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On Global Real Analytic Solutions of the Degenerate Kirchhoff Equation

KUNIHICO KAJITANI - KAORU YAMAGUTI

1. - Introduction

We shall consider the problem of existence and uniqueness of real analytic solutions of the Cauchy problem for the degenerate Kirchhoff equation

$$(1.1) \quad \begin{cases} \partial_t^2 u + M((Au, u))Au = f(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^n \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $Au(t, x) = \sum_{i,j=1}^n D_j(a_{ij}(x)D_i u(t, x))$, $D_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$, $(Au(t, \cdot), u(t, \cdot))$ is an inner product of $Au(t, x)$ and $u(t, x)$ in $L^2(\mathbb{R}_x^n)$ and $M(\eta)$ is a non-negative function in $C^1([0, \infty))$.

When $A = \sum_{j=1}^n D_j^2$ the equation (1.1) is called the Kirchhoff equation, which has been studied by many authors (cf. [1], [2], [3], [8], [9] and [10]). In this paper, we shall treat the case where A is degenerate elliptic, that is, $[a_{ij}(x); i, j = 1, \dots, n]$ is a real symmetric matrix and

$$(1.2) \quad a(x, \xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0$$

for $x \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Moreover we assume that there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$(1.3) \quad |D_x^\alpha a_{ij}(x)| \leq c_0 \rho_0^{-|\alpha|} |\alpha|!$$

for $x \in \mathbb{R}_x^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $i, j = 1, \dots, n$, and that $M(\eta) \in C^1([0, \infty))$ and

$$(1.4) \quad M(\eta) \geq 0$$

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for $\eta \in [0, \infty)$. We introduce some functional spaces. For a topological space X and an interval $I \subset \mathbb{R}^1$, we denote by $C^k(I; X)$ the set of functions from I to X which are k times continuously differentiable with respect to $t \in I$ in X . For $s \in \mathbb{R}$ and $\rho > 0$ we define a Hilbert space $H_\rho^s = \{u(x) \in L^2(\mathbb{R}_x^n); \langle \xi \rangle^s e^{\rho(\xi)} \hat{u}(\xi) \in L^2(\mathbb{R}_\xi^n)\}$, where $\hat{u}(\xi)$ stands for Fourier transform of u and $\langle \xi \rangle = (1 + \xi_1^2 + \dots + \xi_n^2)^{1/2}$. For $\rho < 0$ we define H_ρ^s as the dual space of $H_{-\rho}^{-s}$. For $\rho = 0$ we denote by $H^s = H_0^s$ the usual Sobolev space. Then note that the dual space of H_ρ^s becomes $H_{-\rho}^{-s}$ for any $s, \rho \in \mathbb{R}$.

For $\rho \in \mathbb{R}$ define an operator $e^{\rho(D)}$ from H_ρ^s to H^s as follows:

$$e^{\rho(D)}u(x) = \int_{\mathbb{R}_\xi^n} e^{ix \cdot \xi + \rho(\xi)} \hat{u}(\xi) d\tilde{\xi}$$

for $u \in H_\rho^s$, where $d\tilde{\xi} = (2\pi)^{-n} d\xi$. Note that $(e^{\rho(D)})^{-1} = e^{-\rho(D)}$ is a mapping from H^s to H_ρ^s .

We prove the following result:

MAIN THEOREM. *Assume that (1.2) through (1.4) are valid. Let $0 < \rho_1 < \rho_0/\sqrt{n}$. Put $\rho(t) = \rho_1 e^{-\gamma t}$ for $\gamma > 0$. Then there exists $\gamma > 0$ such that for any $u_0 \in H_{\rho_1}^2, u_1 \in H_{\rho_1}^1$ and for any $f(t, x)$ satisfying $e^{\rho(t)(D)}f \in C^0([0, \infty); H^1)$, the Cauchy Problem (1.1) has the unique solution $u(t, x)$ satisfying $e^{\rho(t)(D)}u \in \bigcap_{j=0}^2 C^{2-j}([0, \infty); H^j)$.*

The idea in the proof of our main theorem is based on the method introduced in [5] in order to find the global real analytic solution of the Cauchy problem for a Kowalevskian system. Roughly speaking, we transform an unknown function u such as $v = e^{\rho(t)(D)}u$ and then change the hyperbolic equation (1.1) of the unknown function u into the parabolic equation of v . Thanks to parabolicity, we can prove local existence of a solution v of the modified problem in the usual Sobolev spaces by the use of the principle of a contraction mapping. Finally we can show the existence of a time global solution of the original equation (1.1) modifying the energy estimate which was introduced in [3] in the case of $A = -\Delta$.

2. - Preliminaries

Let S^m be the class of symbols of pseudo-differential operators of order m whose element $a(x, \xi)$ in $C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ satisfies

$$|a_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|}$$

for $x, \xi \in \mathbb{R}^n$ and for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, where $a_{(\beta)}^{(\alpha)}(x, \xi) =$

$\left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}\right)^\beta a(x, \xi)$. We define a pseudo-differential operator $a(x, D)$ as usual

$$a(x, D)u(x) = \int_{\mathbb{R}_\xi^n} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \bar{d}\xi$$

for $u \in \mathcal{S}$ where \mathcal{S} denotes the Schwartz space of rapidly decreasing functions in \mathbb{R}^n . Then we have the following well-known fact:

PROPOSITION 2.1. (i) For $a(x, \xi) \in S^m$ and $s \in \mathbb{R}$, there is $C_s > 0$ such that

$$(2.1) \quad \|a(x, D)u\|_s \leq C_s \|u\|_{s+m}$$

for $u \in H^{s+m}$.

(ii) Assume $a(x, \xi) \in S^2$ is non-negative. Then there are positive numbers C_1 and C_2 such that

$$(2.2) \quad \Re(a(x, D)u, u)_s \geq -C_1 \|u\|_s$$

and

$$(2.3) \quad \sum_{|\alpha|=1} \{ \|a_{(\alpha)}(x, D)u\|_{s-1}^2 + \|a^{(\alpha)}(x, D)u\|_s^2 \} \leq C_2(2C_1 \|u\|_s^2 + \Re(a(x, D)u, u)_s)$$

for $u \in H^{s+2}$.

For a proof refer to [6] and [4] for (i) and (ii) respectively. Now let us state some properties of the Hilbert space H_ρ^s .

LEMMA 2.2. (i) Let $\rho > 0$. Then it holds that

$$(2.5) \quad \|D_x^\alpha w\|_{H^\rho} \leq \|w\|_{H_\rho^s} \rho^{-|\alpha|} |\alpha|!$$

and

$$(2.6) \quad |D_x^\alpha w(x)| \leq C_n \|w\|_{H_\rho^s} \rho^{-(|\alpha|+n+|s|)} (|\alpha| + n + |s|)!$$

for $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$ and $w \in H_\rho^s$.

