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Regularity of Free Boundaries in Two Dimensions

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1. - Introduction

In this paper we discuss regularity of free boundaries in two dimensions which appear in an obstacle problem. Let u be a non-negative function defined in the unit disk B_1 of the complex z-plane such that:

(i) $\Gamma(u) = (\partial\Omega(u)) \cap B_1$ contains the origin 0, where $\Omega(u) = \{z \in B_1 : u(z) > 0\}$;

(ii) u is of class C^1 in B_1;

(iii) $\Delta u(z) = 1$ in $\Omega(u)$ in the sense of distributions.

What can we say about regularity of the free boundary $\Gamma(u)$?

An accurate description of $\Gamma(u)$ was given by Caffarelli and Rivière in [1] and [2]. They showed that either:

(1) 0 is a regular point; namely, for a small disk B_δ with radius $\delta > 0$ and center 0, $\Omega(u) \cap B_\delta$ is simply connected and $\Gamma(u) \cap B_\delta$ is a regular analytic simple arc passing through 0;

or

(2) $B_\delta \backslash \Omega(u)$ is arranged along a straight line for small B_δ; more precisely, there is an increasing function η defined on a half-open interval $[0, \delta)$ such that $\eta(0) = 0$ and

$$B_\delta \backslash \Omega(u) \subset e^{i\alpha}\{z = x + iy \in B_\delta : |y| \leq \eta(|x|)\},$$

where α denotes a real number and $e^{i\alpha}E$ for a set E denotes $\{e^{i\alpha}z : z \in E\}$.

Furthermore they proved in [1] that in case (2) it follows that:

(α) if 0 is not an isolated point of $\Gamma(u)$ and if the interior of $B_1 \backslash \Omega(u)$ is empty, then $\Gamma(u) \cap B_\delta$ is a real analytic simple arc, and

(β) the boundary of each connected component of the interior of $B_\delta \backslash \Omega(u)$ is the union of a finite number of real analytic simple arcs.

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Their results are fairly accurate, but there is still a possibility that an infinite number of connected components of the interior of $B_1\backslash\Omega(u)$ exist and cluster around 0.

The purpose of this paper is to give a complete description of the free boundary $\Gamma(u)$ and to exclude such a possibility. Our main result is:

THEOREM 1.1. *Let u, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$ be as above. Then the origin* 0 *is either a* (1) *regular, or a* (2a) *degenerate, or a* (2b) *double or a* (2c) *cusp point of* Γ. *Namely, there is a small disk $B = B_\delta$ such that one of the following occurs:*

(1) *$\Omega \cap B$ is simply connected and $\Gamma \cap B$ is a regular real analytic simple arc passing through* 0;

(2a) *$\Gamma \cap B = \{0\}$ or $\Gamma \cap B$ is a regular real analytic simple arc passing through* 0. *$\Omega \cap B$ is equal to $B\backslash\Gamma$;*

(2b) *$\Omega \cap B$ consists of two simply connected components Ω_1 and Ω_2. $(\partial\Omega_1)\cap B$ and $(\partial\Omega_2)\cap B$ are distinct regular real analytic simple arcs passing through* 0. *They are tangent to each other at* 0;

(2c) *$\Omega \cap B$ is simply connected and $\Gamma \cap B$ is a regular real analytic simple arc except for a cusp at* 0. *The cusp is pointing into $\Omega \cap B$. It is a very special one. There is a holomorphic function T defined on a closed disk $\overline{B_\varepsilon}$ such that:*

i. $T(0) = 0$, $T'(0) = 0$ *and* $T''(0) \neq 0$;

ii. *T is univalent on the closure $\overline{H}$ of a half disk $H = \{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$;*

iii. *T satisfies $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$ and $T(\overline{H}) \subset \Omega \cup \Gamma$, where $i(-\varepsilon, \varepsilon) = \{it : -\varepsilon < t < \varepsilon\}$.*

Furthermore, all the second derivatives of u are continuous up to Γ, on Ω and u is real analytic up to Γ, on Ω except double and cusp points of Γ. If 0 *is a double point, then there is a positive number γ such that*

$$B\backslash\Omega(u) \subset e^{i\alpha}\{z = x + iy \in B : |y| \leq \gamma x^2\}, \tag{1.1}$$

where $e^{i\alpha}$ denotes the unit vector at 0 *tangent to Γ. If* 0 *is a cusp point, then, for some positive number γ, it follows that*

$$B\backslash\Omega(u) \subset e^{i\alpha}\{z = x + iy \in B : x \leq 0 \text{ and } |y| \leq \gamma x^2\}, \tag{1.2}$$

where $e^{i\alpha}$ denotes the unit vector at 0 *tangent to Γ and pointing into Ω. Namely, we can take a quadratic function γt^2 as a function $\eta(t)$ in the argument given by Caffarelli and Rivière.*

This regularity theorem holds also if we replace the constant function with value 1 in (iii) by a positive real analytic function φ defined in B_1. This fact is quite interesting when we compare it with an example of the free boundary for the obstacle problem with C^∞-obstacle due to Schaeffer [9]: if we replace the constant function with value 1 in (iii) by some special positive C^∞-function φ defined in B_1, then there is a non-negative function u satisfying (i) to (iii) such that an infinite number of connected components of the interior of $B_1 \backslash \Omega(u)$ actually cluster around 0.

2. - Proof of Theorem 1.1

We shall first define classes of functions which appear in an obstacle problem. For the free boundary for the obstacle problem, we refer to Chapter V of Kinderlehrer -Stampacchia [4] and Chapter 6 of Rodrigues [6].

DEFINITION 2.1. We say that a non-negative function u in B_ρ is of class $P(1, B_\rho)$ if u satisfies (i) to (iii) of Section 1 with B_ρ replacing B_1. We say that a real-valued function u in B_ρ is of class $R(1, B_\rho)$ if there exists an open subset $\Omega(u)$ of B_ρ such that u and $\Omega(u)$ satisfy (ii) and (iii) of Section 1 with B_ρ replacing B_1 and if

(i′) 0 is contained in $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$, and

(iv) $\dfrac{\partial u}{\partial x}(z) = \dfrac{\partial u}{\partial y}(z) = 0$ on $\Gamma(u)$, where $z = x + iy$.

If u is of class $P(1, B_\rho)$, then $u(z) \geq 0$ in B_ρ and $u(z) = 0$ on $\Gamma(u)$, so u satisfies (iv). Thus, by taking $\Omega(u) = \{z \in B_\rho; u(z) > 0\}$, we see that $P(1, B_\rho) \subset R(1, B_\rho)$.

