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A Liouville-type Theorem for Elliptic Systems

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0. - Introduction

A priori estimates for solutions of superlinear elliptic problems can be established by a blow up technique. Such a method has been used by Gidas-Spruck [GS1] for the case of a single equation. Similar arguments can also be used in the case of systems. We refer to the work of Jie Qing [J] and M.A. Souto [S]. As in the scalar case, the treatment of systems poses the question of the validity of a result which is referred as a Liouville-type theorem for solutions of systems of elliptic equations in $\mathbb{R}^N$. Let us make precise this question.

We consider the elliptic system

$$\begin{cases} -\Delta u = v^\alpha \\ -\Delta v = u^\beta \end{cases} \tag{0.1}$$

in the whole of $\mathbb{R}^N$, $N \geq 3$. The question is to determine for which values of the exponents α and β the only non-negative solution (u, v) of (0.1) is $(u, v) = (0, 0)$. The notion of solution here is taken in the classical sense, i.e., $u, v \in C^2(\mathbb{R}^N)$. In the case of a single equation

$$\Delta u + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \tag{0.2}$$

it has been proved in [GS2] that the only solution of (0.2) is $u = 0$ when

$$1 \leq p < \frac{N+2}{N-2}, \quad N \geq 3. \tag{0.3}$$

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In dimension $N = 2$, a similar conclusion holds for $0 \leq p < \infty$. This is a special case of the result that asserts that any superharmonic function bounded below in the whole plane $\mathbb{R}^2$ is necessarily constant, see [PW, Theorem 29, p. 130]. It is also well known that in the critical case, $p = (N+2)/(N-2)$, problem (0.2) has a two-parameter family of solutions given by:

$$u(x) = \frac{[N(N-2)\lambda^2]^{\frac{N-2}{4}}}{[\lambda^2 + |x - x_0|^2]^{\frac{N-2}{2}}}. \tag{0.4}$$

We see then that the critical exponent, in the scalar case, plays an important role in the validity of the Liouville-type theorem for equation (0.2). It is natural to conjecture that in the case of system (0.1) the condition (0.3) is replaced by

$$1 \geq \frac{1}{\alpha+1} + \frac{1}{\beta+1} > \frac{N-2}{N}. \tag{0.5}$$

The basis of this conjecture lies on the fact that the existence of positive solutions for the Dirichlet problem for system (0.1) in a bounded domain holds true if condition (0.5) is satisfied, see [CFM, FF, HV]. If this conjecture were true we would have a complete analogy with the scalar case. A further argument in favor of this conjecture is a theorem of Mitidieri [M, Theorem 3.2] which states that (0.1) has no nontrivial *radial* positive solutions of class $C^2(\mathbb{R}^N)$, $N \geq 3$, provided $1 < \alpha \leq \beta$ and (0.5) holds. The result proved here is the following:

THEOREM. A) If $\alpha > 0$ and $\beta > 0$ are such that

$$\alpha, \beta \leq \frac{N+2}{N-2}, \quad \text{but not both are equal to} \quad \frac{N+2}{N-2}, \tag{0.6}$$

then the only non-negative C^2 solution of (0.1) in the whole of $\mathbb{R}^N$ is the trivial one: $u = 0,\ v = 0$.

B) If $\alpha = \beta = \dfrac{N+2}{N-2}$, then u and v are radially symmetric with respect to some point of $\mathbb{R}^N$.

REMARK. In [S] it was proved that (0.1) has no nontrivial non-negative C^2 solutions in the whole of $\mathbb{R}^N$ provided

$$\alpha, \beta > 0 \quad \text{and} \quad \frac{1}{\alpha+1} + \frac{1}{\beta+1} \geq \frac{N-2}{N-1}. \tag{0.7}$$

We see then that this establishes the conjecture in a hyperbolic region of the plane (α, β) which is smaller than the region determined by (0.5). However (and this is important to stress) such a region contains points which are not included in the region defined in (0.6). Souto's argument in [S] is based on the non-existence of positive solutions in the whole of $\mathbb{R}^N$ of the inequality

$$\Delta u + u^\sigma \leq 0 \tag{0.8}$$

when $\sigma \leq N/(N-2)$. This fact was proved by Gidas [G]. The conjecture would be proved if we had this non-existence result (for inequality (0.8)) up to $(N+2)/(N-2)$. We do not know if such a result is true. In [J] there are some Liouville-type theorems; though Jie's results apply to more general nonlinearities, they do not cover our result.

