in A] \leq (1+3\varepsilon) P^x[T_B < T_E] \sup_{y \in B \backslash E} P(y, A)$$

$$\leq (1+3\varepsilon)\delta P^x[T_{A_0} < \infty] \sup_{y \in B \backslash E} P(y, A).$$

To finish the proof it suffices to note that, for every $k \in \mathbb{N}_0$,

$$P^x[X_k \notin A_0 \cup B, X_{k+1} \in A, k < T_E] = \int_{[X_k \notin A_0 \cup B, k < T_E]} P^{X_k}[X_1 \in A]\, dP^x = 0$$

since $P(y, A) = 0$ for every $y \in U \backslash (A_0 \cup B \cup E)$, and hence

$$P^x[T_A < T_E] = P^x[1 \leq T_A < T_E]$$

$$\leq \sum_{m=1}^{\infty} (P^x[S_m < \infty, X_{S_m+1} \in A] + P^x[T_m < T_E, X_{T_m+1} \in A]).$$

□

PROPOSITION 1.2. *For every* $n \in \mathbb{N}$,

$$f_{n+1} \leq (1 + 2^{2-n}) f_n \quad \text{on } U \backslash C_{n+1}.$$

PROOF. We know by construction of P that

$$P(y, A_{n+1}) = 0 \quad \text{for every } y \in U \backslash (A_n \cup B_{n+1}),$$

$$P(y, C_n) = \left(\frac{c_n}{\rho_n}\right)^d \leq 2^{-nd} \leq 2^{-(n+1)} \quad \text{for every } y \in A_n,$$

$$P(y, C_{n+1}) \leq \left(\frac{2c_{n+1}}{\rho_{n+1}}\right)^d \leq 2^{-nd} \leq 2^{-(n+1)} \quad \text{for every } y \in B_{n+1}.$$

In order to get the necessary estimates for $P^y[T_{A_n} < \infty]$ and $P^y[T_{B_{n+1}} < \infty]$ we note that every superharmonic function $s \geq 0$ on U is P-supermedian and hence for every Borel subset D of U

$$P^{\cdot}[T_D < \infty] = \inf\{s : s \ P\text{-supermedian}, \ s \geq 1 \text{ on } D\} \leq R_1^D.$$

In particular,

$$P^{\cdot}[T_{A_n} < \infty] \leq R_1^{A_n} \leq R_1^{B_n} \leq 2^{-(n+1)} \qquad \text{on } U\backslash C_n$$

and

$$P^{\cdot}[T_{B_{n+1}} < \infty] \leq R_1^{B_{n+1}} \leq 2^{-(n+2d)}\beta_{n+1}/\gamma_n \qquad \text{on } U\backslash C'_{n+1}.$$

On the other hand, the random walk associated with Q is obtained from the random walk associated with P by stopping on C'. Hence clearly

$$P^{\cdot}[T_{A_0} < \infty] \geq Q^{\cdot}[T_{A_0} < \infty] = g.$$

Moreover, for every $y \in U$,

$$\begin{aligned} P^y[T_{A_n} < \infty] &\geq P^y[T_{A_0} < \infty, X_{T_{A_0}+1} \in A_1, \dots, X_{T_{A_0}+n} \in A_n] \\ &= \int_{[T_{A_0}<\infty]} P^{X_{T_{A_0}}}[X_1 \in A_1, \dots, X_n \in A_n]\, dP^y \\ &= \left(\frac{a_1}{\rho_0}\right)^d \cdot \dots \cdot \left(\frac{a_n}{\rho_{n-1}}\right)^d P^y[T_{A_0} < \infty] \\ &= \gamma_n^{-1} P^y[T_{A_0} < \infty]. \end{aligned}$$

Thus, for every $y \in C_{n+1}\backslash C'_{n+1}$,

$$\begin{aligned} P^y[T_{A_n} < \infty] &\geq \gamma_n^{-1} g(y) \geq \beta_{n+1}/\gamma_n \\ &\geq 2^{n+2d} R_1^{B_{n+1}}(y) \geq 2^{n+2d} P^y[T_{B_{n+1}} < \infty]. \end{aligned}$$

