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Convexly Totally Bounded and Strongly Totally Bounded Sets.
Solution of a Problem of Idzik

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0. - Introduction

A long outstanding problem in fixed point theory is the following

**Problem of Schauder** (Problem 54, [G]). Does every continuous function $f : C \rightarrow C$ defined on a convex compact subset of a Hausdorff topological linear space have a fixed point?

Schauder [S] gave a positive answer to this problem if the linear space is a Banach space. Thychonoff [T] generalized Schauder’s theorem for locally convex spaces. Schauder’s problem is still open even for metrizable (nonlocally convex) spaces. Idzik [11] proved that the answer to Schauder’s problem is “yes” if $C$ is convexly totally bounded. This notion was introduced by Idzik [11]: A subset $K$ of a topological linear space $E$ is called **convexly totally bounded** (ctb for short) if for every $0$-neighbourhood $U$ there are $x_1, \ldots, x_n \in E$ and convex subsets $C_1, \ldots, C_n$ of $U$ such that $K \subset \bigcup_{i=1}^{n} (x_i + C_i)$. Idzik formulated

– comparing his theorem with Schauder’s problem – the following

**Problem** (cf. Problem 4.7 of [12]). Is every convex compact subset of a Hausdorff topological linear space ctb?

A positive answer to this problem would imply a positive answer to Schauder’s problem.

In the first section of this paper we give a negative answer to Idzik’s problem. In Section 2, we introduce the notion of strongly convexly totally bounded (sctb) sets and in Section 3 a parameter, which measures “the lack of strongly convexly total boundedness”. This notion and this parameter is – in contrast to convexly total boundedness – invariant when one passes to the convex hull of a set. That admits the formulation of a fixed point theorem of

Darbo type [D] in nonlocally convex spaces. In Sections 4 and 5, we examine the mentioned parameter in the space $l_p$ ($0 < p < 1$) and in the space $L_0$ of measurable functions.

The notion of scnb sets and the corresponding noncompactness measure is the main tool in [DP/T2] to get a best approximation result of Fan type in nonlocally convex linear spaces. In [W] strongly convexly total boundedness is linked with affine embeddability in locally convex spaces.

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In this paper we use the terminology of [J]. $\mathbb{N}$ and $\mathbb{R}$ stands for the sets of all natural and real numbers, respectively.

1. - A compact convex set not convexly totally bounded

In this section we construct a compact convex set, which is not ctb; this solves Problem 4.7 of Idzik [12]. We obtain such a set by a modification of Robert’s example for a compact convex set without extreme point, see [R], [Ro, Section 5.6]. Robert’s example is based on his notion of needle point; that is a point $x_0 \neq 0$ of an (in the sense of [J]) F-normed linear space $(E, \| \cdot \|)$ with the following property: for every $\varepsilon > 0$, the ball $B_\varepsilon := \{ x \in B : \| x \| \leq \varepsilon \}$ contains a finite set $F$ such that $\text{co } F \subset \text{co} \{0, x_0\} + B_\varepsilon$ and $x_0 \in \text{co } F + B_\varepsilon$. (Hereby co $F$ denotes the convex hull of $F$). In our construction, a stronger property plays an important role:

**DEFINITION 1.1.** We call a point $x_0$ of an F-normed linear space $(E, \| \cdot \|)$ a *strong needle point*, if $x_0 \neq 0$ and, for every $\varepsilon > 0$, there is a natural number $k$ and an infinite subset $M$ of $B_\varepsilon$ such that $\text{co } M \subset \text{co} \{0, x_0\} + B_\varepsilon$ and $x_0 \in \text{co } F + B_\varepsilon$. (Hereby we denote by $|F|$ the cardinality of $F$).

**LEMMA 1.2.** For each $n \in \mathbb{N} \cup \{0\}$, let $F_n$ be a finite subset of an F-normed linear space $E$ and $\varepsilon_n > 0$ such that $\text{co } F_n \subset \text{co } F_0 + B_{\varepsilon_n}$. Assume that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. Then $C := \text{co } \bigcup_{n=0}^{\infty} F_n$ is totally bounded.

**PROOF.** Since $\bigcup_{i=0}^{n} F_i$ is a compact subset of a finite dimensional subspace of $E$, also $C_n := \text{co } \bigcup_{i=0}^{n} F_i$ is compact. Furthermore, we have $r_n := \sum_{i=0}^{n} \varepsilon_i \to 0$ ($n \to \infty$). Therefore it is enough to prove that $C \subset C_n + B_{r_n}$ for all $n \in \mathbb{N}$. 

For $\alpha, \beta \in [0, 1]$, with $\alpha + \beta = 1$ and $n \in \mathbb{N}$, we have
\[
\alpha \cdot C_n + \beta \cdot \text{co} F_{n+1} \subset \alpha \cdot C_n + \beta \cdot \text{co} F_0 + \beta \cdot B_{\varepsilon_{n+1}} \\
\subset \alpha \cdot C_n + \beta \cdot C_n + B_{\varepsilon_{n+1}} \subset C_n + B_{\varepsilon_{n+1}},
\]
hence $C_{n+1} = \bigcup_{\alpha \in [0,1]} \alpha \cdot C_n + (1 - \alpha) \cdot \text{co} F_{n+1} \subset C_n + B_{\varepsilon_{n+1}}$. It follows inductively that $C_{n+k} \subset C_n + \sum_{k=n+1}^{n+k} B_{\varepsilon_k}$ for $n, k \in \mathbb{N}$; therefore $C_{n+k} \subset C_n + B_{\varepsilon_n}$ and
\[
C = \bigcup_{k=1}^{\infty} C_{n+k} \subset C_n + B_{\varepsilon_n}.
\]

**THEOREM 1.3.** Let $E$ be a complete $F$-normed linear space, which contains a strong needle point. Then $E$ contains a compact absolutely convex set, which is not ctb.

