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Introduction

Given $p = (p_1, \dots, p_n)$ with $p_1, \dots, p_n \geq \frac{1}{2}$, define

$$\mathcal{E}(p) := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < 1 \right\}.$$

The aim of this note is to describe all *complex geodesics* (cf. [Ves], [Din]) $\varphi : E \to \mathcal{E}(p)$, where $E := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. So far, only special cases of p and φ have been studied, e.g. [Pol], [Gen], [BFKKMP] — cf. Remark 3.

Recall that a holomorphic mapping $\varphi : E \to \mathcal{E}(p)$ is a complex geodesic if

$$k_{\mathcal{E}(p)}(\varphi(\lambda'), \varphi(\lambda'')) = k_E(\lambda', \lambda''), \quad \lambda', \ \lambda'' \in E,$$

where k stands for the Kobayashi distance. Let us mention that $\mathcal{E}(p)$ is convex and therefore, by the Lempert theorem (cf. [Lem]), the Kobayashi distance $k_{\mathcal{E}(p)}$ coincides with the Carathéodory distance and the Kobayashi-Royden metric $\kappa_{\mathcal{E}(p)}$ coincides with the Carathéodory-Reiffen metric.

Observe that if $\varphi = (\varphi_1, \dots, \varphi_n) : E \to \mathcal{E}(p)$ is a complex geodesic with $\varphi_n \equiv 0$ then the mapping $(\varphi_1, \dots, \varphi_{n-1}) : E \to \mathcal{E}((p_1, \dots, p_{n-1}))$ is a "lower dimensional" complex geodesic. This means that, without loss of generality, we can assume that

(1) $$\varphi_j \not\equiv 0, \ j = 1, \dots, n.$$

Moreover, using a suitable permutation of coordinates, we can always assume that for some $0 \leq s \leq n$:

(2) $\varphi_1, \dots, \varphi_s$ have zeros in E and $\varphi_{s+1}, \dots, \varphi_n$ have no zeros in E.

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The main result of the paper is the following:

THEOREM 1. *A mapping* $\varphi = (\varphi_1, \dots, \varphi_n) : E \to \mathbb{C}^n$ *enjoying* (1) *and* (2) *is a complex geodesic in* $\mathcal{E}(p)$ *if and only if*

$$\varphi_j(\lambda) = \begin{cases} a_j \dfrac{\lambda - \alpha_j}{1 - \bar{\alpha}_j \lambda} \left(\dfrac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right)^{1/p_j}, & j = 1, \dots, s \\ a_j \left(\dfrac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right)^{1/p_j}, & j = s+1, \dots, n \end{cases}, \tag{3}$$

where

(4) $a_1, \dots, a_n \in \mathbb{C} \setminus \{0\}$,

(5) $\alpha_0, \dots, \alpha_s \in E$, $\alpha_{s+1}, \dots, \alpha_n \in \bar{E}$,

(6) $\alpha_0 = \sum_{j=1}^{n} |a_j|^{2p_j} \alpha_j$,

(7) $1 + |\alpha_0|^2 = \sum_{j=1}^{n} |a_j|^{2p_j} (1 + |\alpha_j|^2)$,

(8) *the case* $s = 0$, $\alpha_0 = \alpha_1 = \dots = \alpha_n$ *is excluded.*

Moreover, the complex geodesics in $\mathcal{E}(p)$ *are uniquely determined* mod Aut(E), *i.e. if* φ, ψ *are complex geodesics in* $\mathcal{E}(p)$ *with* $\varphi(0) = \psi(0)$, $\varphi'(0) = \psi'(0)$ *(or* $\varphi(0) = \psi(0)$, $\varphi(\sigma) = \psi(\sigma)$ *for some* $0 < \sigma < 1$*) then* $\varphi \equiv \psi$.

REMARK. (a) In particular, Theorem 1 shows that the components of a geodesic φ satisfying (1) have at most simple zeros in E.

(b) The case $p_1 = \dots = p_n = 1$, i.e. the case of the unit ball $\mathbb{B}_n$, is well-known — cf. Remark 7. Observe that the general case is a simple "transformation" of the case of $\mathbb{B}_n$.

COROLLARY 2. *Let* $\varphi : E \to \mathcal{E}(p)$ *be a complex geodesic.*

(a) *If* $p_1, \dots, p_n \in \left\{ \frac{1}{2}, 1 \right\}$ *then* φ *extends holomorphically to a neighborhood of* $\bar{E}$.