(ii) Let $u(x)$ be a function in H^∞ and $s \in \mathbb{R}$. If u satisfies

$$(2.7) \quad \|D_x^\alpha u\|_{H^\rho} \leq c_0 \rho_1^{-|\alpha|} |\alpha|!$$

for every multi-index $\alpha \in \mathbb{N}^n$, then $u(x)$ belongs to H_ρ^s for $\rho < \rho_1/\sqrt{n}$.

PROOF. (i) It is easy to verify (2.5) using the fact that $|\xi^\alpha| \leq \langle \xi \rangle^{|\alpha|}$. Representing $D_x^\alpha w$ by Fourier transformation, we get

$$\begin{aligned} |D_x^\alpha w(x)| &= \left| \int e^{ix \cdot \xi} \xi^\alpha \hat{w}(\xi) \tilde{d}\xi \right| \\ &\leq \int \langle \xi \rangle^{|\alpha|} |\hat{w}(\xi)| \tilde{d}\xi \\ &\leq \left\{ \int (\rho^{-\rho(\xi)} \langle \xi \rangle^{|\alpha|+|s|})^2 \tilde{d}\xi \right\}^{\frac{1}{2}} \|w\|_{H^s} \end{aligned}$$

which implies (2.6).

(ii) Since

$$\xi^\alpha \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} D_x^\alpha u(x) dx,$$

we have the estimate by virtue of (2.7) that

$$\|\langle \xi \rangle^j \hat{u}\|_{H^s}^2 \leq (c_n c_0 (\sqrt{n} \rho_1^{-1})^j j!)^2$$

for any j . Hence we obtain

$$\|e^{\rho(\xi)} \hat{u}(\xi)\|_{H^s}^2 \leq \sum_{j=0}^{\infty} \left\| \frac{\rho^j \langle \xi \rangle^j}{j!} \hat{u} \right\|_{H^s}^2 \leq \frac{c_n^2 c_0^2}{1 - \sqrt{n} \rho \rho_1^{-1}}$$

if $\sqrt{n} \rho \rho_1^{-1} < 1$. □

Let $a(x)$ be a real analytic function in \mathbb{R}^n satisfying

$$(2.8) \quad |D_x^\alpha a(x)| \leq c_0 \rho_0^{-|\alpha|} |\alpha|!$$

for all $x \in \mathbb{R}^n$ and for all multi-indices $\alpha \in \mathbb{N}^n$. Define a multiplier $a \cdot$ as $(a \cdot u)(x) = a(x)u(x)$. Let us define $a(\rho; x, D)u(x) = e^{\rho(D)} a \cdot e^{-\rho(D)} u(x)$ for $u \in L^2(\mathbb{R}^n)$ and denote its symbol by $a(\rho; x, \xi)$.

PROPOSITION 2.3. *Suppose that $a(x)$ satisfies (2.8).*

(i) *If a function u belongs to the class $H_{\rho_1}^s$ and $0 < \rho_1 < \rho_0$, then $a \cdot u$ belongs to the class H_ρ^s for $0 < \rho < \rho_1/\sqrt{n}$.*

(ii) *$a(\rho; x, D)$ is a pseudodifferential operator of order 0 and its symbol has the representation*

$$(2.10) \quad a(\rho; x, \xi) = a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi),$$

where

$$(2.11) \quad a_1(x, \xi) = - \sum_{j=1}^n D_{x_j} a(x) \xi_j \langle \xi \rangle^{-1},$$

and a_2 and r respectively satisfy

$$(2.12) \quad |a_{2(\beta)}^{(\alpha)}(\rho; x, \xi)| \leq C_{\alpha\beta\rho_0}(\xi)^{-|\alpha|},$$

$$(2.13) \quad |r_{(\beta)}^{(\alpha)}(\rho; x, \xi)| \leq C_{\alpha\beta\rho_0}(\xi)^{-1-|\alpha|}$$

for $x, \xi \in \mathbb{R}^n$, $|\rho| < n^{-1}\rho_0$ and $\alpha, \beta \in \mathbb{N}^n$.

PROOF. (i) Assume $\rho > 0$. Taking into account the fact that

$$\sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha - \alpha'|! |\alpha'|! \rho_0^{-|\alpha'|} \rho_1^{-|\alpha - \alpha'|} \leq \frac{\rho_0}{\rho_0 - \rho_1} \rho_1^{-|\alpha|} |\alpha'|!$$

if $\rho_1 < \rho_0$, we have the estimate that

$$\begin{aligned} \|D_x^\alpha(a \cdot u)\|_{H^s} &= \left\| \sum_{\alpha' < \alpha} \binom{\alpha}{\alpha'} D_x^{\alpha'} a \cdot D_x^{\alpha - \alpha'} u(\cdot) \right\|_{H^s} \\ &\leq c_0 \|u\|_{H_{\rho_1}^s} \sum \binom{\alpha}{\alpha'} \rho_0^{-|\alpha'|} |\alpha'|! \rho_1^{-|\alpha - \alpha'|} |\alpha - \alpha'|! \\ &\leq c_0 \frac{\rho}{\rho_0 - \rho_1} \|u\|_{H_{\rho_1}^s} \rho^{-|\alpha|} |\alpha'|! \end{aligned}$$

from (2.5) and (2.7). Therefore it follows from (ii) of Lemma 2.2 that $a \cdot u \in H_\rho^s$ for $\rho < \rho_1/\sqrt{n}$.

(ii) For $u \in \mathcal{S}$ and $\varepsilon > 0$ we put $\hat{u}_\varepsilon(\xi) = e^{-\varepsilon|\xi|^2} \hat{u}(\xi)$. $u_\varepsilon(x)$ denotes the inverse Fourier transformation of $\hat{u}_\varepsilon(\xi)$. Then $u_\varepsilon(x)$ is in H_τ^s for every $\tau > 0$ and $e^{-\rho(D)}u_\varepsilon(x)$ is also in H_τ^s for all $\tau > 0$ and $\rho \in \mathbb{R}^1$. Therefore it follows from (i) that $a \cdot e^{-\rho(D)}u_\varepsilon$ is in H_τ^s if $\tau < \rho_0/\sqrt{n}$. Note that $a \cdot e^{-\rho(D)}u_\varepsilon \in L^1(\mathbb{R}_x^n)$ and $e^{\rho(\xi)} \mathcal{F}[a \cdot e^{-\rho(D)}u_\varepsilon](\xi) \in L^1(\mathbb{R}_\xi^n)$ for $|\rho| < \rho_0/\sqrt{n}$ and $\varepsilon > 0$. So we can write

$$\begin{aligned} &e^{\rho(D)}(a \cdot e^{-\rho(D)}u_\varepsilon)(x) \\ &= \int e^{ix \cdot \eta + \rho(\eta)} \tilde{d}\eta \int e^{-iy \cdot \eta} (a \cdot e^{-\rho(D)}u_\varepsilon)(y) dy \\ &= \lim_{\delta \rightarrow +0} \int e^{ix \cdot \eta + \rho(\eta) - \delta|\eta|^2} \tilde{d}\eta \int e^{-iy \cdot \eta - \delta|x-y|^2} (a \cdot e^{-\rho(D)}u_\varepsilon)(y) dy \\ &= \lim_{\delta \rightarrow +0} \int \int \int e^{i(x-y) \cdot \eta + \rho(\eta) - \delta|x-y|^2 - \delta|\eta|^2} a(y) e^{iy \cdot \xi - \rho(\xi)} \hat{u}_\varepsilon(\xi) \tilde{d}\eta dy \tilde{d}\xi \\ &= \lim_{\delta \rightarrow +0} \int e^{ix \cdot \xi} a_\delta(x, \xi) \hat{u}_\varepsilon(\xi) \tilde{d}\xi, \end{aligned}$$

where $a_\delta(x, \xi)$ is given by

$$a_\delta(x, \xi) = \int \int e^{-iy \cdot \eta - \delta|y|^2 - \delta|\xi + \eta|^2 + \rho((\xi + \eta) - (\xi))} a(x + y) dy \tilde{d}\eta.$$