**PROOF.** Let $\varepsilon_n > 0$, with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$, and $x_0$ be a strong needle point of $E$. Choose infinite subsets $M_n$ of $B_{\varepsilon_n}$ and $k_n \in \mathbb{N}$ such that $\text{co} M_n \subset \text{co} \{0, x_0\} + B_{\varepsilon_n}$, and $x_0 \in \text{co} F + B_{\varepsilon_n}$, for every finite subset $F$ of $M_n$, with $|F| \geq k_n$. Let $F_0 := \{0, x_0\}$ and $F_n$ be finite subsets of $M_n$ with $|F_n| \geq n \cdot k_n$ for $n \in \mathbb{N}$. Since $C := \bigcup_{n=0}^{\infty} F_n$ is totally bounded by Lemma 1.2, $\overline{C}$ and $\overline{C-C}$ are compact. Therefore the closed absolutely convex hull $K := \overline{\text{co} \bigcup_{n=0}^{\infty} F_n}$ of $C$ is a compact subset of $\overline{C-C}$ containing $C$. We now show that $C$ is not ctb; then neither is $K$. Let $\varepsilon > 0$, with $x_0 \notin B_{4\varepsilon}$. Suppose that $C$ is ctb. Then there are convex subsets $C_i$ of $B_{\varepsilon}$ and $y_i \in E$ such that $C \subset \bigcup_{i=1}^{m} (y_i + C_i)$. Choose $n \geq m$ with $\varepsilon_n \leq \varepsilon$. Since $F_n \subset C \subset \bigcup_{i=1}^{m} (y_i + C_i)$ and consequently
\[
\sum_{i=1}^{m} |F_n \cap (y_i + C_i)| \geq |F_n| \geq n \cdot k_n \geq m \cdot k_n,
\]
for some $j \in \{1, \ldots, m\}$, the set $F := F_n \cap (y_j + C_j)$ has at least $k_n$ elements. Therefore
\[
x_0 \in \text{co} F + B_{\varepsilon_n} \subset y_j + C_j + B_{\varepsilon} \subset y_j + B_{2\varepsilon}.
\]
Let $y \in F$. Then $y_j \in y - C_j \subset F_n - C_j \subset B_{\varepsilon_n} + B_{\varepsilon} \subset B_{2\varepsilon}$. It follows that $x_0 \in y_j + B_{2\varepsilon} \subset B_{2\varepsilon} + B_{2\varepsilon} \subset B_{4\varepsilon}$, a contradiction.

We now show the existence of strong needle points in the same way as Roberts has shown the existence of needle points; see [R], [Ro, Section 5.6].
In the following, let \( \varphi : [0, +\infty] \rightarrow [0, +\infty] \) be a continuous increasing concave function such that 
\[ \varphi(x) = 0 \text{ iff } x = 0 \text{ and } \varphi(x)/x \rightarrow 0 \text{ for } 0 < x \rightarrow +\infty. \]

If \( \mathcal{U} \) is an algebra of sets and \( \mu \) a \([0, 1]\)-valued measure (= finitely additive set function) on \( \mathcal{U} \), we denote by \( S(\mathcal{U}) \) the space of real-valued \( \mathcal{U} \)-simple functions, i.e. the linear hull of the system of all characteristic functions of sets of \( \mathcal{U} \); furthermore we put

\[
\|f\|_\varphi = \int \varphi \circ |f| \, d\mu \quad \text{and} \quad \|f\|_p := \left( \int |f|^p \, d\mu \right)^{1/p}
\]

for \( f \in S(\mathcal{U}) \) and \( 1 \leq p < +\infty \). \( \| \cdot \|_\varphi \) is then a Riesz pseudonorm in the sense of [A/B, p. 39] and \( \|f\|_\varphi \leq \varphi(\|f\|_1) \) for \( f \in S(\mathcal{U}) \).

In the following let \( \mathcal{U}_0 \) be an algebra in the power set \( \mathcal{P}(\Omega_0) \) of a non-empty set \( \Omega_0 \),
\( \mu_0 : \mathcal{U}_0 \rightarrow [0, 1] \) a measure with \( \mu_0(\Omega_0) = 1 \),
\( (A_n) \) a sequence in \( \mathcal{U}_0 \) with \( 0 < \mu_0(A_n) \rightarrow 0 \) \((n \rightarrow \infty)\),
\( (\Omega, \mathcal{U}, \mu) = \bigotimes_{n \in \mathbb{N}} (\Omega_n, \mathcal{U}_n, \mu_n) \), where \( (\Omega_n, \mathcal{U}_n, \mu_n) = (\Omega_0, \mathcal{U}_0, \mu_0) \) for \( n \in \mathbb{N} \).

For \( f : \Omega_0 \rightarrow \mathbb{R} \) and \( i \in \mathbb{N} \), we define \( S_i(f) : \Omega \rightarrow \mathbb{R} \) by \( (S_i(f))(x_n) := f(x_i) \).

**Lemma 1.4.** Let \( f \in S(\mathcal{U}_0) \), with \( \int f \, d\mu_0 = 1 \), and \( 0 \leq t_i \leq t < +\infty \), with \( \sum_{i=1}^n t_i \leq 1 \). Then \( g := \sum_{n=1}^n t_i(S_i(f) - 1) \in S(\mathcal{U}_0) \), \( \|g\|_1 \leq \sqrt{t} \cdot \|f - 1\|_2 \), \( \|g\|_\varphi \leq \varphi(\sqrt{t} \cdot \|f - 1\|_2) \).

For the proof, see [Ro, pp. 244-245] or [R, Lemma 3.8].

**Lemma 1.5.** Let \( 0 < b \leq 1 \) and \( \delta > 0 \). Then there are a non-negative function \( f \in S(\mathcal{U}_0) \) and a number \( a \in ]0, b[ \) with the following properties:

(i) \( \int f \, d\mu_0 = 1 \);
(ii) \( \|f\|_\varphi \leq \delta \);
(iii) if \( n \in \mathbb{N} \) and \( t_i \geq b \), with \( \sum_{i=1}^n t_i \leq 1 \), then
\[
\left\| \sum_{i=1}^n t_i S_i(f) \right\|_\varphi \leq \delta;
\]
For the proof, see [Ro, Lemma 5.6.3] or [R, Lemma 3.10]. Hereby, 1.4 is used to prove (iv) of 1.5. With the aid of 1.5, one can prove the following proposition as [Ro, Proposition 5.6.4].