LEMMA 2.1. *Let $u \in R(1, B_\rho)$. Then $S(z) = \overline{z} - 4\dfrac{\partial u}{\partial z}(z)$, where $\overline{z}$ denotes the complex conjugate of z, is the Schwarz function of $\Omega(u) \cup \Gamma(u)$ in B_ρ; namely, S is a function defined on $\Omega(u) \cup \Gamma(u)$ which is holomorphic in $\Omega(u)$, is continuous on $\Omega(u) \cup \Gamma(u)$ and satisfies $S(\varsigma) = \overline{\varsigma}$ on $\Gamma(u)$.*

PROOF. By (ii), S is continuous on $\Omega(u) \cup \Gamma(u)$. Since by (iii)

$$\partial S(z) \backslash \partial \overline{z} = 1 - 4(\partial^2 u(z))/(\partial z \partial \overline{z}) = 1 - \Delta u(z) = 0 \text{ in } \Omega(u),$$

S is holomorphic in $\Omega(u)$. By (iv), $\partial u/\partial z = 0$ on $\Gamma(u)$, and so $S(z) = \overline{z}$ on $\Gamma(u)$. Hence S is the Schwarz function of $\Omega(u) \cup \Gamma(u)$ in B_ρ. Q.E.D.

The next proposition is just an application of a regularity theorem proved in [8]; nevertheless, it is somewhat surprising when we look at the definition of $R(1, B_\rho)$. In particular, all the second derivatives of u are continuous up to $\Gamma(u)$, on $\Omega(u)$ for every $u \in R(1, B_\rho)$.

PROPOSITION 2.2. *Let $u \in R(1, B_\rho)$, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$. Then the origin 0 is a regular, double, cusp point of Γ in the sense of Theorem 1.1 or a degenerate point in the sense that:*

(2a′) *for a small disk* $B = B_\delta$, $\Gamma \cap B = \{0\}$ *or* $\Gamma \cap B$ *is an infinite set accumulating at* 0 *and is contained in a uniquely determined regular real analytic simple arc passing through* 0. $\Gamma \cap B$ *is a proper subset of the arc or the whole arc.* $\Omega \cap B$ *is equal to* $B \backslash \Gamma$.

Furthermore, all the second derivatives of u *are continuous up to* Γ, *on* Ω *and* u *is real analytic up to* Γ, *on* Ω *except double and cusp points of* Γ.

PROOF. The first assertion follows from Lemma 2.1 and the Regularity Theorem for a boundary having a Schwarz function, see [8]. Let $U(z)$ and $V(z)$ be the real and imaginary parts of $S(z) = \bar{z} - 4\dfrac{\partial u}{\partial z}(z)$, respectively. Then $U(z) = x - 2\partial u/\partial x$, $V(z) = -y + 2\partial u/\partial y$, $\partial U/\partial x = 1 - 2(\partial^2 u)/(\partial x^2) = -1 + 2(\partial^2 u)/(\partial y^2)$, $\partial V/\partial x = 2(\partial^2 u)/(\partial x \partial y)$ and $S' = (\partial U/\partial x) + i(\partial V/\partial x)$ in Ω. Since $\lim\limits_{z \in \Omega, z \to \varsigma} S'(z)$ exists for every ς on Γ by Corollary 5.4 of [8], we see that all the second derivatives of u are continuous up to Γ, on Ω. The final assertion follows from the Regularity Theorem. Q.E.D.

Now we shall give a proof of our Theorem 1.1. Since $P(1, B_\rho) \subset R(1, B_\rho)$, from Proposition 2.2 it follows that the origin 0 falls in one of the situations (1), (2a′), (2b) or (2c). If the origin 0 is in (2a′), then $S(z) = \bar{z} - 4\dfrac{\partial u}{\partial z}(z)$ is holomorphic in B, and so $\dfrac{\partial u}{\partial z}(z) = 0$ on the arc determined by $\Gamma \cap B$ if 0 is not an isolated point of Γ. This means that $u(z) = 0$ on the arc. Hence $\Gamma \cap B$ is the whole arc determined by $\Gamma \cap B$ and (2a) holds. If (2b) holds, then (1.1) holds for some α and γ by Corollary 5.3 of [8]. To show that (1.2) holds if (2c) holds, we need the following lemma:

LEMMA 2.3. *Let* u *and* S *be as in Lemma* 2.1. *Assume that* 0 *is a cusp point of* Γ *and let* $z = T(\tau) = \kappa e^{i\alpha}(\tau^2 + a_3\tau^3 + a_4\tau^4 + \cdots)$ *be a one-to-one conformal mapping of* $\{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$ *into* Ω *such that* $\Gamma \cap B \subset T(i(-\varepsilon, \varepsilon))$, *where* $\kappa > 0$. *Then* $\operatorname{Re} a_3 \leq 0$, *and* $\operatorname{Re} a_3 = 0$ *if* $u \in P(1, B_\rho)$.

PROOF. We can assume that $\kappa e^{i\alpha} = 1$. Set $\alpha_j = \operatorname{Re} a_j$ and $\beta_j = \operatorname{Im} a_j$. If $\tau = it$ and t is real, then $x = \operatorname{Re} T(\tau) = -t^2 + \beta_3 t^3 + \alpha_4 t^4 + \cdots$ and $y = \operatorname{Im} T(\tau) = -\alpha_3 t^3 + \beta_4 t^4 + \cdots$. Hence $x < 0$ for t with small $|t|$. If $\alpha_3 > 0$, then $y < 0$ for small $t > 0$ and $y > 0$ for $t < 0$ with small $|t|$. This contradicts the univalence of the mapping T. Therefore $\alpha_3 \leq 0$.

Since $S(z) = S(T(\tau)) = \overline{T(-\bar{\tau})} = \tau^2 - \overline{a_3}\tau^3 + \overline{a_4}\tau^4 + \cdots$, we obtain

$$\begin{aligned}\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z) = 2\frac{\partial u}{\partial z}(z) &= \frac{1}{2}(\bar{z} - S(z)) \\ &= -i \operatorname{Im}\tau^2 + \overline{a_3}\operatorname{Re}\tau^3 - i\overline{a_4}\operatorname{Im}\tau^4 + \cdots \\ &= (\alpha_3 r^3 \cos 3\theta + \cdots) - i(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta + \cdots),\end{aligned}$$

where $\tau = re^{i\theta}$. For small fixed $r > 0$, $\frac{\partial u}{\partial y}(z) = 0$ has a unique solution $\theta = \theta(r)$ in $(-\pi/2, \pi/2)$ and $\theta(r)$ is close to 0. Hence $u(T(re^{i\theta(r)})) - u(0) = \int_J \frac{\partial u}{\partial x}(z)dx$, where $J = \{T(se^{i\theta(s)}) : 0 \leq s \leq r\}$. If $\alpha_3 < 0$, then $\frac{\partial u}{\partial x}(z) < 0$ on J for small $r > 0$, and so $u(T(re^{i\theta(r)})) < u(0)$. Thus if $u \in P(1, B_\rho)$, namely, if $u(z) \geq 0$ in B_ρ, then $\alpha_3 = 0$. Q.E.D.