Putting

$$\begin{aligned} \langle \xi + \eta \rangle - \langle \xi \rangle &= \sum_{j=1}^n \eta_j \int_0^1 (\xi_j + \theta \eta_j) \langle \xi + \theta \eta \rangle^{-1} d\theta \\ &= \eta \cdot w(\xi, \eta), \end{aligned}$$

we can re-write $a_\delta(x, \xi)$ using Stokes formula:

$$\begin{aligned} a_\delta(x, \eta) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(y - i\rho w(\xi, \eta)) \cdot \eta - \delta |y|^2 - \delta |\xi + \eta|^2} a(x + y) dy d\tilde{\eta} \\ &= \int_{\mathbb{R}^n} d\tilde{\eta} \int_{\mathbb{R}^n - i w(\xi, \eta)} e^{-iz \cdot \eta - \delta (z + i\rho w(\xi, \eta))^2 \delta |\xi + \eta|^2} a(x + z + i\rho w(\xi, \eta)) dz \\ &= \int_{\mathbb{R}^n} d\tilde{\eta} \int_{\mathbb{R}^n} e^{-iy \cdot \eta - \delta (y + i\rho w(\xi, \eta))^2 - \delta |\xi + \eta|^2} a(x + y + i\rho w(\xi, \eta)) dy \end{aligned}$$

for $\rho < \rho_0/n$, where we write $z^2 = \sum_{j=1}^n z_j^2$ for $z \in C^n$. Thus, by Taylor's expansion, we obtain

$$\begin{aligned} \lim_{\delta \rightarrow +0} a_\delta(x, \xi) &= Os - \int \int e^{-y \cdot \eta} a(x + y + i\rho w(\xi, \eta)) dy d\tilde{\eta} \\ &= a(x + i\rho w(\xi, 0)) + r(\rho; x, \xi), \end{aligned}$$

where

$$\begin{aligned} r(\rho; x, \xi) &= \lim_{\delta \rightarrow 0} \int \int \left(e^{-iy \cdot \eta - \delta (y + i\rho w(\xi, \eta))^2 - \delta |\xi + \eta|^2} \right. \\ &\quad \left. \sum_{|\alpha|=1} \partial_\eta^\alpha \{ D_y^\alpha a(x + y + i\rho w(\xi, \eta)) \} \right) dy d\tilde{\eta} \end{aligned}$$

satisfies (2.13) (See Lemma 2.4 in [6]). Another application of Taylor's expansion yields

$$\begin{aligned} &a(x + i\rho w(\xi, 0)) \\ &= a(x + i\rho \xi \langle \xi \rangle^{-1}) \\ &= a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi), \end{aligned}$$

where $a_1(x, \xi)$ and $a_2(\rho; x, \xi)$ satisfy (2.11) and (2.12) respectively. Since

$$u_\varepsilon(x) \rightarrow u(x) \text{ in } \mathcal{S} \text{ as } \varepsilon \rightarrow +0,$$

we have

$$(e^{\rho\langle D \rangle} a \cdot e^{-\rho\langle D \rangle} u)(x) = \lim_{\varepsilon \rightarrow +0} a(\rho; x, D)u_\varepsilon(x),$$

if $u \in \mathcal{S}$. □

Let $P(t) = [p_{ij}(t, x, D)]_{i,j=1,\dots,d}$ be a matrix consisting of pseudo-differential operators whose symbols $p_{ij}(t, x, \xi)$ belong to the class $C([0, T]; S^1)$. Let us consider the following Cauchy problem

$$(2.14) \quad \begin{cases} \frac{d}{dt} U(t) = P(t)U(t) + F(t), & t \in (0, T), \\ U(0) = U_0, \end{cases}$$

where $U(t) = {}^t(U_1(t, x), \dots, U_d(t, \dot{x}))$ is an unknown vector-valued function and $F(t) = {}^t(F_d(t, x), \dots, F_1(t, x))$, $U_0 = {}^t(U_{01}, \dots, U_{0d})$ are known vector-valued functions. Then we have:

PROPOSITION 2.4. *Suppose that $\det(\lambda - p(t, x, \xi)) \neq 0$ for $\lambda \in C^1$ with $\Re \lambda > -c_0\langle \xi \rangle$, $t \in [0, T]$ and $|\xi| \gg 1$. Take an arbitrary real number s . Then for any $U_0 \in (H^{s+1}(\mathbb{R}^n))^d$ and for any $F(t) \in C^0([0, T]; (H^{s+1})^d)$, there exists a unique solution $U(t) \in C^1([0, T]; (H^s)^d \cap C^0([0, T]; (H^{s+1})^d)$ of (2.14).*

This proposition will be used in Section 4 to prove existence of local solutions of the Cauchy problem (1.6). The proof of this proposition is given in Proposition 4.5 in [7].

3. - A priori estimates of solutions for the linear problem

Let $0 < T < \infty$ and $m(t)$ be a non-negative function in $C^0([0, T])$ and $\rho(t)$ a positive function in $C^1([0, T]) \cap C^0([0, T])$ such that $\rho_t(t) < 0$ for $t \in [0, T]$. Consider the following Cauchy Problem,

$$(3.1) \quad \begin{cases} (\partial_t - \Lambda_t)^2 v(t) + m(t)A_\Lambda v(t) = g(t), & t \in (0, T) \\ v(0) = v_0, \\ \partial_t v(0) = v_1, \end{cases}$$

where $\Lambda(t) = \rho(t)\langle D \rangle$, $\Lambda_t(t) = \rho_t(t)\langle D \rangle$ and $A_\Lambda = e^{\Lambda(t)} A e^{-\Lambda(t)}$. Then by (ii) of Proposition 2.3 we have

$$(3.2) \quad A_\Lambda = A + \rho(t)a_1(x, D) + \rho(t)^2 a_2(\rho(t); x, D) + r(\rho(t); x, D),$$

where

$$\begin{aligned}
 a(x, \xi) &= \sum_{i,j} a_{ij}(x) \xi_i \xi_j, \\
 a_1(x, \xi) &= - \sum_{|\alpha|=1} a_{(\alpha)}(x, \xi) \xi^\alpha \langle \xi \rangle^{-1} \in C^0([0, T]; S^2), \\
 a_2(t; x, \xi) &\in C^0([0, T]; S^2),
 \end{aligned}$$

and

$$r(\rho(t); x, \xi) \in C^0([0, T]; S^1).$$

Let $\tilde{m}(t)$ and $\lambda(t)$ be positive functions in $C^1([0, T])$ and assume $\lambda'(t) \leq 0$ for $t \geq 0$. Define

$$(3.3) \quad E(t)^2 = \frac{1}{2} \{ \|(\partial_t - \Lambda_t)v(t)\|_s^2 + \lambda(t) \|v(t)\|_{s+1}^2 + \tilde{m}(t)(A\langle D \rangle^s v(t), \langle D \rangle^s v(t))_{L^2} \}$$

for $t \in [0, T]$, where $(\cdot, \cdot)_s$ and $\|\cdot\|_s$ stand for an inner product and a norm of H^s respectively.

