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On the Stationary Motion of Compressible Viscous Fluids

PAOLO SECCHI

1. - Introduction

In this paper we continue our study, see [5], about the stationary motion of a compressible, viscous and heat-conductive fluid in a bounded domain Ω of $\mathbb{R}^3$, in the presence of self-gravitation, with the velocity field satisfying a slip boundary condition instead of the usual adherence condition. The corresponding Navier-Stokes equations for the unknown velocity field $u(x) = (u_1(x), u_2(x), u_3(x))$, density $\rho(x)$ and absolute temperature $\Theta(x)$ are

$$(1.1) \qquad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla p(\rho, \Theta) = \rho[f - (u \cdot \nabla)u - \nabla U], \\ \operatorname{div}(\rho u) = 0, \\ -\chi\Delta\Theta + c_v \rho u \cdot \nabla\Theta + \Theta p'_\Theta \operatorname{div} u = \rho g + \alpha(u) \quad \text{in } \Omega. \end{cases}$$

Here the pressure $p = p(\rho, \Theta)$ is a known smooth function of ρ and Θ; U is the Newtonian gravitational potential given by

$$(1.2) \qquad U(x) = -\gamma \int_\Omega \frac{\rho(y)}{|x - y|}\, dy,$$

where γ is the constant of gravitation; μ is the shear viscosity and $\nu = \mu + \mu'$, where μ' is the bulk viscosity; χ is the coefficient of heat conductivity and c_v is the specific heat at constant volume. In order to avoid technicalities we will assume that the coefficients μ, ν, χ, c_v are constant. In general, μ and μ' must satisfy the physical constraints $\mu \geq 0$, $2\mu + 3\mu' \geq 0$; the latter implies $\nu \geq \mu/3$. Since the fluid is viscous we will assume $\mu > 0$ and also $\nu > \frac{\mu}{3}$, $\chi > 0$, $c_v > 0$

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(see Remark (iii) at the end of Section 3). Finally, f denotes the given external force field, g the given heat supply and $\alpha = \alpha(u)$ the dissipation function

$$\alpha(u) = 2\mu T(u) : T(u) + (\nu - \mu)(\operatorname{div} u)^2$$

where $T(u) = \frac{1}{2}(D_i u_j + D_j u_i)_{1\le i,j\le 3}$ is the deformation tensor and

$$T(u) : T(v) = \frac{1}{4}\sum_{i,j=1}^{3}(D_i u_j + D_j u_i)(D_i v_j + D_j v_i)$$

with $D_i = \dfrac{\partial}{\partial x_i}$. Since the total mass of the fluid is given, we impose the condition

$$\frac{1}{|\Omega|}\int_\Omega \rho(x)dx = m_0 \tag{1.3}$$

where $m_0 > 0$ is given. On the boundary $\Gamma \equiv \partial\Omega$, instead of the usual adherence condition $u = 0$, we impose for u the slip boundary condition

$$\begin{aligned} &u \cdot n = 0, \\ &t_i \cdot T(u) \cdot n = 0 \qquad \text{on } \Gamma,\ i = 1, 2, \end{aligned} \tag{1.4}$$

where n is the unit outward normal vector to Γ and t_1, t_2 span the tangent plane. For Θ we impose the Dirichlet condition

$$\Theta = \Theta_e \qquad \text{on } \Gamma \tag{1.5}$$

(for other boundary conditions for Θ see Remark (ii) at the end of Section 3).

In our previous paper [5] we proved the existence of a unique solution (u, ρ, Θ) in the Sobolev spaces $W^{j+2,p} \times W^{j+1,p} \times W^{j+2,p}$, for any integer $j \geq 1$ and real $p > 3$, provided that the data $(f, g, \Theta_e) \in W^{j,p} \times W^{j,p} \times W^{j+2-\frac{1}{p},p}(\Gamma)$ belong to a suitable neighbourhood of $(0, 0, \Theta_0)$, $\Theta_0 = \text{const} > 0$, and that γ is sufficiently small. The purpose of the present paper is to cover also the case $j = 0$, that is we prove the existence of a solution (u, ρ, Θ) in $W^{2,p} \times W^{1,p} \times W^{2,p}$ for small enough data $(f, g, \Theta_e - \Theta_0) \in L^p \times L^p \times W^{2-\frac{1}{p},p}(\Gamma)$ and small enough γ (for a different regularity of the temperature Θ see Remark (i) at the end of Section 3).

As in [5] the core of the paper is the study of the linearized system (2.1) for (u, σ), $\sigma = \rho - \rho_0$, where ρ_0 is the equilibrium state. In order to solve it we introduced an equivalent formulation of (2.1). Such a formulation, in the present context of a solution (u, σ) in $W^{2,p} \times W^{1,p}$, looses its meaning because of the lower regularity (see in particular (2.24), (2.25) in [5]). We overcome this difficulty by introducing a different approach which gives us the density as solution of a linear transport equation, obtained in turn as solution of a Neumann problem. The result is obtained without resorting to weak formulations of the equations for σ.