(b) *If* $t := \max\{p_1, \dots, p_n\} > 1$ *then* φ *extends to a mapping of the class* $C^{0,1/t}(\bar{E}) := \left\{ \frac{1}{t}\text{-Hölder continuous mappings on } \bar{E} \right\}$.

(c) *If* $u := \max\{p_j : p_j \neq 1\} < 1$ *then* φ *extends to a mapping of the class* $C^{1,(1/u)-1}(\bar{E})$.

REMARK 3. (a) The case $p_1 = \dots = p_n > 1$ has been studied in [Pol].

(b) The case $p_1 = \dots = p_n = \frac{1}{2}$ under the additional assumption that φ extends continuously on $\bar{E}$ has been discussed in [Gen].

(c) The case $n = 2$, $p_1 = 1$ can be found in [BFKKMP].

(d) Notice that in many cases Theorem 1 gives a tool to calculate $\kappa_{\mathcal{E}(p)}$ effectively. For example, if $n = 2$ and $p_1 = 1$ then, using Theorem 1, one can easily verify the formulas for $\kappa_{\mathcal{E}(p)}$ obtained in [BFKKMP].

PROOF OF THEOREM 1. The proof will be based on the following criterion due to H. Royden and P.-M. Wong:

THEOREM 4. *Let $G \subset \mathbb{C}^n$ be a bounded convex domain with $0 \in G$. Then a holomorphic mapping $\varphi : E \to G$ is a complex geodesic if and only if:*

$$\varphi^*(\lambda) \in \partial G \quad \textit{for almost all} \quad \lambda \in \partial E$$

and there exists $h \in H^1(E, \mathbb{C}^n)$, $h \not\equiv 0$, such that

$$\operatorname{Re}\left(\frac{1}{\lambda}\varphi^*(\lambda) \bullet h^*(\lambda)\right) = \hat{q}_G\left(\frac{1}{\lambda}h^*(\lambda)\right) \quad \textit{for almost all} \quad \lambda \in \partial E,$$

where:

H^1 denotes the Hardy space,

φ^*, h^* stand for non-tangential boundary values,

$w \bullet z := \sum_{j=1}^{n} w_j z_j$ and

$\hat{q}_G$ is the dual subnorm for the Minkowski function q_G of G, i.e.

$$\hat{q}_G(w) := \max\{\operatorname{Re}(w \bullet z) : z \in \partial G\}, \quad w \in \mathbb{C}^n.$$

REMARK. The proof presented in [Roy-Won] contains an error in the proof of Proposition 1. Complete proofs may be found in [Aba, Section 2.6] and in [Jar-Pfl, Section 8.2].

REMARK 5. (a) Suppose that $z_0 \in \partial G$ is such that there is a uniquely determined unit outer normal vector $\nu(z_0)$ to ∂G at z_0. Then for $w_0 \in \mathbb{C}^n \setminus \{0\}$ one has:

$$[\operatorname{Re}(z_0 \bullet w_0) = \hat{q}_G(w_0)] \Leftrightarrow [\exists t > 0 : w_0 = t\overline{\nu(z_0)}].$$

(b) Observe that any point $z_0 \in \partial\mathcal{E}(p)$ has a local peak function. In particular, if $\varphi : E \to \overline{\mathcal{E}(p)}$ is a non-constant holomorphic mapping then $\varphi : E \to \mathcal{E}(p)$.

Theorem 4, Remark 5 and the identity theorem for H^1-functions imply the following useful

COROLLARY 6. *Let $\varphi = (\varphi_1, \dots, \varphi_n) : E \to \mathbb{C}^n$ be a non-constant bounded holomorphic mapping enjoying* (1). *Then φ is a complex geodesic in $\mathcal{E}(p)$ if and only if:*

$$\sum_{j=1}^{n} |\varphi_j^*|^{2p_j} = 1 \quad \textit{a.e. on} \quad \partial E \tag{9}$$

and there exist $h \in H^1(E, \mathbb{C}^n)$ *and* $\rho : \partial E \to \mathbb{R}_{>0}$ *such that*

$$\frac{1}{\lambda} h_j^* = \rho p_j |\varphi_j^*|^{2(p_j-1)} \overline{\varphi_j^*} \quad a.e.\ on \quad \partial E,\ j = 1, \dots, n. \tag{10}$$

REMARK 7. Let $p_1 = \dots = p_n = 1$ and φ be given by (3) with conditions (4)–(8). Then φ is a complex geodesic in $\mathbb{B}_n$. In fact:

— condition (8) assures that φ is non-constant,

— conditions (6) and (7) imply (9),

— if we put

$$h_j(\lambda) := \begin{cases} \overline{a}_j(1 - \bar{\alpha}_j\lambda)(1 - \bar{\alpha}_0\lambda), & j = 1, \dots, s \\ \bar{a}_j(\lambda - \alpha_j)(1 - \bar{\alpha}_0\lambda), & j = s+1, \dots, n \end{cases}, \tag{11}$$

then also condition (10) is fulfilled with $\rho(\lambda) := |1 - \bar{\alpha}_0\lambda|^2$.