Now we shall show that (1.2) holds if (2c) holds. We use the notation of Lemma 2.3. From the lemma we see that $\alpha_3 = \operatorname{Re} a_3 = 0$. Hence, for $\tau = it$, $y = \beta_4 t^4 + \cdots$. Since $x = -t^2 + \beta_3 t^3 + \cdots$ for $\tau = it$, $x \leq 0$ and $|y| \leq 2|\beta_4| x^2$ for $z \in \Gamma \cap B_\delta$ with small $\delta > 0$. This completes the proof of Theorem 1.1.

3. - A function associated with a Schwarz function

In Section 2, to each $u \in R(1, B_1)$, we have assigned a Schwarz function S by $S(z) = \bar{z} - 4\frac{\partial u}{\partial z}(z)$. In this section we shall discuss the converse. For a given Schwarz function S defined on $\Omega \cup \Gamma$, we shall construct a function $u \in R(1, B_\rho)$ such that $S(z) = \bar{z} - 4\frac{\partial u}{\partial z}(z)$ on $(\Omega \cup \Gamma) \cap B_\rho$.

In contrast with the arguments given in Chapter V of Kinderlehrer-Stampacchia [4], Lewy-Stampacchia [5], Schaeffer [9] and others, we do not assume that the free boundary Γ is a simple arc or a continuum.

Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1, namely, let S be a function which is holomorphic in an open subset Ω of B_1, is continuous on $\Omega \cup \Gamma$ and satisfies $S(\zeta) = \bar{\zeta}$ on Γ, where $\Gamma = (\partial\Omega) \cap B_1$ and $0 \in \Gamma$. Let D be a connected component of Ω. By the Regularity Theorem of [8], every point of $(\partial D) \cap B_1$ is an accessible boundary point of D. Let ζ be a fixed point of $(\partial D) \cap B_1$ and let $z \in D$. The integral $\int_J S(w)dw$ may depend on the choice of path J in $D \cup \{\zeta\}$ joining ζ and z. We shall show that $\operatorname{Re} \int_J S(w)dw$ does not depend on J. Let J' be another path in $D \cup \{\zeta\}$ joining ζ and z. To show that $\operatorname{Re} \int_{J-J'} S(w)dw = 0$, we may assume that $J - J'$ is a simple closed curve.

If $J - J'$ does not surround any part of $B_1 \backslash \Omega$, then, by the Cauchy theorem, $\int_{J-J'} S(w)dw = 0$. If $J - J'$ surrounds a part E of $B_1 \backslash \Omega$, then, by the Regularity Theorem, we may assume that ∂E consists of a finite number of analytic simple closed curves having possibly double and cusp points. Hence

$$\int_{J-J'} S(w)dw = \int_{\partial E} S(w)dw = \int_{\partial E} \overline{w}dw = \int_E d\overline{w} \wedge dw = 2i \int_E dm,$$

and therefore the real part of this integral is 0. Thus $\mathrm{Re} \int_{\varsigma}^{z} S(w)dw$ is single-valued and harmonic in D.

Next, for ρ with $0 < \rho < 1$, we shall define a function h harmonic in $\Omega \cap B_\rho$ and continuous on $\overline{\Omega \cap B_\rho}$. By the Regularity Theorem, we may assume that $\partial(\overline{\Omega \cap B_\rho})$ consists of a finite number of piecewise analytic simple closed curves. Take a connected component F_k of $\overline{\Omega \cap B_\rho}$ and let ς_k be a fixed point on $(\partial F_k)\backslash(\partial B_\rho)$. Set

$$h_k(z) = 2\,\mathrm{Re} \int_{\varsigma_k}^{z} S(w)dw + |\varsigma_k|^2$$

on F_k. Then h_k is harmonic in the interior of F_k, continuous on F_k and

$$\begin{aligned} h_k(z_2) - h_k(z_1) &= 2\,\mathrm{Re} \int_{z_1}^{z_2} \overline{w}dw = \int_{z_1}^{z_2} (\overline{w}dw + wd\overline{w}) \\ &= \int_{z_1}^{z_2} d|w|^2 = (|z_2|^2 - |z_1|^2) \end{aligned} \tag{3.1}$$

if z_1 and z_2 belong to the same connected component of $(\partial F_k)\backslash(\partial B_\rho)$. The function h_k depends on the choice of $\varsigma_k \in (\partial F_k)\backslash(\partial B_\rho)$ and is uniquely determined up to a real additive constant. Now we define a function h on $\overline{\Omega \cap B_\rho}$ by $h(z) = h_k(z)$ on F_k and set

$$u(z) = \frac{1}{4}(|z|^2 - h(z))$$

on $\overline{\Omega \cap B_\rho}$. Then $\Delta u(z) = 1$ in $\Omega \cap B_\rho$, u is continuous on $\overline{\Omega \cap B_\rho}$ and, by (3.1), u is constant on each connected component of $(\partial(\overline{\Omega \cap B_\rho}))\backslash\partial B_\rho$. We extend u onto B_ρ so that the extension, which we denote by u again, is continuous in B_ρ, is of class C^1 in the interior I of $B_\rho\backslash\Omega$ and satisfies

$$\lim_{z\in I, z\to\varsigma} (\partial u/\partial x)(z) = \lim_{z\in I, z\to\varsigma} (\partial u/\partial y)(z) = 0 \tag{3.2}$$

for every $\varsigma \in (\partial(\overline{\Omega \cap B_\rho}))\backslash\partial B_\rho$. This is possible, because u is constant on each connected component of $(\partial(\overline{\Omega \cap B_\rho}))\backslash\partial B_\rho$.

LEMMA 3.1. *The function* u *defined above satisfies* $S(z) = \bar{z} - 4\dfrac{\partial u}{\partial z}(z)$ *on* $(\Omega \cup \Gamma) \cap B_\rho$ *and is of class* $R(1, B_\rho)$ *if we set* $\Omega(u) = \Omega \cap B_\rho$.