Moreover, this new approach applies as well to the case $j \geq 1$ already considered in [5], without additional difficulties (see the Remark at the end of Section 2).

Before stating our main result let us introduce some notation. By $c, C, C_i, k_i, i \geq 0$, we denote positive constants depending at most on Ω, j, p, unless explicitly stated otherwise.

We denote by $W^{j,p}$, j a positive integer, $1 < p \leq +\infty$, the Sobolev space $W^{j,p}(\Omega)$, endowed with the usual norm $\|\cdot\|_{j,p}$. For real $s > 0$, $W^{s,p}$ denotes the Sobolev space $W^{s,p}(\Omega)$ of fractional order s with norm $\|\cdot\|_{s,p}$ (for the definition see [1]). The norm in $L^p = L^p(\Omega)$ is denoted by $|\cdot|_p$, $1 \leq p \leq +\infty$. If $p = 2$ we write $W^{j,2} = H^j$ whose norm is simply denoted by $\|\cdot\|_j$; the norm of $L^2 = H^0$ is denoted by $\|\cdot\| (= |\cdot|_2)$. On the boundary we use trace spaces $W^{j-\frac{1}{p},p}(\Gamma)$ with norm $\|\cdot\|_{j-\frac{1}{p},p,\Gamma}$. For convenience we use the same symbols for spaces of vector-valued functions. We denote by $\overline{W}^{j,p}$ the space of scalar functions $\{\sigma \in W^{j,p} : \overline{\sigma} = 0\}$ where $\overline{\sigma}$ is the mean value of σ over Ω. We denote by $W_b^{j,p}$ the space of vector-valued functions u in $W^{j,p}$ such that $u \cdot n = 0$, $t_i \cdot T(u) \cdot n = 0$ on $\Gamma, i = 1, 2$ (here $j \geq 2$). Let us introduce the space $H = \{u \in H^1 : u \cdot n = 0$ on $\Gamma\}$ endowed by the norm $\|u\|_H^2 = \|\nabla u\|^2 = \sum_{i,j=1}^{3} \|D_i u_j\|^2$. In H this norm is equivalent to the H^1-norm since

$$\|u\| \leq k_0 \|u\|_H \qquad \text{for all } u \in H,$$

see [8]. Let us denote by H' its dual space with norm $\|\cdot\|_{H'}$.

Associated to the linear problem

$$\begin{cases} \mu \Delta u - \nu \nabla \operatorname{div} u = f & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \end{cases} \tag{1.6}$$

let us consider the following variational problem: find $u \in H$ such that

$$a(u, v) := 2\mu \int_\Omega T(u) : T(v) + (\nu - \mu) \int_\Omega \operatorname{div} u \operatorname{div} v = \int_\Omega f \cdot v \tag{1.7}$$

for any $v \in H$. Observe that the bilinear form $a(u, v)$ is bicontinuous in H.

To obtain the coerciveness in H of $a(u, v)$ we must exclude the rigid body motions

$$S = \{u \in H : T(u) = 0\} = \{u = b \wedge (x - x_0) : u \cdot n = 0 \text{ on } \Gamma\}$$

from H provided that $S \neq \emptyset$, i.e. Ω is a body of revolution around its axis of symmetry $b \in \mathbb{R}^3$. If $S \neq \emptyset$ we let $\mathbb{H}$ denote the subspace of vectors in H which are orthogonal to rigid motions. If $S = \emptyset$, then $\mathbb{H} = H$. For each $u \in \mathbb{H}$ we have Korn's inequality

$$\|u\|_H^2 \leq k_1 \int_\Omega T(u) : T(u), \tag{1.8}$$

see [8]. For the sake of simplicity we assume in our main Theorem 1 that $S = \emptyset$. Partial results in the case of domains Ω with symmetry can be obtained as in Section 4 of [5].

Let us now introduce the equilibrium solutions. By an equilibrium solution we mean a regular solution (u, ρ, Θ) of (1.1)-(1.5) in the case $f \equiv 0$, $g \equiv 0$ in Ω, such that $u \equiv 0$ in Ω, $\Theta \equiv \Theta_0 = \text{const} > 0$ in Ω and $\rho > 0$ in $\overline{\Omega}$. Hence ρ solves

$$\begin{cases} \rho \nabla U + \nabla p(\rho, \Theta_0) = 0 & \text{in } \Omega, \\ U(x) = -\gamma \displaystyle\int_\Omega \frac{\rho(y)}{|x-y|}\, dy & x \in \Omega. \end{cases} \tag{1.9}$$

From [5] we have:

PROPOSITION 1. *Let $p > 3$ and assume that $\Gamma \in C^2$, $p'_\rho(\rho, \Theta_0) > 0$ for $\rho > 0$. Then, given $\varepsilon > 0$ there exists $\gamma_0 > 0$ such that for any $0 \leq \gamma \leq \gamma_0$ there exists a solution $\rho_0 \in W^{2,p}$ of (1.9) such that $\rho_0 > 0$ in $\overline{\Omega}$ and*

$$\|\nabla \rho_0\|_{1,p} \leq \varepsilon. \tag{1.10}$$

Let us state now our main result.