Assume that $v(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{j+s})$ is a solution of (3.1). Differentiating (3.3) we have

$$\begin{aligned}
 2E'(t)E(t) &= \Re(-m(t)A_\Lambda v + g, (\partial_t - \Lambda_t)v)_s \\
 &\quad + \rho_t(t) \|(\partial_t - \Lambda_t)v\|_{s+\frac{1}{2}}^2 \\
 &\quad + \tilde{m}_t(t)(A\langle D \rangle^s v, \langle D \rangle^s v)_{L^2} \\
 &\quad + \Re((\partial_t - \Lambda_t)v, v)_{s+1} \lambda(t) + \lambda'(t) \|v(t)\|_{s+1}^2 \\
 &\quad + \tilde{m}(t) \{ \Re(\langle D \rangle^{-s} A\langle D \rangle^s v, (\partial_t - \Lambda_t)v)_s + \Re(\Lambda_t \langle D \rangle^{-s} A v, v)_s \} \\
 &\quad + \rho_t(t) \|v\|_{s+\frac{3}{2}}^2 \lambda(t) \\
 (3.4) \quad &\leq \Re(g, (\partial_t - \Lambda_t)v)_s + \tilde{m}(t) \Re(\Lambda_t \langle D \rangle^{-s} A\langle D \rangle^s v, v)_s \\
 &\quad + |\tilde{m}_t(t)| (A\langle D \rangle^s v, \langle D \rangle^s v)_s \\
 &\quad + \Re((\tilde{m}(t)\langle D \rangle^{-s} A\langle D \rangle^s - m(t)A_\Lambda)v, (\partial_t - \Lambda_t)v)_s \\
 &\quad + \frac{1}{2} \rho_t(t) \|(\partial_t - \Lambda_t)v\|_{s+\frac{1}{2}}^2 + \lambda(t) \left\{ \rho_t + \frac{\lambda}{|\rho_t|} \right\} \|v\|_{s+\frac{3}{2}}^2 \\
 &\leq \|g(t)\|_s E(t) + \tilde{m}(t) \Re(\Lambda_t \langle D \rangle^{-s} A\langle D \rangle^s v, v)_s \\
 &\quad + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)} E(t)^2 \\
 &\quad + \| |\Lambda_t|^{-\frac{1}{2}} (\tilde{m}(t)\langle D \rangle^{-s} A\langle D \rangle^s - m(t)A_\Lambda)v \|_s^2 \\
 &\quad + \frac{1}{4} \rho_t(t) \|(\partial_t - \Lambda_t)v\|_{s+\frac{1}{2}}^2 + \lambda \left\{ \rho_t + \frac{\lambda}{|\rho_t|} \right\} \|v\|_{s+\frac{3}{2}}^2
 \end{aligned}$$

for $t \in [0, T)$. Since A is a positive operator, by taking into account (2.2) we have

$$\begin{aligned}
 & \Re(\Lambda_t \langle D \rangle^{-s} A \langle D \rangle^s v, v)_s \\
 &= \rho_t(t) \Re(\langle D \rangle^{1-s} A \langle D \rangle^s v, v)_s \\
 (3.5) \quad & \leq \rho_t(t) (A \langle D \rangle^{s+\frac{1}{2}} v, \langle D \rangle^{s+\frac{1}{2}} v)_{L^2} + c |\rho_t(t)| \|v\|_{s+1}^2 \\
 & \leq c \frac{|\rho_t|}{\lambda(t)} E(t)^2
 \end{aligned}$$

where c is a positive constant depending only on s and A .

The equality

$$\begin{aligned}
 & \tilde{m}(t) \langle D \rangle^{-s} A \langle D \rangle^s - m(t) A_\Lambda \\
 (3.6) \quad &= (\tilde{m}(t) - m(t)) \langle D \rangle^{-s} A \langle D \rangle^s + m(t) (A - A_\Lambda) + m(t) (\langle D \rangle^{-s} A \langle D \rangle^s - A)
 \end{aligned}$$

and (3.2) lead us to the estimate

$$\begin{aligned}
 & \| |\Lambda_t(t)|^{-\frac{1}{2}} (\tilde{m}(t) \langle D \rangle^{-s} A \langle D \rangle^s - m(t) A_\Lambda) v \|_s \\
 & \leq |\tilde{m}(t) - m(t)| \| |\Lambda_t(t)|^{-\frac{1}{2}} \langle D \rangle^{-s} A \langle D \rangle^s v \|_s \\
 & \quad + m(t) \{ \rho(t) \| |\Lambda_t(t)|^{-\frac{1}{2}} a_1 v \|_s + \rho(t)^2 \| |\Lambda_t(t)|^{-\frac{1}{2}} a_2 v \|_s \\
 (3.7) \quad & \quad + \| |\Lambda_t(t)|^{-\frac{1}{2}} r v \|_s + c m(t) \| v \|_{s+1} \} \\
 & \leq |\rho_t(t)|^{-\frac{1}{2}} \left\{ c |\tilde{m}(t) - m(t)| \| v \|_{s+\frac{3}{2}} + m(t) \rho(t) \| a_1 v \|_{s-\frac{1}{2}} \right. \\
 & \quad \left. + c m(t) \rho(t)^2 \| v \|_{s+\frac{3}{2}} + \frac{c m(t)}{\sqrt{\lambda(t)}} E(t) \right\}
 \end{aligned}$$

for $t \in [0, T)$. Besides, by virtue of (2.3) we have

$$\begin{aligned}
 & \| a_1 v \|_{s-\frac{1}{2}}^2 \leq c \{ 2c \| v \|_{s+\frac{1}{2}}^2 + \Re(A v, v)_{s+\frac{1}{2}} \} \\
 (3.8) \quad & \leq c \{ 3c \| v \|_{s+\frac{1}{2}}^2 + (A \langle D \rangle^{s+\frac{1}{2}} v, \langle D \rangle^{s+\frac{1}{2}} v) \},
 \end{aligned}$$

where c is a positive constant depending only on s and the coefficients of A . Therefore, from (3.4) through (3.8), we come to the conclusion that