An important feature of the present paper is to show how the Method of Moving Planes, as developed by Gidas-Ni-Nirenberg [GNN], and more recently by Berestycki-Nirenberg [BN], can be used to produce rather simple and elegant proofs of Liouville-type theorems for systems. The main difficulty stems from the fact that the domain where the solutions are considered is $\mathbb{R}^N \backslash \{0\}$. In such a case it is not clear where we can start the procedure, see details in Section 2. The present work has been motivated by recent papers by Caffarelli-Gidas-Spruck [CGS] and Chen-Li [CL], where Liouville-type theorems for a single equation (results formerly proved by Gidas-Spruck [GS2]) have received a treatment by the Method of Moving Planes.

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1. - Some general facts about superharmonic (subharmonic) functions

Let us recall the so called Hadamard Three Spheres Theorem, see [PW, p. 131]:

"Let Ω be an open set containing the set

$$\{x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2\}, \quad N \geq 3$$

and $u \in C^2(\Omega)$ with $\Delta u \geq 0$. For $r_1 \leq r \leq r_2$, let

$$M(r) = \max \{u(x) : |x| = r\}.$$

Then

$$M(r) \leq \frac{M(r_1)(r^{2-N} - r_2^{2-N}) + M(r_2)(r_1^{2-N} - r^{2-N})}{r_1^{2-N} - r_2^{2-N}}, \tag{1.1}$$

for any r in $[r_1, r_2]$."

The proof of this result is very simple. Let $\varphi(r) = a + br^{2-N}$. Choose a and b such that $\varphi(r_1) = M(r_1)$ and $\varphi(r_2) = M(r_2)$. Let $v(x) = u(x) - \varphi(|x|)$, and use the maximum principle for subharmonic functions.

LEMMA 1.1. Let $u \in C^2(\mathbb{R}^N \backslash \{0\})$ be such that $u < 0$ and $\Delta u \geq 0$. Then, for each $\varepsilon > 0$, one has

$$u(x) \leq M(\varepsilon), \quad 0 < |x| \leq \varepsilon \tag{1.2}$$

$$u(x) \leq \frac{M(\varepsilon)\varepsilon^{N-2}}{|x|^{N-2}}, \quad |x| \geq \varepsilon. \tag{1.3}$$

PROOF. Letting $r_1 \to 0$ in (1.1) we obtain $M(r) \leq M(r_2)$ for all $0 < r < r_2$, which implies (1.2) by taking $r_2 = \varepsilon$. Letting $r_2 \to +\infty$ in (1.1) we get $M(r)r^{N-2} \leq M(r_1)r_1^{N-2}$ for all $r \geq r_1$, which gives (1.3) by taking $r_1 = \varepsilon$. □

COROLLARY 1.1. Let $u \in C^2(R^N)$ be such that $u > 0$ and $\Delta u \leq 0$ in the whole of $\mathbb{R}^N$. Let

$$w(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right)$$

be its Kelvin transform. Then, for each $\varepsilon > 0$, there are positive constants b_ε and c_ε such that

$$w(x) \geq c_\varepsilon, \quad 0 < |x| \leq \varepsilon \tag{1.4}$$

$$\frac{\varepsilon^{N-2}c_\varepsilon}{|x|^{N-2}} \leq w(x) \leq \frac{b_\varepsilon}{|x|^{N-2}}, \quad |x| \geq \varepsilon. \tag{1.5}$$