THEOREM 1. *Let $p > 3$. Let us assume that $\Gamma \in W^{4-\frac{1}{p},p}$ and that Ω has no axis of simmetry, i.e. $S = \emptyset$. Let $p \in C^3$ with $p'_\rho(\rho, \Theta_0) > 0$ for $\rho > 0$. Let $(f, g, \Theta_e) \in L^p \times L^p \times W^{2-\frac{1}{p},p}(\Gamma)$. There exist positive constants c_0, γ_0 such that if*

$$0 \leq \gamma \leq \gamma_0, \quad |f|_p + |g|_p + \|\Theta_e - \Theta_0\|_{2-\frac{1}{p},p,\Gamma} \leq c_0,$$

then there exists a unique solution $(u, \rho, \Theta) \in W^{2,p} \times W^{1,p} \times W^{2,p}$ of problem (1.1)-(1.5).

Let (ρ_0, Θ_0) be an equilibrium solution with $\overline{\rho}_0 = m_0$ and let U_0 denote the gravitational potential corresponding to ρ_0; define $\sigma = \rho - \rho_0$, $\theta = \Theta - \Theta_0$. Let us write

$$p(\rho, \Theta) = p(\rho_0 + \sigma, \Theta_0 + \theta) = p(\rho_0, \Theta_0) + \pi\sigma + \pi_0\theta + \omega(\sigma, \theta)$$

where $\pi \equiv p'_\rho(\rho_0, \Theta_0) > 0$, $\pi_0 \equiv p'_\Theta(\rho_0, \Theta_0)$, $\omega(0,0) = 0$, $\omega(\sigma, \theta) = 0(|\sigma|^2 + |\theta|^2)$ as

$|\sigma| + |\theta| \to 0$. Problem (1.1)-(1.5) can be written as

$$
\tag{1.11}
\begin{cases}
\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla(\pi\sigma) = F(u,\sigma,\theta) & \text{in } \Omega,\\
\operatorname{div}(m_0 + \sigma)u = E(u) & \text{in } \Omega,\\
\chi\Delta\theta = G(u,\sigma,\theta) & \text{in } \Omega,\\
u\cdot n = 0 & \text{on } \Gamma,\\
t_i\cdot T(u)\cdot n = 0 & \text{on } \Gamma, i = 1,2,\\
\theta = \theta_e & \text{on } \Gamma,\\
\overline{\sigma} = 0,
\end{cases}
$$

where, by definition

$$
\tag{1.12}
\begin{cases}
F(u,\sigma,\theta) = (\rho_0 + \sigma)[f - (u\cdot\nabla)u - \nabla U] + \rho_0\nabla U_0\\
\qquad\qquad - \nabla[\pi_0\theta + \omega(\sigma,\theta)],\\
U(x) = -\gamma \displaystyle\int_\Omega \frac{\rho_0(y) + \sigma(y)}{|x-y|}\,dy,\\
E(u) = \operatorname{div}(m_0 - \rho_0)u,\\
G(u,\sigma,\theta) = -c_v(\rho_0 + \sigma)u\cdot\nabla\theta\\
\qquad\qquad + \dfrac{\Theta_0 + \theta}{\rho_0 + \sigma}p'_\Theta(\rho_0 + \sigma, \Theta_0 + \theta)u\cdot\nabla(\rho_0 + \sigma)\\
\qquad\qquad + (\rho_0 + \sigma)g + \alpha(u),\\
\theta_e = \Theta_e - \Theta_0.
\end{cases}
$$

Observe that $(1.11)_2$ is used to deduce the expression of G.

The plan of the paper is the following: in Section 2 we study the linearized system (2.1) while in Section 3 we consider the nonlinear problem (1.11) and prove Theorem 1.

2. - The linearized system

In this section we study the linear system

$$
\tag{2.1}
\begin{cases}
\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla(\pi\sigma) = F & \text{in } \Omega,\\
\operatorname{div}(m_0 u + \sigma v) = E & \text{in } \Omega,\\
u\cdot n = 0 & \text{on } \Gamma,\\
t_i\cdot T(u)\cdot n = 0 & \text{on } \Gamma, i = 1,2,\\
\overline{\sigma} = 0.
\end{cases}
$$

Here we assume that the given vector field v satisfies

$$v \cdot n = 0 \quad \text{on } \Gamma, \tag{2.2}$$

and that the given function E satisfies the necessary compatibility condition $\overline{E} = 0$.