In addition, for every $y \in A_n$,

$$P^y[T_{A_n} < \infty] = 1 \geq 2^{n+2d} R_1^{B_{n+1}}(y) \geq 2^{n+2d} P^y[T_{B_{n+1}} < \infty].$$

Let

$$R := T_{A_n \cup C_{n+1}} = \inf(T_{A_n}, T_{C_{n+1}})$$

and fix $x \in U\backslash C_{n+1}$. Since $P(y, C'_{n+1}) = 0$ for every $y \in U\backslash(A_n \cup C_{n+1})$ we know that

$$X_R \in A_n \cup (C_{n+1}\backslash C'_{n+1}) \quad P^x\text{-a.s. on } [R < \infty].$$

(Indeed,

$$\begin{aligned}
P^x[R<\infty, X_R\in C'_{n+1}] &= P^x[1\le R<\infty, X_R\in C'_{n+1}]\\
&\le \sum_{k=0}^{\infty} P^x[X_k\in \complement(A_n\cup C_{n+1}), X_{k+1}\in C'_{n+1}]\\
&= \sum_{k=0}^{\infty} \int\limits_{[X_k\in\complement(A_n\cup C_{n+1})]} P^{X_k}[X_1\in C'_{n+1}]\,dP^x = 0).
\end{aligned}$$

Obviously, $T_{A_n} = R + T_{A_n}\circ\theta_R$ and $T_{C_{n+1}} = R + T_{C_{n+1}}\circ\theta_R$, hence the strong Markov property implies that

$$P^x[T_{A_n}<\infty] = \int\limits_{[R<\infty]} P^{X_R}[T_{A_n}<\infty]\,dP^x$$

and

$$P^x[T_{B_{n+1}}<\infty] = \int\limits_{[R<\infty]} P^{X_R}[T_{B_{n+1}}<\infty]\,dP^x.$$

Since

$$P^y[T_{A_n}<\infty] \ge 2^{n+2d} P^y[T_{B_{n+1}}<\infty]$$

for every $y\in A_n\cup(C_{n+1}\backslash C'_{n+1})$ we conclude that

$$P^x[T_{A_n}<\infty] \ge 2^{n+2d} P^x[T_{B_{n+1}}<\infty].$$

Furthermore, $P(y, A_{n+1}) = \left(\frac{a_{n+1}}{\rho_n}\right)^d$ for every $y\in A_n$, whereas, for every $y\in B_{n+1}$,

$$P(y, A_{n+1}) \le \left(\frac{2a_{n+1}}{\rho_{n+1}}\right)^d \le 2^{2d}\left(\frac{a_{n+1}}{\rho_n}\right)^d.$$

Therefore by our Lemma

$$\begin{aligned}
&f_{n+1}(x) = \gamma_{n+1} P^x[T_{A_{n+1}}<\infty]\\
&\le (1+3\cdot 2^{-(n+1)})(1+2^{-n})\gamma_{n+1}\left(\frac{a_{n+1}}{\rho_n}\right)^d P^x[T_{A_n}<\infty] \le (1+2^{2-n}) f_n(x).
\end{aligned}$$

□

Thus we have proven the following result which shows that Veech's conjecture is wrong:

THEOREM 1.3. *For every* $0<\alpha\le 1$ *there exist Borel functions* $r, f>0$ *on* U *such that* $r\le\alpha\,\mathrm{dist}(\cdot,\partial U)$, $\inf r(K)>0$ *for every compact subset* K *of* U, f *is locally bounded and* r*-median, but not harmonic on* U.