PROOF. The function $-w$ satisfies the hypothesis of Lemma 1.1. Choosing $c_\varepsilon = \min\{w(x) : |x| = \varepsilon\}$ we obtain (1.4) and the lower bound in (1.5). Choosing $b_\varepsilon = \max\{u(y) : |y| \leq \varepsilon^{-1}\}$ we obtain the upper bound in (1.5). □

2. - Some auxiliary facts

Let $u > 0$ and $v > 0$ be C^2 solutions of system (0.1) with both α and $\beta \leq (N+2)/(N-2)$. Let us introduce their Kelvin transforms

$$w(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right), \quad z(x) = \frac{1}{|x|^{N-2}} v\left(\frac{x}{|x|^2}\right)$$

which are defined for $x \neq 0$. One verifies that w and z satisfy the system

$$\Delta w + \frac{1}{|x|^{N+2-\alpha(N-2)}} z^\alpha = 0 \tag{2.1}$$

$$\Delta z + \frac{1}{|x|^{N+2-\beta(N-2)}} w^\beta = 0. \tag{2.2}$$

We shall use the method of moving planes. Let us start by considering planes parallel to $x_1 = 0$, coming from $-\infty$. For each real λ let us define

$$\Sigma_\lambda = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < \lambda\}, \quad T_\lambda = \partial\Sigma_\lambda$$

and x^λ as the reflection of x about the plane T_λ. Let $e_\lambda = (2\lambda, 0, \dots, 0)$. In what follows we consider $\lambda \leq 0$ and set $\tilde{\Sigma}_\lambda = \Sigma_\lambda \backslash \{e_\lambda\}$. In $\tilde{\Sigma}_\lambda$ we define the following functions

$$w_\lambda(x) = w(x^\lambda), \qquad z_\lambda(x) = z(x^\lambda),$$
$$W_\lambda(x) = w_\lambda(x) - w(x), \qquad Z_\lambda(x) = z_\lambda(x) - z(x).$$

The first step in the use of the method of moving planes is to show that we can start the process. That is, there exists a $\lambda < 0$ such that $W_\lambda(x) \geq 0$ and $Z_\lambda(x) \geq 0$ for all $x \in \tilde{\Sigma}_\lambda$. Unfortunately, this cannot be shown as we shall see shortly. However this can be achieved with a slight modification of the functions. Let us explain how.

For $x \in \tilde{\Sigma}_\lambda$ we have

$$\Delta w(x) + \frac{1}{|x|^{N+2-\alpha(N-2)}} z^\alpha(x) = 0 \tag{2.3}$$

$$\Delta w(x^\lambda) + \frac{1}{|x^\lambda|^{N+2-\alpha(N-2)}} z^\alpha(x^\lambda) = 0. \tag{2.4}$$

Using the invariance of the Laplacian under a reflection and the fact that $|x^\lambda| \leq |x|$, it follows from (2.4)

$$\Delta w_\lambda(x) + \frac{1}{|x|^{N+2-\alpha(N-2)}} z_\lambda^\alpha(x) \leq 0. \tag{2.5}$$

Hence we get from (2.3) and (2.5)

$$\Delta W_\lambda(x) + \frac{1}{|x|^{N+2-\alpha(N-2)}} (z_\lambda^\alpha(x) - z^\alpha(x)) \leq 0$$

which, using the mean value theorem, can be written as

$$\Delta W_\lambda(x) + c(x;\lambda) Z_\lambda(x) \leq 0 \tag{2.6}$$

where

$$c(x;\lambda) = \frac{\alpha}{|x|^{N+2-\alpha(N-2)}} (\psi(x;\lambda))^{\alpha-1}$$

and $\psi(x;\lambda)$ is a real number between $z_\lambda(x)$ and $z(x)$. Similarly, we obtain

$$\Delta Z_\lambda(x) + \hat{c}(x;\lambda) W_\lambda(x) \leq 0 \tag{2.7}$$

where

$$\hat{c}(x;\lambda) = \frac{\beta}{|x|^{N+2-\beta(N-2)}} (\hat{\psi}(x;\lambda))^{\beta-1}$$