THEOREM 2. *Let* $p > 3$, $\Gamma \in W^{4-\frac{1}{p},p} \subset C^3$, $S = \emptyset$, $p(\rho, \Theta) \in C^2$. *Assume that* $\rho_0 \in W^{2,p}$ *with* $\rho_0 > 0$ *in* $\overline{\Omega}$, $F \in L^p$, $E \in \overline{W}^{1,p}$ *and let* $v \in W^{2,p}$ *satisfy* (2.2). *There exist positive constants* k_2, k_3 *such that if*

$$\|\nabla \rho_0\|_{1,p} \leq k_2, \qquad \|v\|_{2,p} \leq k_3, \tag{2.3}$$

then there exists a unique solution $(u, \sigma) \in W_b^{2,p} \times \overline{W}^{1,p}$ *of problem* (2.1). *Moreover*

$$\|u\|_{2,p} + \|\sigma\|_{1,p} \leq C_0 \left(|F|_p + \|E\|_{1,p}\right) \tag{2.4}$$

where C_0 *depends on* $\Omega, p, \mu, \nu, \pi, \|\rho_0\|_{2,p}$.

PROOF. We prove this result by the continuity method. The first step consists in proving an a priori estimate for a solution (u, σ) in $H^1 \times L^2$.

LEMMA 2.1. *If* v *is sufficiently small, see* (2.11), *then a solution* (u, σ) *in* $H^2 \times H^1$ *of* (2.1) *satisfies*

$$\|u\|_H \leq A \left(\|F\|_{H'} + \|E\|\right), \tag{2.5}$$

$$\|\sigma\| \leq C_1 \left|\frac{1}{\pi}\right|_\infty \left(\|F\|_{H'} + \|u\|_H\right), \tag{2.6}$$

where

$$A = \frac{2}{\mu_*} \max\left\{\frac{2}{\mu_*} + \frac{\mu_*}{4}, \frac{4}{m_0^2 \mu_*} C_1^2 |\pi|_\infty^2 \left|\frac{1}{\pi}\right|_\infty^2\right\} + \frac{C_1}{m_0 \mu_*} |\pi|_\infty \left|\frac{1}{\pi}\right|_\infty,$$

$$\mu_* = \frac{1}{k_1} \min\{2\mu, 3\nu - \mu\} > 0.$$

PROOF. We first multiply $(2.1)_1$ by u and integrate over Ω. Integrating by parts and using Korn's inequality give

$$\mu_* \|u\|_H^2 \leq \int F \cdot u - \int \nabla(\pi\sigma) \cdot u \tag{2.7}$$

where $\int$ denotes integration over Ω. Using $(2.1)_2$ gives

$$\int \nabla(\pi\sigma) \cdot u = \frac{1}{2} \int (\pi\sigma)^2 \operatorname{div}\left(\frac{v}{\pi m_0}\right) - \int \frac{\pi\sigma}{m_0} E; \tag{2.8}$$

hence from (2.7), (2.8) we obtain

$$\frac{\mu_*}{2}\|u\|_H^2 \le \frac{1}{\mu_*}\|F\|_{H'}^2 + \frac{1}{2}\left|\operatorname{div}\left(\frac{v}{\pi m_0}\right)\right|_\infty \|\pi\sigma\|^2 + \frac{1}{m_0}\|E\|\,\|\pi\sigma\|. \tag{2.9}$$

Since from $(2.1)_1$ we have

$$\begin{aligned}\|\sigma\| &= \sup_{\psi\in H}\frac{\left|\int \sigma \operatorname{div}\psi\right|}{\|\operatorname{div}\psi\|}\\ &\le \left|\frac{1}{\pi}\right|_\infty \sup_{\psi\in H}\frac{\left|\int F\cdot\psi - a(\psi,u)\right|}{\|\operatorname{div}\psi\|} \le C_1\left|\frac{1}{\pi}\right|_\infty(\|F\|_{H'}+\|u\|_H),\end{aligned} \tag{2.10}$$

which gives (2.6), from (2.9), (2.10) we obtain (2.5) if

$$2C_1^2|\pi|_\infty^2\left|\frac{1}{\pi}\right|_\infty^2\left|\operatorname{div}\left(\frac{v}{\pi m_0}\right)\right|_\infty \le \frac{\mu_*}{4} \tag{2.11}$$

(see [5] for details). □

The next step consists in proving an a priori estimate of a solution in $W^{2,p}\times W^{1,p}$.

LEMMA 2.2. *If v is small enough, see* (2.23), *then a solution* $(u,\sigma)\in W^{2,p}\times W^{1,p}$ *of* (2.1) *satisfies* (2.4).