Using Corollary 6, one can easily prove the following "transport lemma" for complex geodesics.

LEMMA 8. *Let* $\varphi = (\varphi_1, \dots, \varphi_n) : E \to \mathcal{E}(p)$ *be a complex geodesic enjoying* (1) *and let* $h = (h_1, \dots, h_n)$ *be as in* (10). *Write*

$$\varphi_j = B_j \psi_j,$$

where B_j *is the Blaschke product for* φ_j *and* ψ_j *is nowhere vanishing (if* φ_j *has no zeros in* E *then we put* $B_j :\equiv 1$*),* $j = 1, \dots, n$. *Let* $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_n)$*,* $\tilde{p}_1, \dots, \tilde{p}_n \geq \frac{1}{2}$ *and define:*

$$\tilde{\varphi}_j := B_j \psi_j^{p_j/\tilde{p}_j},$$

$$\tilde{h}_j := \frac{\tilde{p}_j}{p_j} h_j \frac{\varphi_j}{\tilde{\varphi}_j}.$$

Then the mappings $\tilde{\varphi}$ *and* $\tilde{h}$ *satisfy* (9) *and* (10) *with respect to* $\mathcal{E}(\tilde{p})$ *(with the same function* ρ*).*

In particular, if $\tilde{h} \in H^1(E, \mathbb{C}^n)$ *(e.g. if* $p_j \leq \tilde{p}_j$*,* $j = 1, \dots, n$*) then* $\tilde{\varphi} : E \to \mathcal{E}(\tilde{p})$ *is a complex geodesic.*

Consequently, in view of Remark 7, we get:

COROLLARY 9. *If* φ *is given by* (3) *with* (4)–(8) *then* $\varphi : E \to \mathcal{E}(p)$ *is a complex geodesic.*

PROOF. In view of (11), the new mapping $\tilde{h}$ belongs to H^1. □

COROLLARY 10. *If all complex geodesics* $\varphi : E \to \mathcal{E}(p)$ *with* (1) *and* (2) *are of the form* (3) *then the same is true for all complex geodesics* $\tilde{\varphi} : E \to \mathcal{E}(\tilde{p})$ *with* $\tilde{p}_j \leq p_j$*,* $j = 1, \dots, n$.

Thus, in order to finish the proof of the first part of Theorem 1, it is sufficient to verify the following:

LEMMA 11. *If* $p_1 = \ldots = p_n =: p_0$ *then any complex geodesic* $\varphi : E \to \mathcal{E}(p)$ *with* (1) *and* (2) *is of the form* (3) *with* (4)–(8).

PROOF. Let $h = (h_1, \ldots, h_n) \in H^1(E, \mathbb{C}^n)$ and ρ be as in (10). Then

$$\frac{1}{\lambda}\varphi_j^* h_j^* \in \mathbb{R}_{>0} \quad \text{a.e. on} \quad \partial E, \ j = 1, \ldots, n.$$

Hence, by [Gen, Lemma 2 and proof of Theorem 5] (see also [Pol, Statement 6]), there exist

$$r_0, \ldots, r_n > 0, \ \alpha_0, \ldots, \alpha_n \in \bar{E}$$

such that

$$\varphi_j(\lambda) h_j(\lambda) = r_j(\lambda - \alpha_j)(1 - \bar{\alpha}_j \lambda), \ \lambda \in E, \ j = 1, \ldots, n, \tag{12}$$

$$\varphi(\lambda) \bullet h(\lambda) = r_0(\lambda - \alpha_0)(1 - \bar{\alpha}_0 \lambda), \ \lambda \in E. \tag{13}$$

Substituting h by $\frac{1}{r_0} h$ we can always assume that $r_0 = 1$. In view of (2),

$$\alpha_1, \ldots, \alpha_s \in E.$$

Moreover, (12) and (13) imply that

$$\alpha_0 = \sum_{j=1}^{n} r_j \alpha_j \quad \text{and}$$

$$1 + |\alpha_0|^2 = \sum_{j=1}^{n} r_j (1 + |\alpha_j|^2).$$

Observe that, by (9), (10) and (13), we get

$$p_0 \rho(\lambda) = |1 - \bar{\alpha}_0 \lambda|^2 \quad \text{for almost all} \quad \lambda \in \partial E. \tag{14}$$