PROOF. Since $0 \in \Gamma$, (i′) of Definition 2.1 is satisfied. We have already seen that $\Delta u(z) = 1$ in $\Omega(u)$. To show (iv) of Definition 2.1, let $\Sigma_k(z) = \int_{\varsigma_k}^{z} S(w)dw$ on a connected component F_k of $\overline{\Omega \cap B_\rho}$. Then

$$S(z) = \Sigma_k'(z) = \frac{\partial \Sigma_k(z)}{\partial z} + \frac{\partial \overline{\Sigma_k(z)}}{\partial z} = \frac{\partial 2 \operatorname{Re} \Sigma_k(z)}{\partial z}$$

in $\Omega \cap B_\rho$. Hence $\partial u(z)/\partial z = (1/4)(\bar{z} - S(z))$ in $\Omega \cap B_\rho$. Since $S(\varsigma) = \bar{\varsigma}$ on Γ,

$$\lim_{z \in \Omega \cap B_\rho, z \to \varsigma} (\partial u/\partial x)(z) = \lim_{z \in \Omega \cap B_\rho, z \to \varsigma} (\partial u/\partial y)(z) = 0 \tag{3.3}$$

for every $\varsigma \in \Gamma \cap B_\rho = \Gamma(u)$. By (3.2) and (3.3),

$$\lim_{z \to \varsigma} (\partial u/\partial x)(z) = \lim_{z \to \varsigma} (\partial u/\partial y)(z) = 0 \tag{3.4}$$

for every $\varsigma \in (\partial(\overline{\Omega \cap B_\rho})) \backslash \partial B_\rho$. If $\varsigma \in \Gamma(u) \backslash \partial(\overline{\Omega \cap B_\rho})$, then ς is a degenerate point of Γ. If ς is an isolated degenerate point of Γ, then $S(\varsigma) = \bar{\varsigma}$ and (3.4) holds. If ς is a non-isolated degenerate point of Γ, there exists a real analytic simple arc passing through ς such that $S(z) = \bar{z}$ on the arc. Hence u is constant on the arc and (3.4) holds. Thus (iv) of Definition 2.1 holds and u is of class C^1 in B_ρ. Q.E.D.

To discuss the case $u(z) \geq 0$ in $\Omega \cap B_\rho$ for some $\rho > 0$, we shall have a more detailed discussion. We first note that, by the Regularity Theorem, $\overline{\Omega \cap B_\rho}$ is connected for sufficiently small $\rho > 0$. For such a small ρ, we see that $h(\varsigma) = |\varsigma|^2$ on $\Gamma \cap B_\rho$, and so $u(\varsigma) = 0$ on $\Gamma \cap B_\rho$. We set $u_S(z) = u(z)$ on $\overline{\Omega \cap B_\rho}$ and $u_S(z) = 0$ on $B_\rho \backslash \overline{\Omega \cap B_\rho}$.

LEMMA 3.2. *Let* S *be the Schwarz function of* $\Omega \cup \Gamma$ *in* B_1. *If* 0 *is a regular, non-isolated degenerate or double point of* Γ, *then* $u_S \in P(1, B_\rho)$ *for sufficiently small* $\rho > 0$.

PROOF. If 0 is a non-isolated degenerate point of Γ, then S is holomorphic in B_δ for small $\delta > 0$. Hence u_S is real analytic in B_δ, $\Delta u_S(z) = 1$ in B_δ and $u_S(z) = (\partial u_S/\partial x)(z) = (\partial u_S/\partial y)(z) = 0$ on the arc J determined by Γ. Let $(\partial u_S/\partial n)(z)$ be the derivative along the direction normal to J. Then $(\partial^2 u_S/\partial n^2)(z) = \Delta u_S(z) = 1$ on J. Since $u_S(z) = (\partial u_S/\partial n)(z) = 0$ on J, we see that $u_S(z) > 0$ on $B_\rho \backslash J$ and $u_S(z) = 0$ on $J \cap B_\rho$ for some small $\rho > 0$. Hence $u_S \in P(1, B_\rho)$ and $\Gamma(u_S) = J \cap B_\rho$.

The same argument works for the case that 0 is a regular or double point of Γ. Q.E.D.

REMARK. If 0 is a regular or double point of Γ, then $\Omega(u_S) \equiv \{z \in B_\rho : u_S(z) > 0\} = \Omega \cap B_\rho$ for some $\rho > 0$. If 0 is a non-isolated degenerate point of Γ, then $\Omega(u_S) = B_\rho \backslash J$ and $\Omega \cap B_\rho = B_\rho \backslash \Gamma$. Hence $\Omega(u_S) \subset \Omega \cap B_\rho$ and the equality does not hold in general.

Next we shall discuss the case that 0 is a cusp point of Γ. To do so, we shall define the index of a cusp point. Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1 and assume that 0 is a cusp point. Let $e^{i\alpha}$ be the unit vector at 0 tangent to Γ and pointing into Ω. Let C_ε be a half circle defined by $C_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon, |\arg z - \alpha| \leq \pi/2\}$. C_ε is oriented counterclockwise. Let $v(z) = S(z) - \bar{z}$. It is known that $\int_{C_\varepsilon} d \arg v(z) = \pm\pi + 0(\varepsilon)$.

DEFINITION 3.1. We call the origin 0 a cusp with *index* $-1/2$ (resp. $+1/2$) if $\int_{C_\varepsilon} d \arg v(z) = \pi + 0(\varepsilon)$ (resp. $\pi + 0(\varepsilon)$).

Let J be the arc starting from 0 and defined by $J = \{z \in \Omega : \operatorname{Im} v(z) = 0\}$. Then J intersects ∂B_ε once for small $\varepsilon > 0$ and $\operatorname{Re} v(z) = v(z) \neq 0$ on $J \cap B_\varepsilon$. The index of the cusp point 0 is $-1/2$ if $\operatorname{Re} v(z) < 0$ on $J \cap B_\varepsilon$ and $+1/2$ if $\operatorname{Re} v(z) > 0$ on $J \cap B_\varepsilon$. For further details, see [7]. From arguments in Section 4 of [7] and Section 5 of Kinderlehrer-Nirenberg [3], we obtain:

LEMMA 3.3. *Let S be the Schwarz function of $\Omega \cup \Gamma$ in B_1. Let 0 be a cusp point of Γ and let u_S be the function defined before Lemma 3.2. Then $u_S \in P(1, B_\rho)$ for sufficiently small $\rho > 0$ if and only if the index of the cusp point is equal to $-1/2$.*

PROOF. We can assume that $e^{i\alpha} = 1$. Let J be the arc starting from 0 defined by $J = \{z \in \Omega : \operatorname{Im} v(z) = 0\}$. Then $(\partial u_S/\partial y)(z) = 0$ on J, because $(\partial u_S/\partial z)(z) = (1/4)(\bar{z} - S(z)) = -(1/4)v(z)$ is real on J. Let $z(s)$ be a point on J, where s denotes the arc length of J from 0 to $z(s)$. Then, by the mean value theorem, $u_S(z(s)) = s(\partial u_S/\partial s)(z(\lambda s))$ for some λ with $0 < \lambda < 1$. Since $(\partial u_S/\partial s)(z(\lambda s)) = (\partial u_S/\partial x)(z(\lambda s))(\partial x/\partial s)(\lambda s)$, we obtain

$$\begin{aligned}(1/4)\operatorname{Re} v(z(\lambda s)) &= (1/2)(\partial u_S/\partial x)(z(\lambda s)) \\ &= (1/2)((\partial x/\partial s)(\lambda s))^{-1}(\partial u_S/\partial s)(z(\lambda s)) \\ &= (1/2)(s(\partial x/\partial s)(\lambda s))^{-1} u_S(z(s)).\end{aligned}$$

Since $(\partial x/\partial s)(\lambda s) > 0$ for small $s > 0$, $\operatorname{Re} v(z(\lambda s)) < 0$ if $u_S(z) > 0$ in $\Omega \cap B_\rho$. Hence the index is equal to $-1/2$ if $u_S \in P(1, B_\rho)$.