**PROPOSITION 1.6.** Denote by $f_0$ the constant function equal to 1 on $\Omega$. Let $\varepsilon > 0$. Then there is a non-negative function $f \in S(U_0)$ with the following properties:

(i) $\int f \, d\mu_0 = 1$;

(ii) $M := \{S_i(f) : i \in \mathbb{N}\}$ is contained in $B_\varepsilon := \{h \in S(U) : \|h\|_\varphi \leq \varepsilon\}$;

(iii) there is a $k \in \mathbb{N}$ such that, for every $I \subset \mathbb{N}$ with $|I| \geq k$, we have $f_0 \in \text{co}\{S_i(f) : i \in I\} + B_\varepsilon$;

(iv) $\text{co} \, M \subset \text{co}\{0, f_0\} + B_\varepsilon$.

For small $\varepsilon$, a function $f$ satisfying (i) and (ii) of 1.6 cannot be constant. The functions $S_i(f)$ are then different for different indices $i$; therefore $M$ is an infinite set. It follows:

**THEOREM 1.7.** Denote by $N$ the space $N := \{f \in S(U) : \|f\|_\varphi = 0\}$ of all $U$-simple null functions. Then $(S(U), \| \cdot \|_\varphi)/N$ contains strong needle points.

**COROLLARY 1.8.** Let $\lambda$ be the Lebesgue measure on $[0, 1]$ and $\varphi : [0, +\infty[ \to [0, +\infty[ \to [0, +\infty[$ be a continuous increasing concave function such that $\varphi(x) = 0$ iff $x = 0$ and $\varphi(x)/x \to 0$ ($0 < x \to +\infty$). Then the Orlicz space $L_\varphi(\lambda)$ contains a compact absolutely convex set, which is not ctb.

**PROOF.** Since the measure algebras induced by $\lambda$ and by $\lambda \otimes \lambda \otimes \lambda \otimes \cdots$ are isomorphic, the space $L_\varphi(\lambda)$ is isomorphic to the completion of a space $(S(U), \| \cdot \|_\varphi)/N$ considered in 1.7. Since the last-named space contains strong needle points, so does $L_\varphi(\lambda)$. Now the assertion follows from 1.3.

**COROLLARY 1.9.** Let $\lambda$ be the Lebesgue measure on $[0, 1]$ and $0 \leq p < 1$. Then $L_p(\lambda)$ contains a compact absolutely convex set, which is not ctb.

**PROOF.** Apply 1.8 with $\varphi(x) = x^p$ for $0 < p < 1$ and $\varphi(x) = x/(1 + x)$ for $p = 0$. 

(iv) if $n \in \mathbb{N}$ and $0 \leq t_i \leq a$, with $\sum_{i=1}^{n} t_i \leq 1$, then

\[
\left\| \sum_{i=1}^{n} t_i (S_i(f) - 1) \right\|_\varphi \leq \delta.
\]
2. - Strongly convexly totally bounded sets

In this section, let \((E, \tau)\) be a topological linear space. An important question in fixed point theory is: under which condition for a set \(K \subseteq E\), the convex hull of any totally bounded subset of \(K\) is totally bounded, see \([K, p. 10]\), \([H1, p. 31]\), cf. also 3.1 (ii). Easy examples show that the convex hull of a ctb set does not need to be ctb, see 5.3 (b). Therefore we introduce in 2.1 a property stronger than convexly total boundedness, which is conserved passing to the convex hull, see 2.2 (b).

**DEFINITION 2.1.** A subset \(K\) of \(E\) is said to be strongly convexly totally bounded (scrb for short) if, for every 0-neighbourhood \(U\), there is a convex subset \(C\) of \(U\) and a finite subset \(F\) of \(E\) such that \(K \subseteq F + C\).

A subset of a locally convex linear space is scrb iff it is totally bounded.

**PROPOSITION 2.2.** (a) If \(K_1\) and \(K_2\) are scrb subsets of \(E\), then the sets \(K_1 \cup K_2\), \(K_1 + K_2\) and \(\alpha \cdot K_1\) for any \(\alpha \in \mathbb{R}\) are scrb. (b) If \(K\) is a scrb subset of \(E\), then the closed absolutely convex hull \(\overline{\text{aco}} K\) of \(K\) is scrb.

**PROOF.** Let \(U\) be a circled closed 0-neighbourhood in \(E\).

(i) There are finite sets \(F_i \subseteq E\) and convex sets \(C_i \subseteq U\) such that \(K_i \subseteq F_i + C_i\) for \(i = 1, 2\). Since \(\text{co}(C_i \cup \{0\}) \subseteq U\), we may assume that \(0 \in C_i\). Then the convex set \(C := C_1 + C_2\) contains \(C_1\) and \(C_2\). It follows that \(K_1 \cup K_2 \subseteq (F_1 + F_2) + C\), \(K_1 + K_2 \subseteq (F_1 + F_2) + C\), \(C \subseteq U + U\). Hence \(K_1 \cup K_2\) and \(K_1 + K_2\) are scrb.

(ii) Since \(\alpha \cdot K_1 \subseteq \alpha \cdot F_1 + \alpha \cdot C_1\), \(\alpha \cdot C_1 \subseteq \alpha \cdot U\) and \(\overline{K_1} \subseteq F_1 + \overline{C_1}\), \(\overline{C_1} \subseteq \overline{U} = U\), the sets \(\alpha \cdot K_1\) and \(\overline{K_1}\) are scrb.