In particular, (9), (10) and (14) show that

$$\text{the functions } h_1, \ldots, h_n \text{ are bounded} \tag{15}$$

and, in view of (12), that

$$|\varphi_j^*(\lambda)| = r_j^{1/2p_0} \left| \frac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right|^{\frac{1}{p_0}} \quad \text{for almost all} \quad \lambda \in \partial E, \ j = 1, \ldots, n.$$

Consequently, since $(1-\bar{\alpha}_j\lambda)^{1/p_0}$ is an outer function (cf. [Rud, Ch. 17]), we get:

$$\varphi_j(\lambda)(1-\bar{\alpha}_0\lambda)^{1/p_0} = r_j^{1/2p_0} e^{i\theta j} B_j(\lambda) S_j(\lambda)(1-\bar{\alpha}_j\lambda)^{1/p_0}, \quad \lambda \in E, \tag{16}$$

where

$$\theta j \in \mathbb{R},$$

$$B_j(\lambda) := \begin{cases} \dfrac{\lambda-\alpha_j}{1-\bar{\alpha}_j\lambda}, & j=1,\dots,s \\ 1, & j=s+1,\dots,n \end{cases},$$

$$S_j(\lambda) := \exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}\, d\sigma_j(\theta) \right)$$

and where σ_j is a singular non-negative Borel measure, $j=1,\dots,n$.

Now, in order to complete the proof of the lemma, we only need to show that $\sigma_1 = \dots = \sigma_n = 0$. First, observe that, in view of (12), (15) and (16), we have:

$$|\lambda-\alpha_j|^{p_0} |1-\bar{\alpha}_j\lambda|^{2p_0-1} |1-\bar{\alpha}_0\lambda|\, |S_j(\lambda)|^{-p_0} \le M_j, \quad \lambda \in E, \tag{17}$$

for some $M_j > 0$, $j=1,\dots,n$. On the other hand, by [Gar, Ch. II],

$$S_j^*(\lambda) = 0 \quad \text{for } \sigma_j\text{-almost all } \lambda \in \partial E, \; j=1,\dots,n.$$

Moreover, for any $\beta \in \mathbb{R}$, $b > 0$, the function

$$E \ni \lambda \to |\lambda-1|^{\beta} \exp\left(b\, \frac{1-|\lambda|^2}{|\lambda-1|^2} \right)$$

is unbounded. Consequently, condition (17) is fulfilled only when $\sigma_j = 0$ for all $j=1,\dots,n$. □

PROOF OF THE UNIQUENESS IN THEOREM 1. Let φ, $\psi : E \to \mathcal{E}(p)$ be two complex geodesics with

$$\varphi(0) = \psi(0) \quad \text{and} \quad \varphi'(0) = \psi'(0). \tag{18}$$

$$\varphi(0) = \psi(0) \quad \text{and} \quad \varphi(\sigma) = \psi(\sigma) \quad \text{with} \quad 0 < \sigma < 1, \tag{18'}$$

respectively.

So far, Theorem 1 (see also Remark (a) after Theorem 1) shows that: $\varphi_j \equiv 0$ if and only if $\psi_j \equiv 0$. Thus, without loss of generality, we assume that φ and ψ satisfy (1). Moreover, we assume that φ fulfils condition (2). Put $I_0 := \{j : \psi_j \text{ has a zero in } E\}$.

Now, observe that $\chi := (\varphi + \psi)/2$ is again a complex geodesic in $\mathcal{E}(p)$ ($\mathcal{E}(p)$ is convex!). In particular:

$$\chi(\lambda),\ \varphi(\lambda),\ \psi(\lambda) \in \partial\mathcal{E}(p) \quad \text{for all } \lambda \in \partial E.$$

Therefore we obtain for $\lambda \in \partial E$:

$$\text{(19)} \qquad \begin{aligned} \arg\varphi_j(\lambda) = \arg\psi_j(\lambda) = \arg\chi_j(\lambda) \quad &\text{if} \quad \varphi_j(\lambda)\psi_j(\lambda) \neq 0 \quad \text{and} \\ |\varphi_j(\lambda)| = |\psi_j(\lambda)| = |\chi_j(\lambda)| \quad &\text{if} \quad p_j > 1/2. \end{aligned}$$

Hence, if $p_j > 1/2$ then $\varphi_j = \psi_j$ on ∂E and so $\varphi_j \equiv \psi_j$ on E.