2. - A continuous counterexample

Having constructed a measurable counterexample to Veech's conjecture the question arises if perhaps a weakened version is true where r and f are supposed to be continuous. In this section we shall see that this is not the case:

THEOREM 2.1. *Given $0 < \alpha \leq 1$, there exist continuous strictly positive functions r and f on U such that $r \leq \alpha\, \mathrm{dist}(\cdot, \partial U)$, f is r-median, but not harmonic.*

In order to get this result it suffices to modify the measurable counterexample removing the discontinuities at $\partial A_n \cup \partial B_n$, $n \in \mathbb{N}$. To that end we shall use a general property of random walks given by means having a locally bounded density with respect to the Lebesgue measure (cf. a similar argument in [HN3]):

LEMMA 2.2. *Let P be the transition kernel of a random walk given by an admissible function r on U which is locally bounded away from zero, let K be a compact subset of U, fix $x \in U$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $P^x[0 < T_A < \infty] < \varepsilon$ for every Borel subset A of K satisfying $\lambda_U(A) < \delta$.*

PROOF. Define $q(y) = 1 - |y|$, $y \in U$, and let $\gamma := \inf q(K) = \mathrm{dist}(K, \partial U)$. Since q is a continuous potential on U, we know by Lemma 1 in [HN5] that $\lim_{m\to\infty} P^m q = 0$. So there exists $m \in \mathbb{N}$ such that

$$P^m q(x) < \frac{\varepsilon^2}{9}\gamma.$$

Then

$$P^x\left[q(X_m) \geq \frac{\varepsilon}{3}\gamma\right] \leq \left(\frac{\varepsilon}{3}\gamma\right)^{-1} P^m q(x) < \frac{\varepsilon}{3}.$$

Moreover, for every $y \in \left\{q < \frac{\varepsilon}{3}\gamma\right\}$,

$$P^y[T_{\{q\geq\gamma\}} < \infty] \leq R_1^{\{q\geq\gamma\}}(y) \leq \frac{1}{\gamma}\, q(y) < \frac{\varepsilon}{3}.$$

Therefore

$$\begin{aligned} &P^x[X_i \in K \text{ for some } i \geq m] \leq P^x[q(X_i) \geq \gamma \text{ for some } i \geq m] \\ &\leq P^x\left[q(X_m) \geq \frac{\varepsilon}{3}\gamma\right] + P^x\left[q(X_m) < \frac{\varepsilon}{3}\gamma, q(X_i) \geq \gamma \text{ for some } i \geq m\right] \\ &\leq \frac{\varepsilon}{3} + \int_{[q(X_m)<\frac{\varepsilon}{3}\gamma]} P^{X_m}\left[T_{\{q\geq\gamma\}} < \infty\right] dP^x < \frac{2}{3}\varepsilon. \end{aligned}$$

Let

$$\eta := \inf\{r(y) : q(y) \geq \gamma/2\}.$$

If $y \in U$ such that $B(y, r(y)) \cap K \neq \emptyset$ then $q(y) > \gamma/2$ and hence $P(y, \cdot) = \lambda_{B(y,r(y))} \leq \eta^{-d} \lambda_U$. This implies that, for every $i \in \mathbb{N}$,

$$P^x_{X_i}\big|_K \leq \eta^{-d} \lambda_U .$$

Take

$$\delta := \frac{\eta^d}{3m} \varepsilon$$

and let A be a Borel subset of K such that $\lambda_U(A) < \delta$. Then

$$\begin{aligned} P^x[0 < T_A < \infty] &\leq \sum_{i=1}^{m} P^x[X_i \in A] + P^x[X_i \in K \text{ for some } i \geq m] \\ &\leq m \cdot \eta^{-d} \delta + \frac{2}{3} \varepsilon = \varepsilon. \end{aligned}$$

□

Let us now return to the situation considered in the previous section. Defining

$$\varepsilon_n := 2^{-(n+1)} \gamma_{n+1}^{-1}$$

we know that for every $n \in \mathbb{N}$,

$$\sum_{j=n}^{\infty} \varepsilon_j \leq \gamma_{n+1}^{-1} \sum_{j=n}^{\infty} 2^{-(j+1)} = 2^{-n} \gamma_{n+1}^{-1}.$$