(iii) Let \(K\) be scrb; replacing \(K\) by \(K \cup \{0\}\), we may assume that \(0 \in K\). Let \(F\) be a finite subset of \(E\) and \(C\) a convex subset of \(U\) such that \(K \subseteq F + C\), and \(V\) be a 0- neighbourhood in \(E\). Since \(E_0 := \text{span} F\) is finite dimensional, \(V\) contains a convex 0- neighbourhood \(V_0\) in \(E_0\); moreover \(\text{co} F\) is a compact subset of \(E_0\). It follows that there is a finite subset \(F_0\) of \(E_0\) such that \(\text{co} F \subseteq F_0 + V_0\). Since \(K\) is a subset of the convex set \(\text{co} F + C\), we get \(\text{co} K \subseteq \text{co} F + C \subseteq F_0 + (V_0 + C)\), where \(V_0 + C\) is a convex subset of \(V + U\). We have proved that \(\text{co} K\) is scrb. Since \(\overline{\text{aco}} K \subseteq \text{co} K - \text{co} K\), the set \(\text{aco} K\) is scrb by (i). It now follows from (ii) that the closure \(\overline{\text{aco}} K\) is scrb.

Obviously every scrb set is ctb. In 5.3 (b), we give an example for a ctb set, the convex hull of which is not ctb; by 2.2 (b), such a set is ctb, but not scrb. But we do not know whether there are also convex ctb sets, which are not scrb.

In \([W, \text{Section 2}]\) it is proved that a compact convex subset \(K\) of \(E\) is scrb iff there is a locally convex linear topology \(\sigma\) on \(E\) such that the
relative topologies \( \sigma|K \) and \( \tau|K \) coincide. Using the easy implication (\( \Rightarrow \)) of this equivalence for \( \sigma = \sigma(E, E') \) one obtains:

**Proposition 2.3.** If the continuous dual \( E' \) of \( E \) separates the points, then every compact convex subset of \( E \) is scb.

Other examples for scb sets are the compact convex order-bounded subsets of Orlicz function spaces, as mentioned at the end of [W, Section 2]. In 3.5 we will use the following fact:

**Proposition 2.4.** Let \((\tau_i)_{i \in I}\) be a family of linear topologies on \( E \) and \( \tau = \sup \tau_i \). Then a set \( K \subset E \) is scb, ctb or totally bounded in \((E, \tau)\) iff \( K \) is scb, ctb or totally bounded in \((E, \tau_i)\), respectively, for every \( i \in I \).

**Proof** (of the non-obvious implication \( \Rightarrow \) in the scb case). Since every 0-neighbourhood in \((E, \tau)\) is a 0-neighbourhood in \((E, \sup \tau_i)\) for some finite subset \( J \) of \( I \), we may assume that \( I \) is finite. Inductively we can reduce the assertion to the case that \( \tau \) is the supremum of two linear topologies \( \tau_1 \) and \( \tau_2 \).

Let \( U \) be a 0-neighbourhood in \((E, \tau)\) and \( V_i \) be a 0-neighbourhood in \((E, \tau_i)\) such that \((V_1 - V_1) \cap (V_2 - V_2) \subset U \). By assumption, there are convex sets \( C_i \subset V_i \) and elements \( x_r, y_s \in E \) such that \( K \subset \bigcup_{r=1}^m (x_r + C_1) \) and \( K \subset \bigcup_{s=1}^n (y_s + C_2) \). Choose \( z_{rs} \in C_{rs} := (x_r + C_1) \cap (y_s + C_2) \) if \( C_{rs} \neq \emptyset \) and \( z_{rs} \) arbitrary of \( E \) if \( C_{rs} = \emptyset \). Then \( C_{rs} \subset z_{rs} + C \), where \( C := (C_1 - C_1) \cap (C_2 - C_2) \) is a convex subset of \( U \). Moreover \( K \subset \bigcup_{r,s} C_{rs} \subset \bigcup_{r,s} z_{rs} + C \).

3. - Noncompactness measures

One of the main tools in fixed point theory, after the pioneering work of Darbo [D], is the noncompactness measure. There are many axiomatic approaches to this concept (cf. [B/G], [B/R], [R2], [H1] and references therein). All the approaches try to describe the minimal properties for a fixed point result. An axiomatic approach can be useful also in the frame of nonlocally convex spaces. Let \( I \) be a non-empty set and \( V = [0, +\infty]^I \) the set of all functions from \( I \) to \([0, +\infty] \). \( V \) will be equipped with the usual algebraic operations, the usual order, and with the topology of pointwise convergence.

Let \( E \) be a Hausdorff topological linear space.

**Definition 3.1.** We call a set function \( \varphi : 2^E \rightarrow V \) a noncompactness measure in \( E \), if \( \varphi \) has the following properties.

1. If \( A \) is a convex complete subset of \( E \) and \( \varphi(A) = 0 \), then \( A \) has the fixed point property.
2. \( \varphi(A) = \varphi(\overline{A}) \).
(3) $A \subseteq B$ implies $\varphi(A) \leq \varphi(B)$.

(4) If $(A_n)$ is a decreasing sequence of complete non-empty subsets of $E$ with $\varphi(A_n) \to 0$ $(n \to \infty)$, then $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty.

Classical examples of noncompactness measures with $V = [0, +\infty]$ are the Hausdorff measure $[H/G/M]$ and the Kuratowski measure $[Ku]$.

**Definition 3.2.** Let $C$ be a non-empty subset of $E$ and $\varphi$ a noncompactness measure in $E$. A continuous function $f : C \to C$ is said to be a $\varphi$-contraction if there exists $0 \leq q < 1$ such that $\varphi(f(A)) \leq q \cdot \varphi(A)$ for every subset $A$ of $C$.

**Theorem 3.3.** Let $C$ be a non-empty complete convex subset of $E$, $\varphi$ a noncompactness measure and $f : C \to C$ a $\varphi$-contraction. If $\varphi(C) \in [0, +\infty]^I$, then $f$ has a fixed point.

**Proof.** It is classic. We define inductively a sequence of sets by $C_0 := C$ and $C_{n+1} := \overline{\text{co}} f(C_n)$. We have

$$\varphi(C_n) \leq q^n \cdot \varphi(C) \to 0 \quad (n \to \infty) \quad \text{in } V.$$  

By Axiom 4 in Definition 3.1, the set $C_\infty = \bigcap_{n \in \mathbb{N}} C_n$ is non-empty, moreover complete and convex. Since $f(C_\infty) \subseteq C_\infty$ and $\varphi(C_\infty) = 0$, the statement follows by Axiom 1 in Definition 3.1.