and $\hat{\psi}(x;\lambda)$ is a real number between $w_\lambda(x)$ and $w(x)$. We would like to infer that $W_\lambda(x) \geq 0$ and $Z_\lambda(x) \geq 0$, for some $\lambda < 0$ and all $x \in \Sigma_\lambda$ but we observe that system (2.6)-(2.7) is not amenable for application of maximum principles. In order to obtain a system satisfying the hypothesis of maximum principles (see, for instance, [PW] or [FM]) we introduce the function g and show that the functions

$$\overline{W}_\lambda(x) = \frac{W_\lambda(x)}{g(x)}, \quad \overline{Z}(x) = \frac{Z_\lambda(x)}{g(x)},$$

where $g(x) = \sqrt{1 - x_1}$ for all x such that $x_1 < 0$, satisfy the following system of inequalities

$$\Delta \overline{W}_\lambda + 2\frac{\nabla g}{g} \cdot \nabla \overline{W}_\lambda + \frac{\Delta g}{g} \overline{W}_\lambda + c(x;\lambda)\overline{Z}_\lambda \leq 0 \tag{2.8}$$

$$\Delta \overline{Z}_\lambda + 2\frac{\nabla g}{g} \cdot \nabla \overline{Z}_\lambda + \frac{\Delta g}{g} \overline{Z}_\lambda + \hat{c}(x;\lambda)\overline{W}_\lambda \leq 0. \tag{2.9}$$

The latter system is cooperative, since both $c(x;\lambda)$ and $\hat{c}(x;\lambda)$ are positive. Thus we can apply the maximum principle (Thm. 1.1 in [FM]), provided

$$\frac{\Delta g(x)}{g(x)} + c(x;\lambda) \leq 0, \qquad \frac{\Delta g(x)}{g(x)} + \hat{c}(x;\lambda) \leq 0. \tag{2.10}$$

Our next purpose is to establish.

PROPOSITION 2.1. There exists $\lambda^* \leq 0$ such that for all $\lambda \leq \lambda^*$, $\overline{W}_\lambda(x) \geq 0$ and $\overline{Z}_\lambda(x) \geq 0$ for all $x \in \tilde{\Sigma}_\lambda$.

We shall use in the proof of the above proposition some properties of $\overline{W}_\lambda$ (and a similar set of properties of $\overline{Z}_\lambda$), which are collected in the lemmas below. We consider $\lambda < 0$.

LEMMA 2.1. (i) If $\overline{W}_\lambda(x) > 0$ for all x in some punctured ball $\dot{B}_\varepsilon(e_\lambda) \subset \tilde{\Sigma}_\lambda$, and if

$$c_\lambda =: \inf\{\overline{W}_\lambda(x) : x \in \tilde{\Sigma}_\lambda\} < 0$$

then $\overline{W}_\lambda$ attains its infimum in $\tilde{\Sigma}_\lambda$.

(ii) There exists $\overline{\lambda} < 0$, such that if $\lambda < \overline{\lambda}$ and $c(\lambda) < 0$, then $\overline{W}_\lambda$ attains its infimum in $\tilde{\Sigma}_\lambda$.

PROOF. (i) Since $\overline{W}_\lambda(x) \to 0$ as $|x| \to \infty$, we can find $r_1 > 0$ such that $W_\lambda(x) \geq \frac{1}{2}c(\lambda)$ for $|x| > r_1$. Thus W_λ attains its infimum on the compact set $\overline{B}_{r_1}(0) \cap \overline{(\tilde{\Sigma}_\lambda \backslash B_\varepsilon(e_\lambda))}$. Since W_λ vanishes on T_λ, the result follows.