Let $V = U \setminus \bigcup_{j=1}^{\infty} (B_j \setminus A_j)$, fix $n \in \mathbb{N}$, and suppose that we have already defined a continuous function $\tilde{r}$ on $V \cup \bigcup_{j=1}^{n-1} (B_j \setminus A_j)$ such that $\tilde{r} = r$ on V and $0 < \tilde{r} \leq \rho_j$ on $B_j \setminus A_j$, $j = 0, 1, \ldots, n-1$. Let P_n denote the transition kernel on U given by $\tilde{r}$ on $V \cup \bigcup_{j=1}^{n-1} (B_j \setminus A_j)$ and by r on $\bigcup_{j=n}^{\infty} (B_j \setminus A_j)$, i.e.,

$$P_n(x, \cdot) = \begin{cases} \lambda_{B(x,\tilde{r}(x))}, & x \in U \setminus \bigcup_{j=n}^{\infty} (B_j \setminus A_j), \\ \lambda_{B(x,r(x))}, & x \in \bigcup_{j=n}^{\infty} (B_j \setminus A_j). \end{cases}$$

By Lemma 2.2 we know that, for each $x \in \overline{B}_{n-1} \cup (\overline{C}_n \setminus C'_n)$, there exist b'_n, $b''_n \in]a_n, b_n[$ such that the set

$$E_n = \{y \in B_n \setminus A_n : |y - x_n| < b'_n \text{ or } |y - x_n| > b''_n\}$$

satisfies

$$e_n(x) := P_n^x[T_{E_n} < \infty] < \varepsilon_n.$$

By the strong Markov property

$$P_n e_n = e_n \qquad \text{on } U \backslash B_n.$$

Since $\tilde{r}$ is continuous on $\overline{B}_{n-1} \cup (\overline{C}_n \backslash C_n')$ we conclude that e_n is continuous on $\overline{B}_{n-1} \cup (\overline{C}_n \backslash C_n')$. So a simple compactness argument shows that we may choose b_n', b_n'' such that $e_n < \varepsilon_n$ on $\overline{B}_{n-1} \cup (\overline{C}_n \backslash C_n')$. In fact,

$$e_n < \varepsilon_n \qquad \text{on } U \backslash C_n'$$

since $T_{\overline{B}_{n-1} \cup (\overline{C}_n \backslash C_n')} \leq T_{B_n} \leq T_{E_n}$ P_n^x-a.s. for every $x \in U \backslash C_n$.

We now extend $\tilde{r}$ to a continuous function on $V \cup \bigcup_{j=1}^{n} (B_j \backslash A_j)$ such that

$$\tilde{r} = r = \frac{\rho_n}{2} \quad \text{on } (B_n \backslash A_n) \backslash E_n, \qquad 0 < \tilde{r} \leq \rho_n \quad \text{on } E_n.$$

By induction we obtain an admissible function $\tilde{r}$ on U such that

$$\tilde{r} = r \qquad \text{on } U \backslash \bigcup_{n=1}^{\infty} E_n.$$

Define the Markov kernel $\tilde{P}$ on U by

$$\tilde{P}(x, \cdot) = \lambda_{B(x, \tilde{r}(x))}, \qquad x \in U.$$

Using the corresponding random walk we obtain functions $\tilde{f}_n$ on U, $n \in \mathbb{N}_0$, by

$$\tilde{f}_n(x) := \gamma_n \tilde{P}^x[T_{A_n} < \infty], \qquad x \in U.$$

Repeating the argument we used for the sequence (f_n) we obtain that the sequence $(\tilde{f}_n)$ is increasing (note that $\tilde{r} = \rho_n$ on A_n, $n \in \mathbb{N}_0$) and that

$$\tilde{f} := \sup \tilde{f}_n$$

satisfies

$$\tilde{P}\tilde{f} = \tilde{f}.$$

PROPOSITION 2.3. *For every $n \in \mathbb{N}$,*

$$\tilde{f}_{n+1} \leq (1 + 2^{2-n})\tilde{f}_n + 2^{-n} \quad \text{on } U \backslash C_{n+1}.$$