The crucial problem in the previous approach is that Axiom 2 in Definition 3.1 in general does not hold in the nonlocally convex case for the noncompactness measures usually used in locally convex spaces, cf. [H2], [DP/T1].

We will see in 3.8 that the set function $\gamma_\alpha$ introduced below, which measures the “nonstrongly convexly total boundedness”, is a noncompactness measure in the sense of Definition 3.1.

In the following, let $(|| \cdot ||_i)_{i \in I}$ be a family of $F$-seminorms inducing the topology in $E$.

**Notation 3.4.** For $i \in I$ and $A \subseteq E$ we put

$B_{i, \varepsilon} := \{x \in E : ||x||_i \leq \varepsilon\}$ for $\varepsilon > 0$,

$\gamma_i(A) := \inf\{\varepsilon \in [0, +\infty] : \text{there is a finite subset } F \subseteq E \text{ such that } A \subseteq F + B_{i, \varepsilon}\}$,

$\overline{\gamma}_i(A) := \inf\{\varepsilon \in [0, +\infty] : \text{there are } x_1, \ldots, x_n \in E \text{ and convex subsets } C_1, \ldots, C_n \text{ of } B_{i, \varepsilon} \text{ such that } A \subseteq \bigcup_{i=1}^n x_i + C_i\}$,

$\overline{\gamma}_i(A) := \inf\{\varepsilon \in [0, +\infty] : \text{there is a finite subset } F \subseteq E \text{ and a convex subset } C \text{ of } B_{i, \varepsilon} \text{ such that } A \subseteq F + C\}$.

Furthermore, $\gamma := (\gamma_i)_{i \in I}$, $\overline{\gamma} := (\overline{\gamma}_i)_{i \in I}$, $\bar{\gamma}_\alpha := (\bar{\gamma}_i)_{i \in I}$. 


\( \gamma \) is the noncompactness measure of Hausdorff. The set function \( \overline{\gamma} \) is in some sense equivalent to the “measure of the lack of convex precompactness” introduced in the [DP/T1], Definition 2.1.

**Proposition 3.5.** Let \( A \subset E \).

(a) \( \gamma(A) \leq \overline{\gamma}(A) \leq \overline{\gamma}_s(A) \). The three measures coincide if the \( F \)-seminorms \( \| \cdot \|_i \) are even seminorms.

(b) \( A \) is sctb, ctb or totally bounded iff \( \overline{\gamma}_s(A) = 0 \), \( \overline{\gamma}(A) = 0 \) or \( \gamma(A) = 0 \), respectively.

**Proof.** (a) of 3.5 is obvious; (b) follows from 2.4. Hereby observe that e.g. \( \overline{\gamma}_{i,s}(A) = 0 \) iff \( A \) is sctb in the \( \| \cdot \|_i \)-topology.

**Lemma 3.6.** \( \gamma, \overline{\gamma} \) and \( \overline{\gamma}_s \) have property (3) and (4) of 3.1.

**Proof.** Obviously, \( \gamma, \overline{\gamma} \) and \( \overline{\gamma}_s \) are monotone. By 3.5 (a), it is enough to show that \( \gamma \) has property (4). Let \( (A_n) \) be a decreasing sequence of complete non-empty subsets of \( E \) with \( \varphi(A_n) \to 0 \) \( (n \to \infty) \) and \( x_n \in A_n \) for \( n \in \mathbb{N} \). Then

\[
\gamma(\{x_i : i \in \mathbb{N}\}) = \gamma(\{x_i : i \geq n\}) \leq \gamma(A_n) \to 0 \quad (n \to +\infty),
\]

hence \( \gamma(\{x_i : i \in \mathbb{N}\}) = 0 \) and \( \{x_i : i \in \mathbb{N}\} \) is relatively compact. Therefore \( \{x_i : i \in \mathbb{N}\} \) has cluster points, which obviously belong to the intersection \( \bigcap_{n \in \mathbb{N}} A_n \).

**Remark 3.7.** An obvious modification of a part of the proof of 2.2 shows that \( \overline{\gamma}_s \) has property (2) of 3.1, whereas \( \gamma \) and \( \overline{\gamma} \) do not as Example 5.3 (b) shows. \( \overline{\gamma} \) and therefore \( \overline{\gamma}_s \) satisfy (1) of 3.1 by Idzik’s fixed point theorem [11, Theorem 2.4]. Whether \( \gamma \) satisfies (1) of 3.1, is exactly the problem of Schauder mentioned in the introduction.

**Corollary 3.8.** \( \overline{\gamma}_s \) is a noncompactness measure in the sense of 3.1.

An obvious generalization of the proof of 2.2 (a) yields:

**Proposition 3.9.** Let \( A_1, A_2, A \subset E \) and \( \alpha \in \mathbb{R} \). Then

\[
\overline{\gamma}_s(A_1 \cup A_2) \leq \overline{\gamma}_s(A_1) + \overline{\gamma}_s(A_2),
\]
\[
\overline{\gamma}_s(A_1 + A_2) \leq \overline{\gamma}_s(A_1) + \overline{\gamma}_s(A_2),
\]
\[
\overline{\gamma}_s(\alpha A) \leq n \cdot \overline{\gamma}(A) \text{ if } |\alpha| \leq n \in \mathbb{N}.
\]
4. - Noncompactness measures in $l_p$, $0 < p < 1$

Let $0 < p < 1$. For any real sequence $x = (x_n)$, we put $\|x\|_p := \sum_{n \in \mathbb{N}} |x_n|^p$. The space $(l_p, \|\cdot\|_p)$ is an example for an $F$-normed, nonlocally convex space, in which $\gamma(A) = \gamma_s(A) = \gamma_s(A)$ holds for every convex subset $A$.

**Theorem 4.1.** (a) If $A$ is a pointwise bounded subset of $l_p$, then

$$
\gamma(A) = \inf_{n \in \mathbb{N}} \sup_{x \in A} \sum_{i=n}^{\infty} |x_i|^p.
$$

(b) If $A$ is a convex subset of $l_p$, then $\gamma(A) = \gamma_s(A) = \gamma_s(A)$.