(ii) Using Corollary 1.1 we find positive numbers c_1 and r such that $w(x) \geq c_1$, for $0 < |x| \leq 1$, and $w(x) \leq \frac{c_1}{2}$ for $|x| \geq r$. Choose $\overline{\lambda} = \min\{-r, -1\}$. Then for $\lambda \leq \overline{\lambda}$.

$$\overline{W}_\lambda(x) \geq \frac{1}{g(x)}\left(c_1 - \frac{c_1}{2}\right) > 0 \quad \text{if} \quad x \in B_1(e_\lambda). \qquad \square$$

LEMMA 2.2. (i) There is a positive constant d_λ, depending only on w and λ, such that

$$\hat{c}(x;\lambda) \le \frac{\beta d_\lambda^{\beta-1}}{|x|^4} \tag{2.11}$$

for all $x \in \tilde{\Sigma}_\lambda - B_1(e_\lambda)$, where $\overline{W}_\lambda(x) < 0$.

(ii) There is an $R_\lambda > 0$, depending only on λ and w, such that

$$\frac{\Delta g(x)}{g(x)} + \hat{c}(x;\lambda) < 0 \tag{2.12}$$

for all $x \in \tilde{\Sigma}_\lambda \backslash B_{R_\lambda}(0)$, where $\overline{W}_\lambda(x) < 0$.

PROOF. (i) If $\beta \ge 1$, it follows from $\overline{W}_\lambda(x) < 0$ that $w_\lambda(x) \le \hat{\psi}(x;\lambda) \le w(x)$. We then estimate $w(x)$ using inequality (1.5) of Corollary 1.1. We see that d_λ can be taken as $\max\{u(y) : |y| \le |\lambda|^{-1}\}$. Then $\hat{\psi}(x;\lambda) \le \frac{d_\lambda}{|x|^{N-2}}$. Using the definition of $\hat{c}(x;\lambda)$ we obtain (2.11) readily. Actually, inequality (2.11) holds for all $x \in \tilde{\Sigma}_\lambda$.

If $0 < \beta < 1$ then we estimate $w_\lambda(x) = w(x^\lambda)$ using the other part of inequality (1.5) to obtain

$$w_\lambda(x) \ge \frac{d_\lambda}{|x_\lambda|^{N-2}} \ge \frac{d_\lambda}{|x|^{N-2}}$$

where $d_\lambda = \min\{w(x) : |x| = 1\}$. From here we obtain (2.11) since $\beta - 1 < 0$.

(ii) Observe that

$$\frac{\Delta g(x)}{g(x)} = -\frac{1}{4(1-x_1)^2} \le -\frac{1}{8(1+x_1^2)}.$$

So (2.12) follows immediately from the above inequality together with (2.11). □

LEMMA 2.3. There exists $\lambda^* \le \overline{\lambda}$ such that, for all $\lambda \le \lambda^*$, inequality (2.12) holds for all $x \in \tilde{\Sigma}_\lambda$, where $\overline{W}_\lambda(x) < 0$.

PROOF. We first observe that $d_\lambda \le d_\mu$ if $\lambda \le \mu \le 0$. So for all $\lambda \le \overline{\lambda}$ we estimate

$$\frac{\Delta g(x)}{g(x)} + \hat{c}(x;\lambda) \le -\frac{1}{8(1+x_1^2)} + \frac{\beta d_{\overline{\lambda}}^{\beta-1}}{|x|^4} \tag{2.13}$$

for all $x \in \tilde{\Sigma}_\lambda$, where $\overline{W}_\lambda(x) < 0$. Hence the right side of (2.13) is negative if $|x| > R_{\overline{\lambda}}$ (as in Lemma 2.2(ii)). Now take λ^* such that $\tilde{\Sigma}_{\lambda^*} \subset \{x \in R^N : |x| > R_{\overline{\lambda}}\}$. □

LEMMA 2.4. Assume that there is $x_0 \in \overline{\Sigma}_\lambda$ such that

$$\overline{W}_\lambda(x_0) = \inf\{\overline{W}_\lambda(x) : x \in \tilde{\Sigma}_\lambda\} < 0.$$

Then $\overline{Z}_\lambda(x_0) < 0$. If moreover $|x_0| > R_\lambda$ (as in the version of Lemma 2.2(ii) for $\overline{Z}_\lambda$), then $\overline{Z}_\lambda(x_0) < \overline{W}_\lambda(x_0)$.