$$\begin{aligned}
 & 2E'(t)E(t) \leq \| g(t) \|_s E(t) \\
 & \quad + \left\{ \frac{c |\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2 \rho^2}{|\rho_t(t)| \lambda(t)} + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)} + \frac{m(t)^2}{\lambda(t) |\rho_t(t)|} \right\} E(t)^2 \\
 (3.9) \quad & + \frac{\rho_t(t)}{4} \| (\partial_t - \Lambda_t) v \|_{s+\frac{1}{2}}^2
 \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \lambda(t) \left(\rho_t(t) + \frac{\lambda(t)}{|\rho_t(t)|} \right) + c \frac{|\tilde{m}(t) - m(t)|^2}{|\rho_t(t)|} + cm(t)^2 \frac{\rho(t)^4}{|\rho_t(t)|} \right\} \|v\|_{s+\frac{3}{2}}^2 \\
 & + \left\{ \tilde{m}(t)\rho_t(t) + cm(t)^2 \frac{\rho(t)^2}{|\rho_t(t)|} \right\} (A\langle D \rangle^{s+\frac{1}{2}}, \langle D \rangle^{s+\frac{1}{2}}v)_{L^2}
 \end{aligned}$$

for $t \in [0, T)$.

PROPOSITION 3.1. Assume that $m(t)$ is a non-negative function in $C^1([0, T])$. Let $\tilde{m}(t) = m(t) + \varepsilon e^{-\gamma t}$, $\lambda(t) = e^{-2\gamma t}$, $\rho(t) = \rho_1 e^{-\gamma t}$, and $v(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{j+s})$. Then there are $\varepsilon > 0$ and $\gamma > 0$ such that if $v(t)$ satisfies (3.1) we have

$$(3.10) \quad E(t) \leq e^0 \int_0^t p(\tau) d\tau + E(0) + \int_0^t e^\tau \int_0^\tau p(\sigma) d\sigma \|g(\tau)\|_s d\tau$$

for $t \in [0, T)$, where

$$(3.11) \quad p(t) = c\gamma e^{\gamma t} + m(t)^2 \frac{\rho_1^2}{\gamma} e^{\gamma t} + \frac{|m_t(t)|}{\tilde{m}(t)} + \frac{m(t)^2 e^{3\gamma t}}{\gamma}.$$

PROOF. It suffices to prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative, if $\varepsilon > 0$ and $\gamma > 0$ are suitably chosen. In fact the third term is negative because of $\rho_t(t) < 0$. The fourth term is

$$\begin{aligned}
 (3.13) \quad & \lambda(t) \left\{ \rho_t(t) + \frac{\lambda(t)}{|\rho_t(t)|} \right\} + c \frac{|\tilde{m}(t) - m(t)|^2}{|\rho_t(t)|} + cm(t)^2 \frac{\rho(t)^4}{|\rho_t(t)|} \\
 & = -\frac{\rho_1 \gamma}{2} e^{-3\gamma t} + c\varepsilon^2 \frac{e^{-\gamma t}}{\rho_1 \gamma} + c \frac{\rho_1^3}{\gamma} m(t)^2 e^{-3\gamma t} < 0
 \end{aligned}$$

if we take

$$(3.14) \quad \gamma^2 > \frac{3}{2\rho_1^2} + c\rho_1^2 m(t)^2, \quad \varepsilon = \rho^{-\gamma T}.$$

Moreover we have the fifth term

$$\begin{aligned}
 (3.15) \quad & \tilde{m}(t)\rho_t(t) + cm(t)^2 \frac{\rho(t)^2}{|\rho_t(t)|} \\
 & \leq -\rho_1 \gamma m(t) e^{-\gamma t} + c \frac{\rho_1}{\gamma} m(t)^2 e^{-\gamma t} < 0
 \end{aligned}$$

if we take $\gamma > 0$ such that

$$(3.16) \quad \gamma^2 > c \max_{0 \leq t \leq T} m(t).$$

Therefore, choosing $\varepsilon > 0$ and $\gamma > 0$ such that (3.14) and (3.16) are valid, we obtain (3.11) from (3.9). \square

For $m(t) \in L^1([0, T])$ and $\varepsilon > 0$, we define

$$(3.17) \quad \tilde{m}(t) = \int_0^T \chi_\varepsilon(t - \tau)m(\tau)d\tau + \varepsilon$$

where $\chi_\varepsilon(t) = \frac{1}{\varepsilon} \chi\left(\frac{t}{\varepsilon}\right)$ and $\chi(t) \in C_0^\infty((0, 1))$ satisfying that $\chi(t) \geq 0$ and $\int_0^1 \chi(t)dt = 1$.

PROPOSITION 3.2. Assume that $m(t)$ is a non-negative function in $C^1([0, T]) \cap L^1([0, T])$. Let $\tilde{m}(t)$ be a function defined by (3.17) and $v(t) \in \bigcap_{j=0}^2 C^{2-j}(0, T); H^{s+j}$. Then there are $\rho(t)$ and $\lambda(t)$ in $C^1([0, T])$ with $\rho_t(t) \in L^1([0, T])$ and $\varepsilon > 0$ such that if $v(t)$ satisfies (3.1) we have

$$(3.18) \quad E(t) \leq e^{\int_0^t p(\tau)d\tau} E(0) + \int_0^t e^{\int_0^\tau p(\sigma)d\sigma} \|g(\tau)\|_s d\tau$$

for $t \in [0, T)$, where $E(t) = E(t, s, \tilde{m}(t), \rho(t))$ is defined by (3.3) and

$$(3.19) \quad p(t) = \frac{c|\rho_t(t)|}{\lambda(t)} + \frac{m(t)^2}{|\rho_t(t)|} (\rho(t)^2 + 1) + \frac{|\tilde{m}_t(t)|}{\tilde{m}(t)},$$

where c depends only on s and A .

PROOF. If we choose $\rho(t)$ and $\varepsilon > 0$ suitably, we can prove that the terms in the right-hand side of (3.9) except for the first one and the second one are negative. We can take $\rho(t)$ with $\rho_t(t) < 0$ such that the first terms of (3.13) and (3.15) are negative respectively. In fact, it suffices to find a function $\rho(t)$ satisfying

$$(3.20) \quad \begin{cases} \rho_t(t) \leq -c \left\{ \frac{|\tilde{m}(t) - m(t)|}{\sqrt{\lambda(t)}} + \frac{m(t)\rho(t)^2}{\sqrt{\lambda(t)}} + \frac{m(t)\rho(t)}{\sqrt{\tilde{m}(t)}} + \sqrt{\lambda(t)} \right\} \\ \rho(0) = \rho_1. \end{cases} \quad (t \in (0, T))$$

Put

$$\begin{aligned}
 (3.21) \quad \rho(t) &= \rho_1 e^{-ct} - \int_0^t \frac{|\tilde{m}(\tau) - m(\tau)|}{\sqrt{\rho_1}} d\tau, \\
 \lambda(t) &= \rho_1^2 e^{-2c \left\{ t + \int_0^t m(\tau)(1+1/\tilde{m}(\tau)) d\tau \right\}}.
 \end{aligned}$$

Here we take $\varepsilon > 0$ sufficiently small such that $\rho(t) > 0$ for $t \in [0, T]$. Since $\rho(t) \leq \sqrt{\lambda(t)}$ and $\lambda(t) \leq \lambda(0)$, we can see easily that $\rho(t)$ defined by (3.21) satisfies (3.20). Hence, we obtain (3.18) from (3.9) defining $p(t)$ by (3.19). \square

4. - Existence of solutions for the linear problem

In this section, we consider the following linear Cauchy problem:

$$(4.1) \quad \begin{cases} \partial_t^2 u(t) + m(t)Au(t) = f(t), & t \in (0, T) \\ u(0) = u_0, \\ \partial_t u(0) = u_1. \end{cases}$$

Following the idea of the proof of the theorem in [5], we shall prove that the Cauchy problem (4.1) has a unique solution.