It remains to discuss j with $p_j = 1/2$. We fix such an j. Because of (19) we have

$$\text{(20)} \qquad \varphi_j\bar{\psi}_j = \psi_j\bar{\varphi}_j \quad \text{on} \quad \partial E.$$

First let us assume that $1 \leq j \leq s$, i.e.

$$\varphi_j(\lambda) = a_j\,\frac{(\lambda - \alpha_j)(1 - \bar{\alpha}_j\lambda)}{(1 - \bar{\alpha}_0\lambda)^2}.$$

Case $j \in I_0$, i.e.

$$\psi_j(\lambda) = b_j\,\frac{(\lambda - \beta_j)(1 - \bar{\beta}_j\lambda)}{(1 - \bar{\beta}_0\lambda)^2}.$$

An application of (20) leads to

$$a_j\bar{b}_j\,\frac{1}{(1 - \bar{\alpha}_0\lambda)^2(\lambda - \beta_0)^2} = \bar{a}_jb_j\,\frac{1}{(1 - \bar{\beta}_0\lambda)^2(\lambda - \alpha_0)^2},\ \ \lambda \in \partial E.$$

So we obtain $\alpha_0 = \beta_0$. Moreover, because of (18) and (18′), we get

$$\begin{aligned} &\alpha_ja_j = \beta_jb_j \quad \text{and} \quad a_j(1 + |\alpha_j|^2) = b_j(1 + |\beta_j|^2), \\ &\alpha_ja_j = \beta_jb_j \quad \text{and} \quad a_j(1 - \bar{\alpha}_j\sigma + |\alpha_j|^2) = b_j(1 - \bar{\beta}_j\sigma + |\beta_j|^2), \quad \text{respectively.} \end{aligned}$$

Hence

$$\Phi(0, \alpha_j) = \Phi(0, \beta_j), \quad \text{resp.} \quad \Phi(\sigma, \alpha_j) = \Phi(\sigma, \beta_j),$$

where

$$\Phi(\tau, \lambda) := \frac{\lambda}{1 - \tau\bar{\lambda} + |\lambda|^2},\ \ \lambda \in \bar{E},\ \ 0 \leq \tau < 1.$$

Since $\Phi(\tau, \cdot)$ is injective on $\bar{E}$, we obtain $\alpha_j = \beta_j$ and so $a_j = b_j$.

Case $j \notin I_0$, i.e.

$$\psi_j(\lambda) = b_j \left(\frac{1-\bar{\beta}_j\lambda}{1-\bar{\beta}_0\lambda}\right)^2.$$

Then it follows from (20) that

$$\bar{b}_j a_j \frac{1}{(1-\bar{\alpha}_0\lambda)^2}\left(\frac{\lambda-\beta_j}{\lambda-\beta_0}\right)^2 \equiv \bar{a}_j b_j \left(\frac{1-\bar{\beta}_j\lambda}{1-\bar{\beta}_0\lambda}\right)^2 \frac{1}{(\lambda-\alpha_0)^2}$$

as meromorphic functions on $\mathbb{C}$.

So we can conclude that $\alpha_0 = \beta_0$ and so

$$\bar{b}_j a_j(\lambda-\beta_j)^2 \equiv \bar{a}_j b_j(1-\bar{\beta}_j\lambda)^2.$$

Hence we have $|\beta_j| = 1$. Again using (18) and (18′) gives:

$$\alpha_j a_j = b_j \quad \text{and} \quad a_j(1+|\alpha_j|^2) = -2b_j\bar{\beta}_j$$
$$\alpha_j a_j = b_j \quad \text{and} \quad a_j(1-\bar{\alpha}_j\sigma+|\alpha_j|^2) = -b_j(2\bar{\beta}_j-\bar{\beta}_j^2\sigma), \quad \text{respectively.}$$

Then

$$\Phi(0,\alpha_j) = \Phi(0,\beta_j), \quad \text{respectively.} \Phi(\sigma,\alpha_j) = \Phi(\sigma,\beta_j)$$

contradiction.

The remaining case $s < j$ can be solved using similar reasonings as above.

□

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Added in proof

See also S. Dineen & R.M. Timoney, *Complex geodesics on convex domains* in "Progress in Functional Analysis", K.D. Bierstedt, J. Bonet, J. Horváth & M. Maestre (eds.), Elsevier Science Publishers B.V., 1992.

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