PROOF. Since $\nabla\overline{W}_\lambda(x_0) = 0$ and $\Delta\overline{W}_\lambda(x_0) \geq 0$, it follows from (2.8) that

$$\frac{\Delta g(x_0)}{g(x_0)}\overline{W}_\lambda(x_0) + c(x_0;\lambda)\overline{Z}_\lambda(x_0) \leq 0, \tag{2.14}$$

which implies that $\overline{Z}_\lambda(x_0) < 0$. Next we can write (2.14) as

$$\left[\frac{\Delta g(x_0)}{g(x_0)} + c(x_0;\lambda)\right]\overline{W}_\lambda(x_0) + c(x_0;\lambda)\left[\overline{Z}_\lambda(x_0) - \overline{W}_\lambda(x_0)\right] \leq 0. \tag{2.15}$$

As we said before an analogue to Lemma 2.2 holds for $\overline{Z}_\lambda$. So the first bracket is negative, and the conclusion follows. □

PROOF OF PROPOSITION 2.1. Let us assume, by contradiction, that for some $\lambda \leq \lambda^*$ (λ^* = least of the two λ^*'s, the one defined in Lemma 2.3 and the one from the version of Lemma 2.3 for $\overline{Z}_\lambda$), we have

$$\begin{aligned} &\overline{W}_\lambda(x) < 0, \quad \text{for some} \quad x \in \tilde{\Sigma}_\lambda \quad \textit{or} \quad \overline{Z}_\lambda(x) < 0, \\ &\qquad\qquad\quad \text{for some} \quad x \in \tilde{\Sigma}_\lambda. \end{aligned} \tag{2.16}$$

Without loss of generality, we may assume that $\overline{W}_\lambda(x) < 0$ for some $x \in \tilde{\Sigma}_\lambda$. The argument below shows that the conjunction *or* in (2.16) can be replaced by *and*. Indeed, since $\lambda^* \leq \overline{\lambda}$, the assumption $\overline{W}_\lambda(x) < 0$, for some $x \in \tilde{\Sigma}_\lambda$, implies via Lemma 2.1(ii) that the infimum of $\overline{W}_\lambda$ is attained at a point x_0 in $\overline{W}_\lambda$; and Lemma 2.4 states that $\overline{Z}_\lambda(x_0) < 0$.

We next use Lemma 2.3 to see that the first bracket in (2.15) is negative, and then

$$\overline{Z}_\lambda(x_0) < \overline{W}_\lambda(x_0). \tag{2.17}$$

Now we do a similar argument with $\overline{Z}_\lambda$ and conclude that

$$\overline{W}_\lambda(x_1) < \overline{Z}_\lambda(x_1) \tag{2.18}$$

where $x_1 \in \tilde{\Sigma}_\lambda$ is the point where $\overline{Z}_\lambda$ attains its infimum. Finally we use (2.17), (2.18) and the fact that $\overline{Z}_\lambda(x_1) \leq \overline{Z}_\lambda(x_0)$ and $\overline{W}_\lambda(x_0) \leq \overline{W}_\lambda(x_1)$ to come to a contradiction. □

We next define

$$\lambda_0 = \sup\{\lambda < 0 : \overline{W}_\lambda(x) \geq 0 \quad \text{and} \quad \overline{Z}_\lambda(x) \geq 0, \quad \forall x \in \tilde{\Sigma}_\lambda\}.$$

PROPOSITION 2.2. If $\lambda_0 < 0$, then $\overline{W}_{\lambda_0}(x) \equiv 0$ and $\overline{Z}_{\lambda_0}(x) \equiv 0$ for all $x \in \tilde{\Sigma}_{\lambda_0}$.