THEOREM 4.1. *Assume that (1.2) and (1.3) are valid. Let $0 < \rho_1 < \rho_0/\sqrt{n}$, $s \in \mathbb{R}$ and $m(t) \in C^0([0, T])$. Then there is $\gamma > 0$ such that for any $u_0 \in H_{\rho_1}^{s+2}$, $u_1 \in H_{\rho_1}^{s+2}$ and $e^{\Lambda(t)}f(t) \in C^0([0, t]; H^{s+1})$, (4.1) has a unique solution $u(t)$ satisfying $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$, where $\Lambda(t) = \rho_1 e^{-\gamma t} \langle D \rangle$. Moreover if $m(t) \in C^1([0, T])$, the solution $u(t)$ satisfies*

$$\begin{aligned}
 (4.2) \quad & \{ \|e^{\Lambda(t)} \partial_t u(t)\|_s^2 + e^{-2\gamma t} \|e^{\Lambda(t)} u(t)\|_{s+1}^2 \}^{1/2} \\
 & \leq e^{\int_0^t p(\sigma) d\sigma} \left[\{ \|e^{\rho_1 \langle D \rangle} u_1\|_s^2 + (m(0) + \varepsilon) (Ae^{\rho_1 \langle D \rangle} \langle D \rangle^s u_0, e^{\rho_1 \langle D \rangle} \langle D \rangle^s u_0)_{L^2} \right. \\
 & \quad \left. + \|e^{\rho_1 \langle D \rangle} u_0\|_{s+1}^2 \}^{1/2} + \int_0^t \|e^{\Lambda(\sigma)} f(\sigma)\|_s d\sigma \right],
 \end{aligned}$$

for $t \in [0, T]$, where $p(t)$, γ and ε are given by Proposition 3.1.

PROOF. Put $v(t) = e^{\Lambda(t)}u(t)$. If $v(t)$ is a solution of (3.1), it is evident that $u(t)$ satisfies (4.1). So it suffices to prove that problem (3.1) has a solution.

Now we put

$$\begin{aligned} V_1(t) &= \langle D \rangle v(t), \\ V_2(t) &= (\partial_t - \Lambda_t)v(t), \\ V(t) &= {}^t(V_1(t), V_2(t)). \end{aligned}$$

Then if $v(t)$ is a solution of (3.1), $V(t)$ satisfies

$$(4.3) \quad \begin{cases} \frac{d}{dt} V(t) = P(t)V(t) + F(t), & t > 0 \\ V(t) = V_0, \end{cases}$$

where $F(t) = {}^t(0, g(t))$, $V_0 = {}^t(v_0, v_1)$ and

$$(4.4) \quad P(t) = \begin{pmatrix} \Lambda_t & \langle D \rangle \\ m(t)A_\Lambda \langle D \rangle^{-1} & \Lambda_t \end{pmatrix}.$$

Conversely, it is evident that if $V(t)$ is a solution of (4.3), then $v(t) = \langle D \rangle^{-1}V_1(t)$ becomes a solution of (3.1). It follows from (4.4) and (ii) of Proposition 2.3 that $P(t)$ is a pseudo-differential operator of order 1 with symbol satisfying

$$\begin{aligned} \det(\lambda I - p(t; x, \xi)) &= (\lambda + \gamma\rho(t)\langle \xi \rangle)^2 - \gamma^2\rho(t)\langle \xi \rangle \\ &\quad + m(t)\{a(x, \xi) + \rho(t)a_1(x, \xi) + \rho(t)^2a_2(a, \xi) + r(x, \xi)\}. \end{aligned}$$

Since $m(t) \geq 0$, $a(x, \xi) \geq 0$ and $r \in S^1$ there are $\gamma_0 > 0$ and $R_0 > 0$ such that $\det(\lambda - p(t, x, \xi)) \neq 0$ for $\Re \lambda \geq -2^{-1}\gamma e^{-\gamma T}\langle \xi \rangle$, $\gamma \geq \gamma_0 \sup_{0 \leq t \leq T} m(t)$ and $|\xi| \geq R_0\gamma^{-1}e^{2\gamma T}$. Therefore it follows from Proposition 2.4 that there exists a unique solution $V(t)$ of (4.3) and consequently $v(t) = \langle D \rangle^{-1}V_1(t)$ satisfies (3.1) and belongs to $\bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$. Put $u(t) = e^{-\Lambda(t)}v(t)$. Then $u(t)$ satisfies (4.1) and $e^{\Lambda(t)}u(t)$ is in $\bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$. If $m(t)$ is in $C^1([0, T])$, it follows from Proposition 3.1 that $v(t)$ satisfies (3.10) so $u(t)$ satisfies (4.2). In particular, if $u_0 = u_1 \equiv 0$, $f(t) \equiv 0$ and $e^{\Lambda(t;\gamma)}u(t) \subset \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$ for some $\gamma > 0$, $u(t)$ identically vanishes. This implies the uniqueness of the the solution of (4.1). Note that $v(t)$ may depend on γ but $u(t) = e^{-\Lambda(t)}v(t)$ does not depend on γ . In fact, $\tilde{u}(t) = u(t; \gamma) - u(t; \gamma')$ satisfies (4.1) with $u_0 = u_1 = f(t) \equiv 0$ and $e^{\Lambda(t;\bar{\gamma})}\tilde{u}(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$, where $\bar{\gamma} = \max(\gamma, \gamma')$. Therefore we have $\tilde{u}(t) \equiv 0$ from the uniqueness of solution of (4.1), and consequently $u(t; \gamma) = u(t; \gamma')$. \square

Finally we remark that it follows from (4.2) that $u(t)$, the solution of (4.1),

satisfies

$$\begin{aligned}
 & \|\partial_t u(t)\|_s + e^{-\gamma t} \|u(t)\|_{s+1} \\
 & \leq \|e^{\Lambda(t)} \partial_t u(t)\|_s + e^{-\gamma t} \|e^{\Lambda(t)} u(t)\|_{s+1} \\
 (4.5) \quad & \leq c e^0 \int_0^t p(\sigma) d\sigma \left\{ \|e^{\rho_1(D)} u_1\|_s + \|e^{\rho_1(D)} u_0\|_{s+1} + \int_0^t \|e^{\Lambda(\sigma)} f(\sigma)\|_s d\sigma \right\}
 \end{aligned}$$

for $t \in [0, T]$, where the positive constant c is independent of γ .