PROOF. Since for the below computations at least one more derivative is needed, we approximate u,σ,F,E by more regular functions. First of all we observe that $W_b^{3,p}$ is dense in $W_b^{2,p}$. Indeed, for $u\in W_b^{2,p}$ let $w_m\in W^{3,p}$ be such that $w_m\to u$ in the topology of $W^{2,p}$. We solve the following trace problem: find $z_m\in W^{3,p}$ such that

$$z_m\cdot n = w_m\cdot n$$

$$t_i\cdot T(z_m)\cdot n = t_i\cdot T(w_m)\cdot n,\ i=1,2,$$

on Γ. We have

$$\|z_m\|_{j,p}\le C(\|w_m\cdot n\|_{j-1/p,p,\Gamma}+\sum_{i=1,2}\|t_i\cdot T(w_m)\cdot n\|_{j-1-\frac{1}{p},p,\Gamma}),\ j=2,3,$$

which implies $z_m\to 0$ in $W^{2,p}$. Hence $u_m = w_m - z_m\in W_b^{3,p}$, $u_m\to u$ in $W^{2,p}$.

Moreover, $\overline{W}^{2,p}$ is dense in $\overline{W}^{1,p}$. Indeed, for $\sigma\in\overline{W}^{1,p}$ let $\tau_m\in W^{2,p}$ be such that $\tau_m\to\sigma$ in $W^{1,p}$. Then $\sigma_m=\tau_m-\overline{\tau}_m\in\overline{W}^{2,p}$ and $\sigma_m\to\sigma$ in $W^{1,p}$.

Given F,E,v as in Theorem 2 and a solution $(u,\sigma)\in W_b^{2,p}\times\overline{W}^{1,p}$ let us consider $u_m\in W_b^{3,p}$ with $u_m\to u$ in $W^{2,p}$, $\sigma_m\in\overline{W}^{2,p}$ with $\sigma_m\to\sigma$ in $W^{1,p}$,

$F_m \in W^{1,p}$ with $F_m \to F$ in L^p, $E_m \in \overline{W}^{2,p}$ with $E_m \to E$ in $W^{1,p}$ as $m \to +\infty$. For F, F_m let us consider the decompositions $F = \varphi + \nabla\psi$, $\varphi \in L^p$ with $\operatorname{div}\varphi = 0$ in Ω, $\varphi \cdot n = 0$ on Γ, $\psi \in \overline{W}^{1,p}$, $F_m = \varphi_m + \nabla\psi_m$, $\varphi_m \in W^{1,p}$ with $\operatorname{div}\varphi_m = 0$ in Ω, $\varphi_m \cdot n = 0$ on Γ, $\psi_m \in \overline{W}^{2,p}$; we have $\psi_m \to \psi$ in $W^{1,p}$ as $m \to +\infty$. From $(2.1)_2$ we deduce that $\operatorname{div}(\sigma v) \in \overline{W}^{1,p}$; let $a_m \in \overline{W}^{2,p}$ be such that $a_m \to \operatorname{div}(\sigma v)$ in $W^{1,p}$. For these approximations let us introduce the differences δ_m, ε_m defined by

$$\mu\Delta u_m - \nu\nabla \operatorname{div} u_m + \nabla(\pi\sigma_m) = F_m + \delta_m, \tag{2.12}$$

$$\operatorname{div}(m_0 u_m) + a_m = E_m + \varepsilon_m, \tag{2.13}$$

$\delta_m \in W^{1,p}$, $\varepsilon_m \in \overline{W}^{2,p}$, $\delta_m \to 0$ and $\varepsilon_m \to 0$ in L^p and $W^{1,p}$ respectively as $m \to +\infty$. Applying the div operator to (2.12) and the laplacian to (2.13) give respectively

$$\begin{aligned}(\nu + \mu)\Delta \operatorname{div} u_m + \Delta(\pi\sigma_m) &= \Delta\psi_m + \operatorname{div}\delta_m,\\ m_0\Delta \operatorname{div} u_m + \Delta a_m &= \Delta(E_m + \varepsilon_m).\end{aligned} \tag{2.14}$$

We eliminate $\Delta \operatorname{div} u_m$ from (2.14) and obtain

$$\Delta W_m = \operatorname{div}\left(\delta_m + \frac{\nu+\mu}{m_0}\nabla\varepsilon_m\right) \quad \text{in } \Omega, \tag{2.15}$$

where $W_m = \pi\sigma_m + \dfrac{\nu+\mu}{m_0}(a_m - E_m) - \psi_m$. Taking the scalar product on Γ of (2.12) times n and applying the normal derivate $\partial/\partial n$ to (2.13) give

$$\begin{aligned}\mu\Delta u_m \cdot n - \nu\frac{\partial}{\partial n}\operatorname{div} u_m + \frac{\partial}{\partial n}(\pi\sigma_m) &= \frac{\partial\psi_m}{\partial n} + \delta_m \cdot n,\\ m_0\frac{\partial}{\partial n}\operatorname{div} u_m + \frac{\partial a_m}{\partial n} &= \frac{\partial}{\partial n}(E_m + \varepsilon_m).\end{aligned} \tag{2.16}$$

Now we observe that the boundary conditions $(2.1)_{3,4}$ imply that $\Delta u_m \cdot n - \dfrac{\partial}{\partial n}\operatorname{div} u_m$ does not contain second order derivatives of u_m; indeed if the boundary is flat this difference is equal to zero, and in the general case this fact can be proved with a long but straightforward computation. Hence we can introduce a vector function h_1 and a matrix function h_2 such that

$$\mu\Delta u_m \cdot n = \mu\frac{\partial}{\partial n}\operatorname{div} u_m + h_1 \cdot u_m + h_2 : \nabla u_m \quad \text{on } \Gamma. \tag{2.17}$$

h_1 contains at most second order derivatives of n, t_1, t_2, hence $h_1 \in W^{1-\frac{1}{p},p}(\Gamma)$; h_2 contains at most first order derivatives, hence $h_2 \in W^{2-\frac{1}{p},p}(\Gamma)$.