REMARK. It follows readily from Proposition 2.2 that both w and z are symmetric with respect to the hyperplane T_{λ_0}.

PROOF OF PROPOSITION 2.2. By continuity we see that $\overline{W}_{\lambda_0}(x) \geq 0$ and $\overline{Z}_{\lambda_0}(x) \geq 0$ for $x \in \tilde{\Sigma}_{\lambda_0}$. Now observe that if $\overline{Z}_{\lambda_0}(x) \equiv 0$, it follows from (2.9) that $\overline{W}_{\lambda_0}(x) \leq 0$, which will imply $\overline{W}_{\lambda_0}(x) \equiv 0$. So if we assume, by contradiction, that the conclusion of Proposition 2.2 is not true, we conclude that *both* $\overline{W}_{\lambda_0}(x)$ and $\overline{Z}_{\lambda_0}(x)$ are not identically zero. It follows from (2.8) that

$$\Delta\overline{W}_{\lambda_0}(x) + 2\,\frac{\nabla g(x)}{g(x)} \cdot \nabla\overline{W}_{\lambda_0}(x) + \frac{\Delta g(x)}{g(x)}\,\overline{W}_{\lambda_0}(x) \leq 0$$

for all $x \in \tilde{\Sigma}_{\lambda_0}$. So by the maximum principle (recall that $\overline{W}_{\lambda_0}(x) \geq 0$) we see that $\overline{W}_{\lambda_0}(x) > 0$ for $x \in \tilde{\Sigma}_{\lambda_0}$. Similarly $\overline{Z}_{\lambda_0}(x) > 0$ for $x \in \tilde{\Sigma}_{\lambda_0}$.

Next using the Hopf maximum principle we obtain that

$$\frac{\partial \overline{W}_{\lambda_0}}{\partial \nu} < 0 \quad \text{and} \quad \frac{\partial \overline{Z}_{\lambda_0}}{\partial \nu} < 0 \tag{2.19}$$

on the boundary T_{λ_0}. (Here $\partial/\partial\nu$ is $\partial/\partial x_1$). We shall see now that this is impossible. From the definition of λ_0, we conclude that there exist a sequence of real numbers $\lambda_k \to \lambda_0$, with $\lambda_k > \lambda_0$, and a sequence of points in $\tilde{\Sigma}_{\lambda_k}$, where $\overline{W}_{\lambda_k}$ or $\overline{Z}_{\lambda_k}$ is negative. Since $W_{\lambda_0}(x)$ satisfies (2.6) and $Z_{\lambda_0}(x) > 0$ for $x \in \tilde{\Sigma}_{\lambda_0}$, we conclude that $W_{\lambda_0}(x)$ is superharmonic in $\tilde{\Sigma}_{\lambda_0}$. In particular W_{λ_0} is superharmonic in $B_{2R}(e_{\lambda_0})\backslash\{e_{\lambda_0}\}$, where $2R = |\lambda_0|$. The argument in the proof of Lemma 1.1 shows that there exists $c_0 > 0$ such that

$$W_{\lambda_0}(x) \geq c_0, \quad \text{if} \quad x \in B_R(e_{\lambda_0})\backslash\{e_{\lambda_0}\}.$$

A similar argument shows that

$$Z_{\lambda_0}(x) \geq c_0, \quad \text{if} \quad x \in B_R(e_{\lambda_0})\backslash\{e_{\lambda_0}\}.$$

(We may choose the same c, or decrease it). So

$$\overline{W}_{\lambda_0}(x) \geq \tilde{c}_0 > 0, \quad \overline{Z}_{\lambda_0}(x) \geq \tilde{c}_0 > 0, \quad \text{if} \quad x \in B_R(e_{\lambda_0})\backslash\{e_{\lambda_0}\}.$$

By continuity we have

$$\overline{W}_{\lambda_k}(x) \geq \frac{\tilde{c}_0}{2}, \quad \overline{Z}_{\lambda_k}(x) \geq \frac{\tilde{c}_0}{2}, \quad \text{if} \quad x \in B_{R/2}(e_{\lambda_k})\backslash\{e_{\lambda_k}\}, \tag{2.20}$$

for k sufficiently large.