Conversely, if the index is equal to $-1/2$, then $\operatorname{Re} v(z(\lambda s)) < 0$, and so $u_S(z) > 0$ on J. Since

$$\lim_{z \in \Omega, z \to 0} S'(z) = e^{-2i\alpha} = 1$$

by Corollary 5.4 of [8], using the same argument as in the proof of Proposition 2.2 we can see that

$$\lim_{z\in\Omega, z\to 0} \frac{\partial^2 u_S}{\partial y^2}(z) = 1.$$

Hence the restriction of u_S to each vertical line $x = x_0$ is a convex function of y in a neighborhood of 0. It attains its minimum at $x_0 + iy_0 \in J$ if $x_0 > 0$ (because $(\partial u_S/\partial y)(z) = 0$ on J) and at $x_0 + iy_0 \in \Gamma$ if $x_0 \leq 0$, where $u_S(x_0 + iy_0) = (\partial u_S/\partial y)(x_0 + iy_0) = 0$. Thus $u_S(z) > 0$ on $\Omega \cap B_\rho$ for some $\rho > 0$. Q.E.D.

REMARK. Using the same argument as in the proof of Lemma 3.3, we can see that if 0 is an isolated point of Γ then $u_S \in P(1, B_\rho)$ for small $\rho > 0$ if and only if the index of $v(z) = S(z) - \bar{z}$ at 0 is equal to -1. For the definition of the index of v at an isolated point of Γ, see [7].

4. - Cusp points

Let $u \in R(1, B_\rho)$ and assume that the origin 0 is a cusp point of $\Gamma(u)$. By Lemmas 2.1 and 3.1, we see that this cusp is the same as the cusp which appears on a boundary having a Schwarz function. It is precisely described by a conformal mapping as that given in (2c) of Theorem 1.1. In contrast with this fact, the cusp point for $u \in P(1, B_\rho)$ is a very special one. Here we shall have a more detailed discussion about the cusp point for $u \in P(1, B_\rho)$, which improves results due to Schaeffer [9] and Kinderlehrer-Nirenberg [3].

Let $u \in P(1, B_\rho)$, $\Omega = \Omega(u)$ and $\Gamma = \Gamma(u)$. Let 0 be a cusp point of Γ and let $e^{i\alpha} = -1$, where $e^{i\alpha}$ denotes the unit tangent vector to Γ at 0 pointing into Ω. For small $\delta > 0$, $\Gamma \cap B_\delta$ is a simple arc with cusp at 0. We divide $\Gamma \cap B_\delta$ into two regular real analytic simple arcs Γ_1 and $\Gamma_2 : \Gamma \cap B_\delta = \Gamma_1 \cup \Gamma_2$. We may assume for $j = 1, 2$ that Γ_j can be represented as the graph of $y = y_j(x)$ on $[0, \delta)$.

PROPOSITION 4.1. *For $j = 1, 2$ let*

$$y_j(x) = \gamma_j x^{n_j/2} + o(x^{n_j/2}),$$

where γ_j denotes a non-zero constant. Then $n_1 = n_2$ and if we write n for $n_1 = n_2$, then n is a natural number such that $n \geq 4$ and $n \not\equiv 3 \pmod 4$; it follows that $\gamma_2 = (-1)^n \gamma_1$. Conversely, all integers n with $n \geq 4$ and $n \not\equiv 3 \pmod 4$ actually occur in this situation for some $u \in P(1, B_\rho)$.

PROOF. To use the notation as in the proof of Lemma 2.3, we assume again that $e^{i\alpha} = 1$ and that $z = T(\tau) = \tau^2 + a_3\tau^3 + a_4\tau^4 + \cdots$ is a one-to-one conformal mapping of $\{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$ into Ω such that $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$. For

$t \in (-\varepsilon, \varepsilon)$, $z = T(it) = -t^2 + \sum_{j=3}^{\infty} a_j i^j t^j$. Hence

$$x = -t^2 + \beta_3 t^3 + \alpha_4 t^4 - \beta_5 t^5 - \alpha_6 t^6 + \cdots$$

and

$$y = -\alpha_3 t^3 + \beta_4 t^4 + \alpha_5 t^5 - \beta_6 t^6 - \alpha_7 t^7 + \cdots ,$$

where $a_j = \alpha_j + i\beta_j$. Thus we see that $n = n_1 = n_2$ is a natural number not less than 3 and $\gamma_2 = (-1)^n \gamma_1$.

Assume by contradiction that $n \equiv 3 \pmod 4$; then $\alpha_j = 0$ for odd j with $3 \le j < n$, $\beta_j = 0$ for even j with $4 \le j < n$ and $\alpha_n \ne 0$. If $\alpha_n > 0$, then $y < 0$ for small $t > 0$ and $y > 0$ for $t < 0$ with small $|t|$. This contradicts the univalence of the mapping T in the half disk. Hence $\alpha_n < 0$. Since

$$S(z) - \overline{z} = v(z) = \operatorname{Re} v(z) = 2(-\alpha_n r^n \cos n\theta + \cdots)$$

on $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$ and $\theta(r)$ is close to 0 as in the proof of Lemma 2.3, we see that $v(z) = \operatorname{Re} v(z) > 0$ on J, and so $u(z) < 0$ on J. This contradicts the hypothesis $u \in P(1, B_\rho)$, and therefore we have proved that $n \not\equiv 3 \pmod 4$ and $n \ge 4$.

Now we shall construct examples. Let $n \equiv 1 \pmod 4$ and $n \ge 5$. Take a small $\alpha_n > 0$ and set $T(\tau) = \tau^2 + \alpha_n \tau^n$. Then $x = -t^2$, $y = \alpha_n t^n$ for $\tau = it$ and T is univalent in $\{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$ for some $\varepsilon > 0$, because $y > 0$ for $t > 0$ and $y < 0$ for $t < 0$. Set $S(z) = \overline{T(-\overline{\tau})}$ and $v(z) = S(z) - \overline{z}$. Then $v(z) = -2\alpha_n r^n \cos n\theta + i2r^2 \sin 2\theta$ and $v(z) = \operatorname{Re} v(z) = -2\alpha_n r^n < 0$ on $J = \{\varsigma : 0 < \varsigma = \operatorname{Re}\varsigma < \delta\}$, where $\tau = re^{i\theta}$. Hence $u_S \in P(1, B_\rho)$ for some $\rho > 0$.