From (2.16), (2.17) we obtain

$$\frac{\partial W_m}{\partial n} = h_1 \cdot u_m + h_2 : \nabla u_m + \left(\delta_m + \frac{\nu+\mu}{m_0}\nabla \varepsilon_m\right) \cdot n \quad \text{on } \Gamma. \tag{2.18}$$

We multiply (2.15) by $\phi \in W^{1,q}$, $\frac{1}{q} + \frac{1}{p} = 1$, integrate over Ω by parts and use (2.18). Passing to the limit as $m \to +\infty$ gives

$$\int \nabla W \cdot \nabla \phi = \int_\Gamma (h_1 \cdot u + h_2 : \nabla u)\phi \qquad \text{for any } \phi \in W^{1,q}, \tag{2.19}$$

where $W = \pi\sigma + \frac{\nu+\mu}{m_0}(\operatorname{div}(\sigma v)E) - \psi \in W^{1,p}$ and $\int_\Gamma$ denotes integration over Γ. Both sides of (2.19) define a linear continuous functional on $W^{1,q}$. The norm of the functional $\phi \mapsto \int \nabla W \cdot \nabla\phi$ is $|\nabla W|_p$. Given s, $\frac{1}{p} < s < 1$, $u \in W^{2,p} \subset W^{1+s,p}$ gives $h_1 \cdot u + h_2 : \nabla u \in W^{s-\frac{1}{p},p}(\Gamma) \subset L^p(\Gamma)$. Hence the norm of the functional $\phi \mapsto \int_\Gamma (h_1 \cdot u + h_2 : \nabla u)\phi$ can be estimated by $c\|u\|_{1+s,p}$. Then equality (2.19) implies

$$|\nabla W|_p \leq c\|u\|_{1+s,p}. \tag{2.20}$$

From the Poincaré inequality and the fact that $\operatorname{div}(\sigma v), E, \psi$ have mean value zero we deduce

$$|W|_p \leq |W - \overline{W}|_p + |\overline{W}|_p \leq c(|\nabla W|_p + |\pi|_\infty \|\sigma\|).$$

Then from the above inequality and (2.20) we obtain

$$\|W\|_{1,p} \leq c(\|u\|_{1+s,p} + |\pi|_\infty \|\sigma\|). \tag{2.21}$$

Consider now the linear transport equation

$$\pi\sigma + \frac{\nu+\mu}{m_0}\operatorname{div}(\sigma v) = \psi + \frac{\nu+\mu}{m_0}E + W \equiv \Lambda \quad \text{in } \Omega. \tag{2.22}$$

From [3], see in particular Theorem 2.3 and part (i) of the proof of Theorem 1.1, we have that, since $p > 3$, if

$$\frac{\nu+\mu}{m_0} C_2 \left\|\frac{v}{\pi}\right\|_{2,p} < \frac{1}{2}, \tag{2.23}$$

where C_2 is a suitable constant depending only on Ω, p, then for any $\Lambda \in W^{1,p}$, there exists a unique solution $\pi\sigma \in W^{1,p}$ of (2.22) and

$$\|\pi\sigma\|_{1,p} \leq 2\|\Lambda\|_{1,p}.$$

Using (2.21) we obtain

$$\|\pi\sigma\|_{1,p} \leq c\left(|F|_p + \frac{\nu+\mu}{m_0}\|E\|_{1,p} + \|u\|_{1+s,p} + |\pi|_\infty\|\sigma\|\right). \tag{2.24}$$

Consider now the elliptic system

$$\begin{cases} \mu\Delta u - \nu\nabla \operatorname{div} u = F - \nabla(\pi\sigma) & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2. \end{cases} \tag{2.25}$$