It follows from Lemma 2.1(i) (using (2.20)) and Lemma 2.4 that for k sufficiently large both $\overline{W}_{\lambda_k}$ and $\overline{Z}_{\lambda_k}$ attain their negative infima in $\tilde{\Sigma}_{\lambda_k}$. [Recall that we are using the contradiction hypothesis that $\overline{W}_{\lambda_k}$ or $\overline{Z}_{\lambda_k}$ is negative somewhere in $\tilde{\Sigma}_{\lambda_k}$]. Let us denote by x_k and y_k, respectively, the points of minima of $\overline{W}_{\lambda_k}$ and $\overline{Z}_{\lambda_k}$. If follows from Lemma 2.4 that at least one of the sequences, $\{x_k\}$ or $\{y_k\}$ is bounded. Assume that $\{x_k\}$ is bounded and passing to a subsequence assume that $x_k \to \overline{x}$. By continuity we have that $\nabla\overline{W}_{\lambda_0}(\overline{x}) = 0$ and $\overline{W}_{\lambda_0}(\overline{x}) \leq 0$. Since $\overline{x} \neq e_{\lambda_0}$ and $\overline{W}_{\lambda_0}(x) > 0$ in $\tilde{\Sigma}_{\lambda_0}$, we conclude that $\overline{x} \in T_{\lambda_0}$, which contradicts the fact observed before that $\dfrac{\partial \overline{W}_{\lambda_0}}{\partial \nu} < 0$ on T_{λ_0}.

A similar argument applies if we assume that $\{y_k\}$ is bounded. □

3. - Proof of the theorem

We first prove **Part A**. Performing the moving plane procedure we have two possibilities:

(i) If $\lambda_0 < 0$, it follows from Proposition 2.2 that w and z are symmetric will respect to the plane T_{λ_0}. But looking at equations (2.1) and (2.2) we realize that this is impossible.

(ii) If $\lambda_0 = 0$, we conclude that $w_0(x) \geq w(x)$ and $z_0(x) \geq z(x)$ for all $x \in \Sigma_0$. We can perform the procedure from the right and we will reach a $\lambda_0^r = 0$, (λ_0^r cannot be positive, in virtue of an argument as in (i) above), from which we get $w(x) \geq w_0(x)$ and $z(x) \geq z_0(x)$ for all $x \in \Sigma_0$. So w and z are symmetric with respect to the plane $x_1 = 0$.

(iii) This reasoning can be made from any direction. And so the only possibility would be that both w and z are radially symmetric with respect to the origin 0. But 0 was chosen arbitrarily when we perform the Kelvin transform. Thus u and v are radially symmetric with respect to any point. Then they would be constant and from the equations we finally obtain $u = v = 0$.

Part B. We show that w and z are symmetric with respect to some plane parallel to $x_1 = 0$. Indeed, if $\lambda_0 < 0$, this follows from Proposition 2.2, and in this case the plane is T_{λ_0}. If $\lambda_0 = 0$, we perform the moving plane procedure from the right and find a corresponding $\lambda_0^r \geq 0$. If $\lambda_0^r > 0$, an analogue to Proposition 2.2 shows that w and z are symmetric with respect to $T_{\lambda_0^r}$. If $\lambda_0^r = 0$ we proceed as in (ii) above to conclude that both w and z are symmetric with respect to $x_1 = 0$. We perform this moving plane procedure taking planes perpendicular to any direction, and for each direction $\gamma \in R^N$, $|\gamma| = 1$ we find a plane T_γ with the property that both w and z are symmetric with respect to T_γ. A simple argument shows that all these planes intersect at a single point, or $w = z = 0$. □

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