**Proof.** (a) For $n \in \mathbb{N}$ denote by $P_n : l_p \to l_p$ the projection $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Let $Q_n = I - P_n$, where $I$ is the identity. Put $\sigma(A) := \sup_{x \in A} \|x\|_p$ and

$$
\tau(A) := \inf_{n \in \mathbb{N}} \sigma(Q_n(A)) = \inf_{n \in \mathbb{N}} \sup_{x \in A} \sum_{i=n}^{\infty} |x_i|^p
$$

for $A \subset l_p$.

(i) $\tau(A) \leq \gamma(A)$: let $\alpha > \gamma(A)$ and $F$ be a finite subset of $l_p$ with $A \subset F + B_\alpha$.

Then $\tau(A) \leq \tau(F) + \tau(B_\alpha) \leq 0 + \sigma(B_\alpha) = \alpha$.

(ii) Let $A$ be pointwise bounded. Then $\gamma(A) \leq \tau(A)$. Let $\alpha > \tau(A)$.

Then $\alpha > \sigma(Q_n(A))$ for some $n \in \mathbb{N}$. Since $P_n(A)$ is a bounded subset of a finite dimensional space, $\gamma(P_n(A)) = 0$ and from $A \subset P_n(A) + Q_n(A)$ follows $\gamma(A) \leq \gamma(P_n(A)) + \gamma(Q_n(A)) \leq 0 + \sigma(Q_n(A)) < \alpha$.

(b) By (a) and 3.5 (a), it is enough to prove that $\gamma_s(A) \leq \tau(A)$, if $A$ is convex and pointwise bounded. Let $\alpha > \tau(A)$. As in (ii), we get $\gamma_s(A) \leq \gamma_s(P_n(A)) + \gamma_s(Q_n(A))$ for some $n \in \mathbb{N}$. Moreover, $\gamma_s(P_n(A)) = \gamma(P_n(A)) = 0$ since $P_n(A)$ is contained in a finite dimensional space and $\gamma_s(Q_n(A)) \leq \sigma(Q_n(A)) < \alpha$, since $Q_n(A)$ is convex.

It follows from 4.1 (b) and 2.2 (b):

**Corollary 4.2.** A subset of $l_p$ is sctb iff its convex hull if totally bounded.

By 4.1 (b) or by 2.3, a convex subset of $l_p$ is totally bounded iff it is ctb iff it is sctb. In [W] it is proved that any subset of $l_p$ is ctb iff it is sctb. On the other hand $l_p$ contains totally bounded subsets, the convex hull of which is not totally bounded; such a set is totally bounded, but by 2.2 (b) not sctb.

A statement analogous to 4.1 (a) is given in [B/G] if $1 \leq p < \infty$. 

5. - The noncompactness measure $\tilde{\gamma}_s$ in $L_0$

In the following, let $\Omega$ be a non-empty set, $\mathcal{U}$ an algebra in the power set $\mathcal{P}(\Omega)$ of $\Omega$ and $\eta : \mathcal{P}(\Omega) \to [0, +\infty[$ a submeasure, i.e. a monotone, subadditive function with $\eta(\emptyset) = 0$. Then $\|f\| := \inf\{a > 0 : \eta(\{|f| \geq a\}) \leq a\}$ defines a group seminorm on $\mathbb{R}^\Omega$. Let $L_0$ be the closure of the space $S := \text{span}\{\chi_A : A \in \mathcal{U}\}$ of $\mathcal{U}$-simple functions in $L^1(\Omega)$. We will identify functions $f, g \in \mathbb{R}^\Omega$, for which $\|f - g\| = 0$. Then the space $(L_0, \|\cdot\|)$ of "measurable functions" becomes an $F$-normed linear space.

In [A/DP], [T/W], the following two parameters $\lambda$ and $\omega$ are used to estimate the noncompactness measure $\gamma$ in $L_0$:

$\lambda(M) := \inf\{\varepsilon > 0 : \text{there is an } a \in [0, +\infty[ \text{ such that } \eta(\{|f| \geq a\}) \leq \varepsilon \text{ for all } f \in M\}$,

$\omega(M) := \inf\{\varepsilon > 0 : \text{there is a partition } A_1, \ldots, A_n \in \mathcal{U} \text{ of } \Omega \text{ such that for every } f \in M \text{ there is a set } D \subset \Omega \text{ with } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(s) - f(t)| : s, t \in A_i \setminus D\} \leq \varepsilon \text{ for } i = 1, \ldots, n\}$.

By [T/W, 2.2.9] we have $\max\{\lambda, \omega/2\} \leq \gamma \leq \lambda + \omega$. To estimate $\tilde{\gamma}_s$ we use the following parameters:

$\tilde{\lambda}(M) := \inf\{\varepsilon > 0 : \text{there is a set } D \subset \Omega \text{ such that } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(x) - f(y)| : x, y \in \Omega \setminus D\} < +\infty\}$,

$\tilde{\omega}(M) := \inf\{\varepsilon > 0 : \text{there is a partition } A_1, \ldots, A_n \in \mathcal{U} \text{ of } \Omega \text{ and a set } D \subset \Omega \text{ such that } \eta(D) \leq \varepsilon \text{ and } \sup\{|f(x) - f(y)| : x, y \in A_i \setminus D\} \leq \varepsilon \text{ for } i = 1, \ldots, n\}$.

Obviously, $\lambda \leq \tilde{\lambda}$ and $\omega \leq \tilde{\omega}$.

**Theorem 5.1.** $\tilde{\gamma}_s(M) \leq \tilde{\gamma}(M) + \tilde{\omega}(M)$ holds for $M \subset L_0$.

**Proof.** It is similar to that of the known inequality $\gamma \leq \lambda + \omega$. Let $\alpha > \tilde{\lambda}(M)$ and $\beta > \tilde{\omega}(M)$. There are sets $D_1, D_2 \subset \Omega$ and a partition $A_1, \ldots, A_n \in \mathcal{U}$ of $\Omega$ such that

$\eta(D_1) < \alpha$ and $s := \sup\{|f(x)| : f \in M, x \in \Omega \setminus D_1\} < +\infty$,

$\eta(D_2) < \beta$ and $\sup\{|f(s) - f(t)| : s, t \in A_i \setminus D_2\} \leq \beta$ for $i = 1, \ldots, n$.