The weak formulation of (2.25) is (1.7) with $f = F - \nabla(\pi\sigma)$, where $a(u, v)$ is a bilinear form, bicontinuous and coercive in H. The boundary conditions are complementing in the sense of Agmon, Douglis and Nirenberg [2]. Hence the solution u belongs to $W^{2,p}$ if $F - \nabla(\pi\sigma) \in L^p$ and moreover

$$\|u\|_{2,p} \leq c(|F|_p + \|\pi\sigma\|_{1,p}) \tag{2.26}$$

holds. From (2.24), (2.26) we obtain that $\|u\|_{2,p}$ can be estimated by the right-hand side of (2.24). Now, from $W^{2,p} \subset W^{1+s,p} \subset H$, since in particular the first imbedding is compact, we deduce that for any positive ε there exists a constant $c(\varepsilon)$ such that

$$\|u\|_{1+s,p} \leq \varepsilon\|u\|_{2,p} + c(\varepsilon)\|u\|_H.$$

For ε small enough, taking account of (2.5), (2.6), we then obtain

$$\|u\|_{2,p} \leq c\left[|F|_p + \frac{\nu+\mu}{m_0}\|E\|_{1,p} + \left(A + |\pi|_\infty\left|\frac{1}{\pi}\right|_\infty(1+A)\right)(\|F\|_{H'} + \|E\|)\right]$$

and from (2.24)

$$\|\sigma\|_{1,p} \leq c\left\|\frac{1}{\pi}\right\|_{1,p}\left[|F|_p + \frac{\nu+\mu}{m_0}\|E\|_{1,p} + \left(A + |\pi|_\infty\left|\frac{1}{\pi}\right|_\infty(1+A)\right)(\|F\|_{H'} + \|E\|)\right]$$

which gives the thesis. □

The rest of the proof is as in [5]; we briefly recall the main steps, see [5] for details.

(i) We first prove the existence of a solution of (2.1) in the particular case of μ/ν sufficiently small. We define $q = \dfrac{1}{\mu}(\pi_1\sigma - \nu \operatorname{div} u)$ where $\pi_1 = p'_\rho(m_0, \Theta_0) > 0$. Then (2.1) is transformed in the following Stokes problem and linear transport equation

$$\begin{cases} \Delta u + \nabla q = \dfrac{1}{\mu}[F + \nabla(\pi_1 - \pi)\sigma] & \text{in } \Omega, \\ \operatorname{div} u = \dfrac{\mu}{\nu}\left(\dfrac{\pi_1}{\mu}\sigma - q\right) & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ t_i \cdot T(u) \cdot n = 0 & \text{on } \Gamma, i = 1, 2, \\ \overline{q} = 0, \end{cases} \tag{2.27}$$

$$\begin{cases} \dfrac{\pi_1}{\nu}\sigma + \operatorname{div}\left(\dfrac{1}{m_0}\sigma v\right) = \dfrac{\mu}{\nu}q + \dfrac{1}{m_0}E & \text{in } \Omega, \\ \overline{\sigma} = 0. \end{cases} \tag{2.28}$$

We solve (2.27), (2.28) by finding a fixed point of the map $\Psi_0 : (\sigma^*, q^*) \mapsto (\sigma, q)$ in the square $\Sigma_0 = \{(\sigma^*, q^*) \in \overline{W}^{1,p} \times \overline{W}^{1,p} : \|\sigma^*\|_{1,p} \leq B, \ \|q^*\|_{1,p} \leq B\}$ where (σ, q) is the solution of (2.27), (2.28) for $(\sigma^*, q^*) \in \Sigma_0$ inserted in the right-hand side and B is chosen large enough. If $|\nabla\rho_0|_p \geq c\|\pi - \pi_1\|_{1,p}$, $\dfrac{\nu}{m_0\pi_1}\|v\|_{2,p}$ and $\dfrac{\mu}{\nu}$ are small enough the map Ψ_0 is a contraction in Σ_0. Then, there exists a unique fixed point, that is a solution of (2.27), (2.28), with u solution of (2.27) corresponding to the fixed point $\sigma^* = \sigma$, $q^* = q$.

(ii) Secondly we consider the general case, with no restriction on the viscosity coefficients μ and ν; we prove the existence of a solution of (2.1) by the continuity method. Choose μ_0, ν_0 such that μ_0/ν_0 is so small that the result proved in (i) holds. For $\tau \in [0, 1]$ define

$$\mu_\tau = (1 - \tau)\mu_0 + \tau\mu, \qquad \nu_\tau = (1 - \tau)\nu_0 + \tau\nu,$$
$$L_\tau(u, \sigma) = (-\mu_\tau\Delta u - \nu_\tau\nabla \operatorname{div} u + \nabla(\pi\sigma), \operatorname{div}(m_0 u + \sigma v)),$$
$$X = W_b^{2,p} \times \overline{W}^{1,p} \qquad Y = L^p \times \overline{W}^{1,p}.$$

Consider the set

$$T = \{\tau \in [0, 1] : \text{for each } (F, E) \in Y \text{ there exists a unique solution } (u, \sigma) \in X \text{ of } L_\tau(u, \sigma) = (F, E)\}.$$

Since $0 \in T, T$ is not empty. Using (2.4) we prove that T is open and closed, i.e. $T \equiv [0, 1]$. Then for each $(F, E) \in Y$ there exists a solution $(u, \sigma) \in X$ of (2.1). From the linearity of the problem and (2.4) the uniqueness of the solution follows. This complete the proof of Theorem 2. □

REMARK. The same approach can be followed also for obtaining solutions $(u,\sigma) \in W^{j+2,p} \times \overline{W}^{j+1,p}$, $j \geq 1$. In that case the proof is simplified since it is not necessary, due to the higher regularity, to introduce the approximations $u_m, \sigma_m, \ldots$ and in (2.20) (and below) it is sufficient to consider $\|u\|_{j+1,p}$ instead of a norm of fractional order. In particular, (2.21) can be substituted by $\|W\|_{j+1,p} \leq c(\|u\|_{j+1,p} + |\pi|_\infty \|\sigma\|)$ (see also [5]).