PROOF. Fix $n \in \mathbb{N}$ and let

$$F_n = \bigcup_{j=n}^{\infty} E_j.$$

By construction of $\tilde{P}$ we know that

$$\tilde{P}(y, A_{n+1}) = 0 \qquad \text{for every } x \in U \setminus (A_n \cup B_{n+1} \cup F_n).$$

Moreover,

$$\tilde{P}(y, C_n) = \left(\frac{c_n}{\rho_n}\right)^d \leq 2^{-(n+1)} \qquad \text{for every } y \in A_n,$$

$$\tilde{P}(y, C_{n+1}) \leq \left(\frac{2c_{n+1}}{\rho_{n+1}}\right)^d \leq 2^{-(n+1)} \text{ for every } y \in B_{n+1} \setminus F_n.$$

As in the proof of Proposition 1.2 we get that

$$\tilde{P}^{\cdot}[T_{A_n} < \infty] \leq 2^{-(n+1)} \qquad \text{on } U \setminus C_n,$$

$$\tilde{P}^{\cdot}[T_{B_{n+1}} < \infty] \leq 2^{-(n+2d)} \beta_{n+1}/\gamma_n \qquad \text{on } U \setminus C'_{n+1},$$

and

$$\tilde{P}^{\cdot}[T_{A_n} < T_{F_n}] \geq 2^{n+2d} \tilde{P}^{\cdot}[T_{B_{n+1}} < \infty] \geq \gamma_n^{-1} \tilde{P}^{\cdot}[T_{A_0} < T_{F_n}] \geq \gamma_n^{-1} g$$
$$\geq 2^{n+2d} \tilde{P}^{\cdot}[T_{B_{n+1}} < T_{F_n}] \quad \text{on } A_n \cup \left(C_{n+1} \setminus C'_{n+1}\right).$$

As before let $R := T_{A_n \cup C_{n+1}}$ and fix $x \in U \setminus C_{n+1}$. Since $\tilde{P}\left(y, C'_{n+1}\right) = 0$ for every $y \in U \setminus (A_n \cup C_{n+1} \cup F_n)$, we know that

$$X_R \in A_n \cup \left(C_{n+1} \setminus C'_{n+1}\right) \qquad \tilde{P}^x\text{-a.s. on } [R < T_{F_n}].$$

By the strong Markov property

$$\tilde{P}^x[T_{A_n} < T_{F_n}] = \int_{[R<T_{F_n}]} \tilde{P}^{X_R}[T_{A_n} < T_{F_n}]\, d\tilde{P}^x$$

and

$$\tilde{P}^x[T_{B_{n+1}} < T_{F_n}] = \int_{[R<T_{F_n}]} \tilde{P}^{X_R}[T_{B_{n+1}} < T_{F_n}] d\tilde{P}^x,$$

hence

$$\tilde{P}^x[T_{A_n} < T_{F_n}] \geq 2^{n+2d} \tilde{P}^x[T_{B_{n+1}} < T_{F_n}].$$

Therefore Lemma 1.1 now yields

$$\begin{aligned}&\gamma_{n+1}\tilde{P}^x[T_{A_{n+1}} < T_{F_n}]\\ &\le (1+3\cdot 2^{-(n+1)})(1+2^{-n})\gamma_{n+1}\left(\frac{a_{n+1}}{\rho_n}\right)^d \tilde{P}^x[T_{A_n} < \infty]\\ &\le (1+2^{2-n})\tilde{f}_n(x).\end{aligned}$$