Let $k, m \in \mathbb{N}$ such that $1/m < \alpha$ and $-s + k/m > s$. We put $Y := \{-s + i/m : i = 0, \ldots, k\}$ and $F := \left\{\sum_{i=1}^{k} y_i \cdot \chi_A : y_i \in Y\right\}$. For $f \in M$, there is a $g \in F$ such that $|f(x) - g(x)| \leq 1/m + \beta/2 \leq \alpha + \beta$ for every $x \in \Omega \setminus (D_1 \cup D_2)$; therefore $f - g \in C := \{h \in L_0 : \sup\{|h(x)| : x \in (\Omega \setminus (D_1 + D_2))\} \leq \alpha + \beta\}$. Since $F$ is finite and $C$ a convex subset of $B_{\alpha+\beta}$, it follows that $\tilde{\gamma}_s(M) \leq \alpha + \beta$.

One immediately sees that $\tilde{\lambda}(\text{co} M) = \tilde{\lambda}(M)$ and $\tilde{\omega}(\text{co} M) = \tilde{\omega}(M)$ for $M \subset L_0$ and that $\lambda$ and $\omega$ are monotone. Therefore it follows from 5.1 and 3.8:

**Corollary 5.2.** $\tilde{\lambda} + \tilde{\omega}$ is a noncompactness measure in $L_0$ (in the sense of 3.1).
We see in 5.3 (a) that in the inequality \( \max\{\lambda, \omega/2\} \leq \gamma \) one cannot replace \( \lambda, \omega, \gamma \) by \( \bar{\omega}, \bar{\lambda}, \bar{\gamma}_{\lambda} \).

**EXAMPLE 5.3.** Let \( U \) be the Borel algebra of \( \Omega = [0, 1] \) and \( \mu = \eta[U \) be the Lebesgue measure. Let \( A_1, A_2, \ldots \) be an enumeration of the intervals \( \left\{ \left( i - 2 \right)/2^n, i/2^n \right\} \) \( (i, n \in \mathbb{N}; i \leq 2^n) \), \( (a_n) \) a sequence of positive numbers such that \( a_n \to +\infty \) \( (n \to \infty) \), \( f_n = a_n \chi_{A_n} \) for \( n \in \mathbb{N} \) and \( M = \{ f_n : n \in \mathbb{N} \} \).

(a) \( \bar{\lambda}(M) = \bar{\omega}(M) = 1 \). But \( \bar{\gamma}_{\omega}(M) = 0 \) if \( a_n \mu(A_n) \to 0 \) \( (n \to \infty) \).

(b) \( M \) is ctb. But \( \text{co} M \) is not bounded and therefore not ctb if \( a_n \mu(A_n) \to +\infty \) \( (n \to \infty) \).

(a) We prove the last assertion. If \( n_0 \in \mathbb{N}, \epsilon > 0 \) and \( a_n \mu(A_n) \leq \epsilon^2 \) for \( n \geq n_0 \), then \( f_n \in C : = \left\{ f \in S : \int |f|\,d\mu \leq \epsilon^2 \right\} \) for \( n \geq n_0 \) and \( C \) is a convex subset of \( B_e \). Hence \( M \) is ctb.

(b) Let \( n \in \mathbb{N}, \epsilon = 2^{-n} \) and \( B_i = \left( i - 1 \right)/2^n, i/2^n \right\} \) \( \text{for} \ i \leq 2^n \). Then \( C_i : = \{ f \chi_{B_i} : f \in L_0 \} \) are convex subsets of \( B_\epsilon \) and \( M \setminus \bigcup_{1 \leq i \leq 2^n} C_i \) is finite.

Hence \( M \) is ctb.

Let \( b_i : = a_k \) if \( B_i = A_k \). Then

\[
\text{co} M \ni \sum_{i \leq 1 \leq 2^n} 2^{-n} b_i \chi_{B_i} \geq \min_{1 \leq i \leq 2^n} b_i \mu(B_i) \cdot \chi_{[0,1]}.
\]

Therefore \( \text{co} M \) is not bounded (in the linear topological sense), if \( a_n \mu(A_n) \to +\infty \) \( (n \to \infty) \).

**EXAMPLE 5.4.** Let \( U, \Omega, \mu, \eta \) be chosen as in 5.3 and \( n \in \mathbb{N} \).

(a) For \( A_{i,n} : = \{ f \chi_{\left\{ (i - 1)/n, i/n \right\} : 0 \leq f \leq 2n \} \), \( A_n : = \bigcup_{i=1}^n A_{i,n} \) we have \( \bar{\lambda}(A_n) = 0 \), \( \omega(A_n) \leq 1/n \), but \( \bar{\gamma}_\omega(A_n) \geq 1/2 \) since \( A : = \{ f \in L_0 : 0 \leq f \leq 2 \} \subset \text{co} A_n \) and therefore

\[
1/2 = \gamma(A) \leq \bar{\gamma}_\omega(\text{co} A_n) = \bar{\gamma}_\omega(A_n).
\]

(b) For \( B_{i,n} : = \{ c \chi_{\left\{ (i - 1)/n, i/n \right\} : c \in \mathbb{R} \} \), \( B_n : = \bigcup_{i=1}^n B_{i,n} \) we have \( \lambda(B_n) = 1/n \), \( \bar{\omega}(B_n) = 0 \), but \( \bar{\gamma}_\omega(B_n) = 1 \) since \( B : = \{ c \chi_{[0,1]} : c \in \mathbb{R} \} \subset \text{co} B_n \) and therefore

\[
1 = \lambda(B) = \gamma(B) \leq \bar{\gamma}_\omega(\text{co} B_n) = \bar{\gamma}_\omega(B_n) \leq 1.
\]

In contrast to 5.4, we will see that, under the assumptions of 5.4, \( \bar{\lambda}(M) = \omega(M) = 0 \) or \( \lambda(M) = \bar{\omega}(M) = 0 \) implies \( \bar{\gamma}_\omega(M) = 0 \).
PROPOSITION 5.5. Let \( M \subset L_0 \) and \( \lambda(M) = 0 \). Then \( \overline{\lambda}(M) \leq \overline{\omega}(M) \) and therefore \( \bar{\gamma}_\mu(M) \leq 2 \cdot \overline{\omega}(M) \).