3. - Proof of Theorem 1

Since the proof is essentially the same as in [5] we give just a sketch of it. We solve (1.11) by finding a fixed point of the map $\Psi : (v, \sigma^*, \theta^*) \mapsto (u, \sigma, \theta)$, where (u, σ, θ) is the solution of (1.11) with $F(v, \sigma^*, \theta^*), G(v, \sigma^*, \theta^*)$ in the right-hand side and the equation $\operatorname{div}(m_0 u + \sigma v) = E(u^*)$ instead of $\operatorname{div}(m_0 + \sigma)u = E(u)$.

We consider the set

$$\Sigma = \left\{ (u, \sigma, \theta) \in W_b^{2,p} \times \overline{W}^{1,p} \times W^{2,p} : \|u\|_{2,p} + \|\sigma\|_{1,p} + \|\theta\|_{2,p} \leq k_4 \right\},$$

where $k_4 \leq k_3$ is such that $|\sigma|_\infty \leq c\|\sigma\|_{1,p} \leq ck_4 \leq \dfrac{1}{2} \min_{\overline{\Omega}} \rho_0(x)$. The first step consists in proving that $\Psi(\Sigma) \subseteq \Sigma$. This follows using Theorem 2, estimate (2.4) and a well-known estimate for the Dirichlet problem $(1.11)_{3,6}$ under the requirement that $\gamma_0, |f|_p, |g|_p, \|\Theta_e - \Theta_0\|_{2-\frac{1}{p},p,\Gamma}, k_4$ are sufficiently small. Observe that the requirement that γ_0 is small implies, by Proposition 1, that $\nabla \rho_0$ is small. The second step consists in estimating the difference $(u_1 - u_2, \sigma_1 - \sigma_2, \theta_1 - \theta_2)$ for $(u_i, \sigma_i, \theta_i) = \Psi(v_i, \sigma_i^*, \theta_i^*), (v_i, \sigma_i^*, \theta_i^*) \in \Sigma$. For such differences we consider the norms $\|u_1 - u_2\|_H, \|\sigma_1 - \sigma_2\|, \|\theta_1 - \theta_2\|_1$ which we estimate using (2.5), (2.6) and $\|\theta_1 - \theta_2\|_1 \leq c\|G_1 - G_2\|_{-1}$ where $G_i = G(v_i, \sigma_i^*, \theta_i^*)$ and $\|\cdot\|_{-1}$ denotes the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$. Again, provided that $\gamma_0, |f|_p, |g|_p, k_4$ are sufficiently small, we prove that Ψ is a contraction in Σ with respect to a suitable norm in $H \times L^2 \times H^1$. Hence there exists a unique fixed point in Σ of the map Ψ, i.e. a solution of (1.11). This completes the proof. □

REMARKS. (i) If in theorem 1 we assume $(g, \Theta_e) \in W^{1,p} \times W^{3-\frac{1}{p},p}(\Gamma)$ (instead of $(g, \Theta_e) \in L^p \times W^{2-\frac{1}{p},p}(\Gamma)$) we obtain $\Theta \in W^{3,p}$.

(ii) Results similar to Theorem 1 can be obtained if we consider for the temperature Θ, instead of a Dirichlet boundary condition, either a Neumann b.c. $\dfrac{\partial \Theta}{\partial n} = \Theta_e$ on Γ, or an oblique b.c. $\dfrac{\partial \Theta}{\partial n} = h(\Theta_e - \Theta)$ on Γ, $h > 0$, where in each case $\Theta_e \in W^{1-\frac{1}{p},p}(\Gamma)$. In the case of the Neumann b.c. the total amount of temperature is also assigned. A different regularity, as in (i), can be obtained also with such boundary conditions.

(iii) If $\mu > 0$, $\nu = \mu/3$ the operator $-\mu\Delta - \nu\nabla \operatorname{div}$ is elliptic, but the coerciveness of the associated bilinear form, under the boundary conditions (1.4), fails. For this reason our method does not apply. The same difficulty was met in [4] for the stationary problem and in [7] (see also [9]) for the evolutionary problem with free boundary.

The fluid is viscous even if we assume that the shear viscosity μ vanishes and the bulk viscosity μ' is strictly positive, namely if $\mu = 0$, $\nu > 0$. In this case the correct boundary condition is $u \cdot n = 0$ on Γ. The motion of a viscous flow under these assumptions on the viscosity coefficients has been studied only in the evolutionary case in [6].

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