If $n \equiv 0 \pmod 4$ and $n \ge 4$ we take small β_3, $\beta_n > 0$ and set $T(\tau) = \tau^2 + i\beta_3\tau^3 + i\beta_n\tau^n$. It follows that $x = -t^2 + \beta_3 t^3$ and $y = \beta_n t^n$ for $\tau = it$. Let

$$\Gamma' = \{-t^2 + i\beta_n t^n : t \in (-\varepsilon, 0)\} = \{-t^2 + i\beta_n t^n : t \in (0, \varepsilon)\}.$$

Then Γ' is a regular real analytic simple arc contained in the second quadrant. We see that z lies on the right of Γ' for small $t > 0$ and on the left of Γ' for $t < 0$ with small $|t|$. Hence T is univalent on $\{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$ for some $\varepsilon > 0$. Defining S and v as above, we obtain $v(z) = 2\beta_n r^n \sin n\theta + i2\,(r^2 \sin 2\theta + \beta_3 r^3 \cos 3\theta)$. Since $\beta_3 > 0$ and $\cos 3\theta(r) > 0$ for $\theta(r)$ close to 0, $\operatorname{Im} v(z) = 0$ only if $\theta(r) < 0$. Hence $\operatorname{Re} v(z) = 2\beta_n r^n \sin n\theta(r) < 0$ on $J = \{z \in \Omega \cap B_\delta : \operatorname{Im} v(z) = 0\}$, and so $u_S \in P(1, B_\rho)$ for some $\rho > 0$.

If $n \equiv 2 \pmod 4$ and $n \ge 6$, we take small β_5, $\beta_n > 0$ and set $T(\tau) = \tau^2 + i\beta_5\tau^5 + i\beta_n\tau^n$. We notice that

$$\Gamma' = \{-t^2 - i\beta_n t^n : t \in (0, \varepsilon)\}$$

is contained in the third quadrant in this case. By repeating the same argument as above, we can see that $u_S \in P(1, B_\rho)$ for some $\rho > 0$. Q.E.D.

5. - Holomorphic functions having real analytic boundary values

Let $\varphi(x, y)$ be a complex-valued real analytic function of x and y in B_ρ, where $z = x + iy$. We note that

$$\varphi(x, y) = \varphi\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right).$$

To discuss the local property of the function $\varphi(x, y)$, we may assume that

$$f(z, w) = \varphi\left(\frac{z+w}{2}, \frac{z-w}{2i}\right)$$

is a holomorphic function of z and w in $B_\rho^2 = B_\rho \times B_\rho$. In what follows we write $f(z, \overline{z})$ for $\varphi(x, y)$.

Let $F(z, w)$ be a holomorphic function of two variables z and w in B_ρ^2. Let Ω be an open subset of B_ρ such that $0 \in \partial\Omega$ and let $\Gamma = (\partial\Omega) \cap B_\rho$.

DEFINITION 5.1. Let S^F be a function defined on $\Omega \cup \Gamma$. We call S^F the *holomorphic function of* $\Omega \cup \Gamma$ *in* B_ρ *having the boundary values* $F(\varsigma, \overline{\varsigma})$ if:

(i) S^F is holomorphic in Ω;

(ii) S^F is continuous on $\Omega \cup \Gamma$;

(iii) $S^F(\varsigma) = F(\varsigma, \overline{\varsigma})$ on Γ.

If 0 is a non-isolated point of Γ then S^F is uniquely determined (see remarks after Definition 3.1 of [8]).

If $F(z, \overline{z})$ is a holomorphic function of z in B_ρ, then $F(z, \overline{z})$ satisfies (i) to (iii) and it is a function S^F of $\Omega \cup \Gamma$ in B_ρ having the boundary values $F(\varsigma, \overline{\varsigma})$. Namely, S^F exists for any open subset Ω of B_ρ if $F(z, \overline{z})$ is a holomorphic in B_ρ. The situation is quite different if $F(z, \overline{z})$ is not holomorphic in B_ρ.

We state the following lemma without proof.

LEMMA 5.1. Let f be a holomorphic function in B_ρ^2. Then:

(1) $f(z, \overline{z})$ is identically equal to zero in B_ρ if and only if $f(z, w)$ is identically equal to zero in B_ρ^2;

(2) $f(z, \overline{z})$ is holomorphic in B_ρ if and only if $\partial f/\partial w$ is identically equal to zero as a function of two variables in B_ρ^2.

In what follows, for the sake of simplicity, we consider the case $(\partial F/\partial w)(0, 0) \neq 0$.

PROPOSITION 5.2. *Let F be a holomorphic function in B_1^2 satisfying $(\partial F/\partial w)(0,0)\neq 0$. Let Ω be an open subset of B_1 such that $0\in\partial\Omega$ and let $\Gamma=(\partial\Omega)\cap B_1$. Then there exists a holomorphic function S^F of $(\Omega\cap B_\rho)\cup(\Gamma\cap B_\rho)$ in B_ρ having the boundary values $F(\varsigma,\overline{\varsigma})$ for some $\rho>0$ if and only if there exists a Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ for some $\delta>0$.*

The proof of this Proposition follows immediately from the following two lemmas.

LEMMA 5.3. *Let F be a holomorphic function in B_1^2. If there exists a Schwarz function S of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ, then $S^F(z)=F(z,S(z))$ is the holomorphic function of $(\Omega\cap B_\rho)\cup(\Gamma\cap B_\rho)$ in B_ρ having the boundary values $F(\varsigma,\overline{\varsigma})$ for some $\rho>0$.*

PROOF. Take $\rho>0$ so that $\rho<\delta$ and $|S(z)|<1$ on $(\Omega\cup\Gamma)\cap B_\rho$. Then the function $F(z,S(z))$ is holomorphic in $\Omega\cap B_\rho$, is continuous on $(\Omega\cap B_\rho)\cup(\Gamma\cap B_\rho)$ and satisfies $F(\varsigma,S(\varsigma))=F(\varsigma,\overline{\varsigma})$ on $\Gamma\cap B_\rho$, because $S(\varsigma)=\overline{\varsigma}$ on $\Gamma\cap B_\rho$. Q.E.D.