PROOF. Let \( \lambda(M) = 0 \). By 5.1, it is enough to show that \( \overline{\lambda}(M) \leq \overline{\omega}(M) \). Let \( \alpha > \overline{\omega}(M) \). Then there is a set \( D \subset \Omega \) with \( \eta(D) < \alpha \) and a partition \( A_1, \ldots, A_n \in \mathcal{U} \) of \( \Omega \) such that

\[
\text{sup} \{|f(s) - f(t)| : s, t \in A_i \backslash D\} < \alpha \quad \text{for } i = 1, \ldots, n.
\]

We may assume that \( \eta(A_i \backslash D) > 0 \) for \( i < m \) and \( A_i \subset D \) for \( i \geq m \), for some \( m \in \mathbb{N} \). Since \( \lambda(M) = 0 \), there is a \( b \in [0, +\infty[ \) and, for every \( f \in M \), a set \( D(f) \subset \Omega \) such that \( \eta(D(f)) < \min_i \eta(A_i \backslash D) \) and \( |f(x)| \leq b \) for \( x \in \Omega \setminus D(f) \).

Let \( f \in M \) and \( x \in \Omega \setminus D \). We show that \( |f(x)| \leq \alpha + b \). In fact, if \( i < m \) with \( x \in A_i \), then \( \eta(D(f)) < \eta(A_i \backslash D) \); therefore there is an \( y \in (A_i \backslash D) \backslash D(f) \) and \( |f(x)| \leq |f(x) - f(y)| + |f(y)| \leq \alpha + b \). It follows that \( \overline{\lambda}(M) \leq \eta(D) < \alpha \).

PROPOSITION 5.6. Assume that \( \eta(B) = \inf \{\eta(A) : B \subset A \in \mathcal{U}\} \) for any \( B \subset \Omega \) and that \( \mu := \eta|\mathcal{U} \) is additive. Let \( M \subset L_0 \) and \( \omega(M) = 0 \). Then \( \overline{\gamma}_\mu(M) \leq \overline{\lambda}(M) \).

PROOF. Let \( \omega(M) = 0 \) and \( \alpha > \overline{\lambda}(M) \). By assumption, there is a set \( D \in \mathcal{U} \), with \( \eta(D) < \alpha \), and a number \( c > 0 \) such that \( |f(x)| \leq c \) for \( f \in M \) and \( x \in \Omega \setminus D \). \( M_1 := L_0 \cdot \chi_D \) is a convex subset of \( B_\alpha \), hence \( \overline{\gamma}_\mu(M_1) \leq \alpha \). The set \( M_2 := M \cdot \chi_{\Omega \setminus D} \) is totally bounded, since \( \omega(M_2) = \lambda(M_2) = 0 \). On \( \{f \in L_0 : |f| \leq c\} \) the \( \| \cdot \|_1 \)-topology coincides with the \( \| \cdot \| \)-topology, where \( \|f\|_1 := \int |f| \, d\mu \). Therefore \( M_2 \) is also totally bounded with respect to the (semi-)norm \( \| \cdot \|_1 \) and therefore scbt, i.e. \( \overline{\gamma}_\mu(M_2) = 0 \). Since \( M \subset M_1 + M_2 \), it follows \( \overline{\gamma}_\mu(M) \leq \overline{\gamma}_\mu(M_1) + \overline{\gamma}_\mu(M_2) \leq \alpha \).

Under the assumption of 5.6, a set \( M \subset L_0 \) is scbt if \( \omega(M) = \overline{\gamma}(M) = 0 \), in particular, if \( M \) is totally bounded and \( \overline{\lambda}(M) = 0 \). The next proposition clarifies the meaning of \( \overline{\lambda}(M) = 0 \).

PROPOSITION 5.7. Assume that \( \eta(B) = \inf \{\eta(A) : B \subset A \in \mathcal{U}\} \) for \( B \subset \Omega \). Then for \( M \subset L_0 \), \( \lambda(M) = 0 \) iff \( M \subset [-\varphi, \varphi] \) for some \( \varphi \in L_0 \), \( \geq 0 \).

PROOF. \( \Rightarrow \): Let \( \overline{\lambda}(M) = 0 \). By assumption, there are \( A_n \in \mathcal{U} \) and \( a_n \in [0, +\infty[ \) such that \( \eta(\Omega \setminus A_n) \leq 1/n \) and \( |f(x)| \leq a_n \) for \( f \in M \) and \( x \in A_n \). Define \( B_n := A_n \setminus \bigcup_{i=1}^{n} A_i \), \( \varphi_n = \sum_{i=1}^{n} a_i \chi_{B_i} \), \( \varphi = \sum_{i=1}^{\infty} a_i \chi_{B_i} \). Then \( \varphi \in S \), \( \|\varphi - \varphi_n\| \leq \eta(\Omega \setminus A_n) \leq 1/n \), hence \( \varphi \in L_0 \). Moreover, \( |f(x)| \leq \varphi(x) \) for \( f \in M \) and \( x \in \Omega \setminus D \), where \( D = \Omega \setminus \bigcup_{n=1}^{\infty} A_n \) and \( \eta(D) = 0 \).

\( \Leftarrow \): Let \( \varphi \) be a positive function of \( L_0 \), \( M \subset [-\varphi, \varphi] \) and \( \varepsilon > 0 \). Then there is a positive number \( c \) such that \( \eta(\{\varphi \geq c\}) \leq \varepsilon \), hence \( \overline{\lambda}(M) \leq \eta(\{\varphi \geq c\}) \leq \varepsilon \).
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