On the other hand by definition

$$\tilde{P}(y,\cdot) = P_j(y,\cdot) \qquad \text{for every } y \in U\backslash F_n \text{ and } j \ge n,$$

hence

$$\begin{aligned}&\tilde{P}^x[T_{F_n} < \infty] = P_n^x[T_{F_n} < \infty]\\ &= \sum_{j=n}^{\infty} P_n^x[T_{E_j} < \infty, T_{E_n} = T_{E_{n+1}} = \ldots = T_{E_{j-1}} = \infty]\\ &= \sum_{j=n}^{\infty} P_j^x[T_{E_j} < \infty, T_{E_n} = T_{E_{n+1}} = \ldots = T_{E_{j-1}} = \infty]\\ &\le \sum_{j=n}^{\infty} P_j^x[T_{E_j} < \infty] \le \sum_{j=n}^{\infty} \varepsilon_j \le 2^{-n}\gamma_{n+1}^{-1}.\end{aligned}$$

Finally, $[T_{A_{n+1}} < \infty] \subset [T_{A_{n+1}} < T_{F_n}] \cup [T_{F_n} < \infty]$, therefore

$$\tilde{f}_{n+1}(x) = \gamma_{n+1}\tilde{P}^x[T_{A_{n+1}} < \infty] \le (1+2^{2-n})\tilde{f}_n(x) + 2^{-n}.$$

□

Proof of Theorem 2.1. By Proposition 2.3, for every $n \in \mathbb{N}$,

$$\tilde{f} \le e^4\tilde{f}_n + 1 \qquad \text{on } U\backslash \bigcup_{j=n+1}^{\infty} C_j.$$

Thus $\tilde{f}$ is locally bounded. Since $\tilde{P}\tilde{f} = \tilde{f}$, since $\tilde{r}$ is continuous and $\tilde{r} \le \frac{1}{2}\,\mathrm{dist}(\cdot, \partial U)$, we conclude that $\tilde{f}$ is a continuous $\tilde{r}$-median function. Because of

$$\tilde{f}(x_2) \ge \tilde{f}_2(x_2) = \gamma_2 > c\left(e^4\gamma_1 + 1\right) = c\left(e^4\tilde{f}_1(x_1) + 1\right) \ge c\tilde{f}(x_1)$$

the function $\tilde{f}$ is not harmonic. □

Remarks 2.4. 1. For every $z \in \partial U$, $z \ne (1,0,\ldots,0)$,

$$\lim_{x\to z} \tilde{f}(x) = 0.$$

Indeed, the proof of (2.3) shows that

$$\tilde{f} \leq e^4 \tilde{f}_1 + \sum_{m=1}^{\infty} m\gamma_{m+1} e_m \qquad \text{on } U \setminus \bigcup_{j=2}^{\infty} C_j .$$

The functions $\tilde{f}_1$ and e_m, $m \in \mathbb{N}$, tend to zero at ∂U (since $\tilde{f}_1 \leq \gamma_1 R_1^{A_1}$ and $e_j \leq R_1^{E_j}$) and, for every $m \in \mathbb{N}$,

$$m\gamma_{m+1} e_m < m\gamma_{m+1} \varepsilon_m = m2^{-(m+1)} \qquad \text{on } U \setminus \bigcup_{j=2}^{\infty} C_j .$$

So our claim follows from the fact that $\sum_{m=1}^{\infty} m2^{-(m+1)} < \infty$ and the point $(1, 0, \ldots, 0)$ is the only limit point of the set $\bigcup_{j=2}^{\infty} C_j$ contained in the boundary ∂U.

2. We could have arranged without difficulty that $\tilde{r}$ is a $\mathcal{C}^\infty$-function and then $\tilde{P}\tilde{f} = \tilde{f}$ implies that $\tilde{f}$ is a $\mathcal{C}^\infty$-function as well.

3. Every $\tilde{r}$-median function on U which is bounded by some harmonic function on U is harmonic, hence every extremal positive harmonic function on U is an extremal $\tilde{r}$-median function. So the euclidean boundary of U is a proper subset of the Martin boundary for the random walk given by $\tilde{r}$. (A close inspection should reveal that the function f we constructed is an extremal $\tilde{r}$-median function and that it is up to constant multiples the only extremal positive $\tilde{r}$-median function which is not harmonic).

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