LEMMA 5.4. *Let S^F be the holomorphic function of $(\Omega\cap B_\rho)\cup(\Gamma\cap B_\rho)$ in B_ρ having the boundary values $F(\varsigma,\overline{\varsigma})$ and assume that $(\partial F/\partial w)(0,0)\neq 0$. Then there exists a Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ for some $\delta>0$.*

PROOF. We introduce a new variable τ and consider a function $F(z,w)-\tau$ which is holomorphic in $B_\rho^2\times B_\rho(\tau_0)$, where $\tau_0=F(0,0)$ and $B_\rho(\tau_0)$ denotes the disk with radius ρ and center τ_0. Since $(\partial/\partial w)(F(z,w)-\tau)(0,0,\tau_0)=(\partial F/\partial w)(0,0)\neq 0$ by the implicit function theorem there exists a unique holomorphic function $g(z,\tau)$ in $B_\varepsilon\times B_\varepsilon(\tau_0)$ for some $\varepsilon>0$ such that $g(0,\tau_0)=0$ and $F(z,g(z,\tau))-\tau=0$ in $B_\varepsilon\times B_\varepsilon(\tau_0)$. Take $\delta>0$ so that $\delta<\varepsilon$ and $|S^F(z)-\tau_0|<\varepsilon$ on $(\Omega\cup\Gamma)\cap B_\delta$. Then $S(z)=g(z,S^F(z))$ is holomorphic in $\Omega\cap B_\delta$ and continuous on $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$. Since $F(\varsigma,\overline{\varsigma})-S^F(\varsigma)=0$ on $\Gamma\cap B_\rho$, we obtain $\overline{\varsigma}=g(\varsigma,S^F(\varsigma))$ on $\Gamma\cap B_\delta$, because $g(z,\tau)$ is uniquely determined by z and τ. Hence $S(\varsigma)=g(\varsigma,S^F(\varsigma))=\overline{\varsigma}$ on $\Gamma\cap B_\delta$ and S is the Schwarz function of $(\Omega\cap B_\delta)\cup(\Gamma\cap B_\delta)$ in B_δ. Q.E.D.

In the previous proof we have applied the implicit function theorem. The same idea can be found in Lewy-Stampacchia [5]. From Proposition 5.2 we see that if $(\partial F/\partial w)(0,0)\neq 0$ and if there exists a holomorphic function of $(\Omega\cap B_\rho)\cup(\Gamma\cap B_\rho)$ in B_ρ having the boundary values $F(\varsigma,\overline{\varsigma})$ for some $\rho>0$, then the origin is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem.

Let $F(z,w)=F(0,0)+z^nF_1(z,w)$, where n denotes a non-negative integer and F_1 denotes a holomorphic function of z and w such that $F_1(0,w)$ is not identically equal to zero as a function of w. By applying the Fuchs theorem (see [8, Sec. 2]), we see that there exists a S^F of $\Omega\cup\Gamma$ in B_ρ if and only if there exists a S^{F_1} of $\Omega\cup\Gamma$ in B_ρ and $S^F(z)=F(0,0)+z^nS^{F_1}(z)$. Hence the same conclusion of Proposition 5.2 holds if $(\partial F_1/\partial w)(0,0)\neq 0$. If $(\partial F_1/\partial w)(0,0)=0$, then the function $g(z,\tau)$ for F_1 in the proof of Lemma 5.4 has a singularity at $(0, F_1(0,0))$ and the conclusion of Lemma 5.4 would be complicated.

6. - A generalization of Theorem 1.1

We first state the following lemma without proof.

LEMMA 6.1. *Let f be a holomorphic function in B_ρ^2. Then:*

(1) *$f(z,\overline{z})$ is real-valued in B_ρ if and only if $\check{f} = f$ in B_ρ^2, where:*

$$\check{f}(z,w) = \overline{f(\overline{w},\overline{z})}.$$

In other words, $a_{jk} = \overline{a_{kj}}$ for every $j,\ k \geq 0$ if f has the Taylor expansion $\Sigma a_{jk} z^j w^k$ in B_ρ^2;

(2) *$f(z,\overline{z})$ is real-valued in B_ρ if and only if $\Phi(z,\overline{z})$ is real-valued in B_ρ, where:*

$$\Phi(z,w) = \int_0^z \left\{ \int_0^w f(s,t)dt \right\} ds.$$

Next we define two classes of real-valued functions in B_ρ.

DEFINITION 6.1. Let f be a holomorphic function in B_ρ^2 such that $f(z,\overline{z})$ is real-valued and satisfies $f(z,\overline{z}) \neq 0$ in B_ρ. We say that a real-valued function u in B_ρ is of class $R(f, B_\rho)$ if there exists an open subset $\Omega(u)$ of B_ρ and if:

(i) 0 is contained in $\Gamma(u) = (\partial\Omega(u)) \cap B_\rho$;

(ii) u is of class C^1 in B_ρ;

(iii) $\Delta u(z) = f(z,\overline{z})$ in $\Omega(u)$ in the sense of distributions;

(iv) $\dfrac{\partial u}{\partial x}(z) = \dfrac{\partial u}{\partial y}(z) = 0$ on $\Gamma(u)$.

DEFINITION 6.2. Let f be a holomorphic function in B_ρ^2 such that $f(z,\overline{z})$ is real-valued and satisfies $f(z,\overline{z}) > 0$ in B_ρ. Let u be a non-negative function in B_ρ and set $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$. We say that u is of class $P(f, B_\rho)$ if u and $\Omega(u)$ satisfy (i) to (iii) of Definition 6.1.

In Definition 6.1, the set $\Omega(u)$ may not be uniquely determined. For $u \in P(f, B_\rho)$, we take $\Omega(u) = \{z \in B_\rho : u(z) > 0\}$; then u satisfies (iv), and so u is of class $R(f, B_\rho)$.

To prove a regularity theorem for functions of class $P(f, B_\rho)$, we start with two lemmas.

LEMMA 6.2. *Let u be a function of class $R(f, B_\rho)$ and set*

$$F(z,w) = \int_0^w f(z,t)dt. \tag{6.1}$$

Then

$$S^F(z) = F(z,\overline{z}) - 4\frac{\partial u}{\partial z}(z)$$

is the holomorphic function of $\Omega(u) \cup \Gamma(u)$ in B_ρ having the boundary values $F(\varsigma, \overline{\varsigma})$.

PROOF. By the chain rule, we obtain

$$\begin{aligned}\frac{\partial S^F}{\partial \overline{z}}(z) &= \frac{\partial F}{\partial z}(z,\overline{z})\frac{\partial z}{\partial \overline{z}} + \frac{\partial F}{\partial w}(z,\overline{z})\frac{\partial \overline{z}}{\partial \overline{z}} - 4\frac{\partial^2 u}{\partial z \partial \overline{z}}(z)\\ &= f(z,\overline{z}) - \Delta u(z)\end{aligned}$$

in $\Omega(u)$. Hence, by (iii) of Definition 6.1, S^F is holomorphic in $\Omega(u)$. It is continuous on $\Omega(u) \cup \Gamma(u)$, because u is of class C^1 in B_ρ. On $\Gamma(u)$, by (iv) of Definition 6.1, we obtain $S^F(\varsigma) = F(\varsigma, \overline{\varsigma})$. This completes the proof. Q.E.D.