5. - Local existence of solutions of the nonlinear problem

Let $0 \leq \tau < T_1 < \infty$. For $T \in (\tau, T_1]$ we consider the Cauchy problem

$$(5.1) \quad \begin{cases} \partial_t^1 u(t) + M((Au(t), u(t)))Au(t) = f(t), & \tau < t < T \\ u(\tau) = u_0, \\ u(\tau) = u_1. \end{cases}$$

THEOREM 5.1. *Assume that the conditions (1.2), (1.3) and (1.4) are valid. Let $s \in \mathbb{R}$ and $0 < \rho_2 < \rho_0/\sqrt{n}$. Then for any $u_0 \in H_{\rho_2}^2$, $u_1 \in H_{\rho_2}^1$ and $e^{\Lambda(t)} f(t) \in C^0([\tau, T_1]; H^1)$ where $\Lambda(t) = \rho_2 e^{-\gamma(t-\tau)} \langle D \rangle$, there are $T \in (\tau, T_1]$ and $\gamma_0 > 0$ such that the Cauchy problem (4.1) has a unique solution $u(t)$ satisfying $e^{\Lambda(t)} u(t) \in \bigcap_{j=0}^2 C^{2-j}([\tau, T]; H^j)$ for any $\gamma \geq \gamma_0$.*

PROOF. We may assume $\tau = 0$ without loss of generality. We shall prove the existence of solutions of (5.1) by the principle of *contraction mapping*. For $T > 0$ and $s \in \mathbb{R}$, we introduce a space of functions

$$X_T^s = C^0([0, T]; H^{s+1}) \cap C^1([0, T]; H^s)$$

equipped with its *norm* $\|\cdot\|_{X_T^s}$ as

$$(5.2) \quad \|w\|_{X_T^s} = \sup_{0 \leq t \leq T} \left\{ \frac{1}{2} (\|\partial_t u(t)\|_s^2 + \|u(t)\|_{s+1}^2) \right\}^{1/2}$$

for every $w \in X_T^s$. We now define two functions

$$\begin{aligned}
 m(t) &= m(t; w) = M(\eta(t; w)), \\
 (5.3) \quad \eta(t; w) &= \sum_{i,j=1}^n (a_{ij} D_j w(t), D_i w(t))_{L_2},
 \end{aligned}$$

for each $w \in X_T^1$. Note that $m(t) \in C^1([0, T])$ if $w \in X_T^1$, and it satisfies

$$(5.4) \quad \sup_{0 \leq t \leq T} \{m(t) + |m'(t)|\} \leq K(\|w\|_{X_T^1}),$$

where K is a positive and continuous function defined in $[0, \infty)$.

Let us consider the Cauchy problem (4.1) with $m(t) = m(t; w)$. Then it follows from Theorem 4.1 that there exists a unique solution $u(t)$ of (4.1) satisfying that $e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^j)$, where $\Lambda(t) = \rho_1 e^{-\gamma t} \langle D \rangle$. So the correspondence with each $w \in X_T^1$ to $u \in X_T^1$ defines a map

$$\Psi : X_T^1 \ni w \mapsto u \in X_T^1$$

such that

$$\Psi(w) = u; \quad \partial_t^2 u + m(t; w)Au = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1.$$

We shall prove that Ψ is a contraction mapping if T is sufficiently small. For $k > 0$, let us define a set

$$B_T(k) = \left\{ e^{\Lambda(t)}u(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^j); \|u\|_{X_T^1} \leq k \right\}.$$

Then we can prove that for every $k \gg 1$ there is a real number $T = T(k) > 0$ such that $\Psi(w) \in B_T(k)$ as long as $w \in B_T(k)$. Actually, we can gain an estimate

$$(5.5) \quad \|\Psi(w)\|_{X_T^1} \leq ce^{\int_0^T (p(t)+\gamma)dt} \quad \text{for } w \in B_T(k),$$

which is deduced from the estimate (4.5) with $s = 1$ and the fact that

$$\Lambda(t, \xi) = \rho_2 e^{-\gamma t} \langle \xi \rangle \leq \rho_2 \langle \xi \rangle.$$

Note that the constant c appearing in (5.5) is independent of T , k and w . Since $p(t)$ is determined by (3.1) and (5.3), we can find a function $\bar{p}(t, k) \in C^0([0, T] \times [0, \infty))$, by virtue of (5.4), such that

$$p(t) + \gamma \leq \bar{p}(t, k) \quad t \in (0, T)$$

if $w \in B_T(k)$. Since the constant c in (5.5) is independent of k and the function $\bar{p}(t, k)$ is continuous in (t, k) , we can find $T = T(k) > 0$ such that

$$ce^{\int_0^T \bar{p}(t, k)dt} = k$$

for every $k > c$. Hence (5.5) implies that $\Psi(w)$ belongs to $B_T(k)$ provided $w \in B_T(k)$.

Next we shall prove Ψ is Lipschitz continuous in X_T^0 , that is, with sufficiently small $T > 0$ we have the inequality

$$(5.6) \quad \|\Psi(w) - \Psi(w')\|_{X_T^0} \leq \frac{1}{2} \|w - w'\|_{X_T^0}$$

for any $w, w' \in B_T(k)$. Since the difference $\Psi(w) - \Psi(w')$ satisfies

$$(\partial_t^2 + m(t; w)A)(\Psi(w) - \Psi(w')) = (m(t; w') - m(t; w))A\Psi(w'), \quad t > 0;$$

$$(\Psi(w) - \Psi(w'))(0) = 0,$$

$$\partial_t(\Psi(w) - \Psi(w'))(0) = 0,$$

we obtain, by virtue of (4.5) with $s = 0$

$$(5.7) \quad \begin{aligned} \|\Psi(w) - \Psi(w')\|_{X_T^0} &\leq c e^{\int_0^T p(\sigma) d\sigma} \\ &\times \int_0^T |m(\sigma; w) - m(\sigma; w')| \|e^{\Lambda(\sigma)} A\Psi(w')\|_{L^2} d\sigma. \end{aligned}$$

On the other hand, an application of Proposition 2.3 to A and the estimate (4.5) with $s = 1$ yield

$$\begin{aligned} \|e^{\Lambda(\sigma)} A\Psi(w')(\sigma)\|_{L^2} &\leq c \|e^{\Lambda(\sigma)} \Psi(w')(\sigma)\|_2 \\ &\leq c_1 e^{\int_0^\sigma \bar{p}(\tau, k) d\tau} \leq C_1(k) \end{aligned}$$

for $w' \in B_T(k)$. Moreover, taking into account (5.4) we gain

$$\begin{aligned} |m(\sigma; w) - m(\sigma; w')| &\leq \|M((Aw, w)) - M((Aw', w'))\| \\ &\leq C_2(k) \|w - w'\|_{X_T^0}. \end{aligned}$$

Hence, from (5.7) we have $C_3(k) > 0$ satisfying

$$\|\Psi(w) - \Psi(w')\|_{X_T^0} \leq C_3(k) T \|w - w'\|_{X_T^0}$$

for $w, w' \in B_T(k)$, which proves assertion (4.6) if $T \leq (2C_3(k))^{-1}$.

Thus once we choose $T = \min\{T(k), (2C_3(k))^{-1}\}$, we can find the solution u of (5.1) with the initial plane $\tau = 0$ which belongs to $B_T(k)$. \square

6. - Existence of time global solutions for the nonlinear problem

In this section we shall prove our *main theorem*. According to D’Ancona and Spagnolo [3], we introduce the following energy,

$$(6.1) \quad e(t)^2 = \frac{1}{2} \{ \|\partial_t u(t) + u(t)\|^2 + \|u(t)\|^2 + F(\eta(t)) \}$$

where $F(\eta) = \int_0^\eta M(\lambda) d\lambda$, $\eta(t) = ((Au(t), u(t)))_{L^2}$ and $\|\cdot\|$ stands for a norm of $L^2(\mathbb{R}^n)$.

PROPOSITION 6.1 ([3]). *Assume that $M(\eta)$ is a non-negative continuous function in $[0, \infty)$ and $f(t) \in C^0([0, T]; L^2)$. If $u(t)$ is a solution of the Cauchy problem of (1.1) in $(0, T)$ such that $u \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^j)$, then we have the energy inequality*

$$(6.2) \quad e(t)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)} M(\eta(\tau)) \eta(\tau) d\tau \leq e^{\frac{5}{2}t} e(0)^2 + \int_0^t e^{\frac{5}{2}(t-\tau)} \|f(\tau)\|^2 d\tau$$

for $t \in [0, T)$.

PROOF. Differentiating (6.1), we get from (1.1)

$$\begin{aligned} \frac{d}{dt}(e(t)^2) &= \Re(f(t) + \partial_t u(t), \partial_t u(t) + u(t)) - M(\eta(t))\eta(t) \\ &\leq \frac{1}{2} \|f(t)\|^2 + \frac{5}{2} e(t)^2 - M(\eta(t))\eta(t) \end{aligned}$$

for $t \in [0, T)$, which yields (6.2). □

PROPOSITION 6.2 ([3]). *If (6.2) holds and $T < \infty$, then $M(\eta(t)) \in L^1([0, T])$.*

PROOF. From (6.2), it is evident that $M(\eta(t))\eta(t) \in L^1([0, T])$. On the other hand

$$\begin{aligned} \int_0^t M(\eta(\tau)) d\tau &= \int_{[0,t] \cap \{\tau: \eta(\tau) > 1\}} M(\eta(\tau)) d\tau + \int_{[0,t] \cap \{\tau: \eta(\tau) \leq 1\}} M(\eta(\tau)) d\tau \\ &\leq \int_0^t M(\eta(\tau)) \eta(t) d\tau + t \sup_{0 \leq \eta \leq 1} M(\eta) \end{aligned}$$

for all $t \in [0, T)$, which implies that $M(\eta(t)) \in L^1([0, T])$, □

Now we can prove our main theorem. Let $\Lambda(t, \gamma) = \rho_1 e^{-\gamma t \langle D \rangle}$, and let T^* be a real number defined by

$$T^* = \max \left\{ T > 0; \text{there exist } \gamma > 0 \text{ and a solution } u(t) \text{ satisfying (1.1)} \right. \\ \left. \text{in } (0, T) \text{ such that } e^{\Lambda(t, \gamma)} u(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^j) \right\}.$$

Theorem 4.1 ensures that $T^* > 0$. We claim that $T^* = \infty$. Suppose that $T^* < \infty$. Then it follows from Proposition 6.2 that $m(t) = M(Au(t), u(t))$ is in $L^1([0, T^*])$. Hence, Proposition 3.2 and the fact that $m(t) \in C^1([0, T^*]) \cap L^1([0, T^*])$ yield that $v(t) = e^{\Lambda(t)} u(t)$ which satisfies (3.18) with $s = 0, 1$ and $T = T^*$, where $\Lambda(t) = \rho(t) \langle D \rangle$ and $\rho(t)$ is what is introduced in (3.21). Let us take $\gamma > 0$ such that $\rho_1 e^{-\gamma t} \leq \rho(t)$ for $t \in [0, T^*]$. Then the definition of T^* and (3.18) imply $e^{\Lambda(t, \gamma)} u(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T^*]; H^j)$, where $\Lambda(t, \gamma) = \rho_1 e^{-\gamma t} \langle D \rangle$. Hence we have the limits $u(T^* - 0)$ and $\partial_t u(T^* - 0)$ which satisfy $e^{\Lambda(T^*, \gamma)} u(T^* - 0) \in H^2$ and $e^{\Lambda(T^*, \gamma)} \partial_t u(T^* - 0) \in H^1$. Therefore, applying Theorem 5.1 with $\rho_2 = \rho_1 e^{-\gamma T^*}$, we have a solution $\tilde{u}(t)$ of the Cauchy problem (5.1) in (T^*, T) ($T > T^*$) with initial data $\tilde{u}(T^*) = u(T^* - 0)$ and $\partial_t \tilde{u}(T^*) = \partial_t u(T^* - 0)$, which satisfies

$$\exp(\rho_2 e^{-\gamma(T-T^*)} \langle D \rangle) \tilde{u}(t) \in \bigcap_{j=0}^2 C^{2-j}([T^*, T]; H^j).$$

Then $\Lambda(t, \gamma) = \rho_2 e^{-\gamma(T-T^*)} \langle D \rangle$ implies that $e^{\Lambda(t, \gamma)} \tilde{u}(t) \in \bigcap_{j=0}^2 C^{2-j}([T^*, T]; H^j)$. Now let us define

$$w(t) = \begin{cases} u(t), & t \in (0, T^*) \\ \tilde{u}(t), & t \in [T^*, T]. \end{cases}$$

Then $w(t)$ has to satisfy (1.1) in $(0, T)$ and $e^{\Lambda(t, \gamma)} w(t) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^j)$. This contradicts the definition of T^* . Thus, we have proved that $T^* = \infty$. Since $M(\eta)$ is of class C^1 , we can prove easily the uniqueness of the solution of (1.1). \square

REFERENCES

- [1] L. AROSIO - S. SPAGNOLO, *Global solution of the Cauchy problem for a nonlinear hyperbolic equation*, Pitman Research Notes in Math. **109** (1984), 1-26.
- [2] S. BERNSTEIN, *Sur une classe d'équations fonctionnelles aux dérivées partielles*. Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), 17-26.
- [3] P. D'ANCONA - S. SPAGNOLO, *Global solvability for the degenerate Kirchhoff equation with real analytic data*. Invent. Math. **108** (1992), 247-246.
- [4] C. FEFFERMAN - D.H. PHONG, *On positivity of pseudo-differential operators*. Proc. Nat. Acad. Sci. USA **75** (1978), 4677-4674.