LEMMA 6.3. *Let S^F be the holomorphic function of $\Omega \cup \Gamma$ in B_1 having the boundary values $F(\varsigma, \overline{\varsigma})$ and set $f = \partial F / \partial w$. If F satisfies* (6.1) *and if $f(z, \overline{z})$ is real-valued in B_1 and $f(0,0) \neq 0$, then*

$$u(z) = \begin{cases} \dfrac{1}{4}\left(\Phi(z,\overline{z}) - 2\,\mathrm{Re}\displaystyle\int_0^z S^F(\tau)d\tau\right) & \text{in } \Omega \cap B_\rho \\ 0 & \text{on } B_\rho \backslash \Omega \end{cases}$$

is of class $R(f, B_\rho)$ for some $\rho > 0$, where

$$\Phi(z,w) = \int_0^z F(s,w)ds.$$

PROOF. Since $(\partial F)/(\partial w)(0,0) = f(0,0) \neq 0$, by Proposition 5.2 and the Regularity Theorem in [8], the origin 0 is a regular, degenerate, double or cusp point in the sense of the Regularity Theorem. If 0 is not a degenerate point, then we can take a small $\rho > 0$ such that each connected component of $\Omega \cap B_\rho$ is simply connected. If 0 is a degenerate point, then we can take ρ such that S^F is holomorphic in B_ρ. In any case $\int_0^z S^F(\tau)d\tau$ is well-defined and single-valued in $\Omega \cap B_\rho$. By (2) of Lemma 6.1, $\Phi(z, \bar{z})$ is real-valued in B_1. Thus u is well-defined and real-valued in B_ρ. Set $\Omega(u) = \Omega \cap B_\rho$.

By definition, (i) of Definition 6.1 is satisfied. By the chain rule, we obtain

$$\frac{\partial \Phi(z, \bar{z})}{\partial z} = \frac{\partial \Phi}{\partial z}(z, \bar{z}) + \frac{\partial \Phi}{\partial w}(z, \bar{z}) \frac{\partial \bar{z}}{\partial z} = F(z, \bar{z}),$$

and so

$$4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z)$$

in $\Omega(u)$. Since u is real-valued, this equality implies (iv) of Definition 6.1. From this fact and condition $u(0) = 0$, we see that $u(z) = 0$ on $\Gamma(u) = \Gamma \cap B_\rho$ and hence (ii) of Definition 6.1 holds. Using the chain rule again, we obtain

$$\Delta u(z) = 4 \frac{\partial^2 u}{\partial \bar{z} \partial z}(z) = \frac{\partial F}{\partial w}(z, \bar{z}) = f(z, \bar{z})$$

in $\Omega(u)$. Hence u satisfies (iii) of Definition 6.1 and therefore it is of class $R(f, B_\rho)$. Q.E.D.

REMARK. We can construct a (complex-valued) u as above also in case $f(z, \bar{z})$ is complex-valued. In fact, $f(z, w)$ can be uniquely decomposed as $f_1(z, w) + i f_2(z, w)$, where f_j, $j = 1, 2$, are holomorphic functions in B_1^2 satisfying $\check{f}_j = f_j$.

Lemmas 6.2 and 6.3 together with Proposition 5.2 imply that there exists a function $u \in R(f, B_\rho)$ with $\Omega(u) = \Omega \cap B_\rho$ for some $\rho > 0$ if and only if there exists a Schwarz function of $(\Omega \cap B_\delta) \cup (\Gamma \cap B_\delta)$ in B_δ for some $\delta > 0$, where Ω denotes an open subset of B_1 such that $0 \in \partial\Omega$ and $\Gamma = (\partial\Omega) \cap B_1$. As a consequence, Proposition 2.2 with $R(f, B_\rho)$ replacing $R(1, B_\rho)$ holds.

Finally we shall show the following theorem which is a generalization of our Theorem 1.1.

THEOREM 6.4. *Let* $u \in P(f, B_\rho)$, $\Omega = \Omega(u)$ *and* $\Gamma = \Gamma(u)$. *Then the same assertion as in Theorem* 1.1 *holds.*

PROOF. Since $P(f, B_\rho) \subset R(f, B_\rho)$, what we have to prove is that if 0 is a degenerate point in the sense of the Regularity Theorem, then (2a) of Theorem 1.1 holds and if 0 is a cusp point, then (1.2) holds.

From Lemmas 5.3 and 6.2 it follows that

$$4 \frac{\partial u}{\partial z}(z) = F(z, \bar{z}) - S^F(z) = F(z, \bar{z}) - F(z, S(z)), \tag{6.2}$$

in $\Omega \cap B_\delta$ for some $\delta > 0$, where F is the function defined by (6.1) and S denotes the Schwarz function of $\Omega \cup \Gamma$ in B_δ. Hence u is real analytic up to Γ on Ω except double and cusp points of Γ.

If 0 is a non-isolated degenerate point in the sense of the Regularity Theorem, then $u(\varsigma) = 0$ on the arc determined by $\Gamma \cap B_\delta$, because $u(0) = 0$ and $\frac{\partial u}{\partial z}(\varsigma) = 0$ on the arc. Hence $\Gamma \cap B_\delta = \{\varsigma \in B_\delta : u(\varsigma) = 0\}$ and $\Gamma \cap B_\delta$ is the whole arc.

Next assume that 0 is a cusp point and that $z = T(\tau) = \tau^2 + a_3\tau^3 + \cdots$ is a one-to-one conformal mapping of $\{\tau \in B_\varepsilon : \operatorname{Re}\tau > 0\}$ into Ω such that $\Gamma \cap B_\delta \subset T(i(-\varepsilon, \varepsilon))$ for some δ and ε. Let $F(z, w) = \sum_{j,k \geq 0} (a_{jk}/(k+1)) z^j w^{k+1}$. Since $S(z) = \overline{T(-\overline{\tau})} = \tau^2 - \overline{a_3}\tau^3 + \cdots$, we obtain

$$\begin{aligned} F(z, \overline{z}) - F(z, S(z)) &= (a_{00}\overline{z} + O(|\tau|^4)) - (a_{00}S(z) + O(|\tau|^4)) \\ &= a_{00}(\overline{z} - S(z)) + O(|\tau|^4). \end{aligned}$$

From Definition 6.2, it follows that

$$a_{00} = \frac{\partial F}{\partial w}(0,0) = f(0,0) > 0.$$

By recalling (6.2) and applying the same argument as in the proof of Lemma 2.3, we see that $\operatorname{Re} a_3 = 0$ if $u(z) > 0$ in Ω, and therefore (1.2) holds. Q.E.D.

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