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and a coupled problem between the Lamé system and the
plate equation. I : regularity of the solutions**

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**About the Lamé System in a Polygonal or a Polyhedral Domain
and a Coupled Problem between the Lamé System and
the Plate Equation
I: Regularity of the Solutions**

SERGE NICAISE

1. - Introduction

This paper is the first of a series of two, devoted to the regularity of the solutions of some problems related to the linear elasticity theory and to the exact controllability of the associated dynamical problems. Part I concerns the regularity, while Part II will study the exact controllability.

On a first step, we study the Lamé system in a polygonal domain of the plane or a polyhedral domain of the space. The interior datum is assumed to be in L^2 , and the boundary conditions are mixed and non-homogeneous. This means that on a part of the boundary we impose Dirichlet boundary conditions (i.e. the displacement vector field is fixed) and on the remainder of the boundary we impose Neumann boundary conditions, in the sense that the normal traction is fixed. We give the singular behaviour of the weak solution of this system near the corners and, in dimension 3, also along the edges. In dimension 2, this is proven by P. Grisvard in [9] (see also [21]). In dimension 3, partial results were given in [9] and [20]; here adapting Dauge's technics of [3], we give vertex singularities and edge singularities up to the vertices but only for small regularity for the regular part (namely $H^{3/2+\varepsilon}$, for some $\varepsilon > 0$).

With these results, it is possible to find geometrical conditions on the domain which ensure the regularity $H^{3/2+\varepsilon}$, for some $\varepsilon > 0$, for the weak solution. As classically (see [22], [8], [16], [11]), to get this regularity result, it suffices to establish the existence of a strip free of pole. When the boundary conditions are purely of Dirichlet type, the study of such strips is wellknown (see [22], [4], [21] in dimension 2 and [16], [11] in dimension 3). When the boundary conditions are mixed, it seems to be new. In dimension 2, our conditions are necessary and sufficient, while in dimension 3, in view of Grisvard's results about the Laplace operator in [8], we think that it is perhaps possible to improve them.

As an application of these results (and this is actually one of our motivations to prove them), we study the regularity of the solution of a coupled problem between the linear elasticity system in the unit cube of \mathbb{R}^3 with a plane crack and the plate equation on a plane domain. This problem differs from the problem obtained by [2] only by the boundary conditions. But as we shall explain at the end of this paper, this is necessary to get positive results. Unfortunately, our problem is perhaps not so realistic from the mechanical point of view. Nevertheless, we may say that we answer to the question of regularity raised in Paragraph 6 of [2]. Our coupled problem is similar to the model problem studied in [19] but the obtained regularity results and the methods of proof are different since the Laplace operator is more convenient than the Lamé system.

In the second part of this paper, we shall consider the exact controllability of the associated dynamical problems. As in [8] and [19], these regularity results will be useful in order to adapt the Hilbert Uniqueness Method of J.-L. Lions [13].

Let us now introduce some notations. Let Ω be a bounded open connected subset of \mathbb{R}^n , $n \in \{2, 3\}$. We suppose that the boundary $\partial\Omega$ of Ω is the union of a finite number of faces $\bar{\Gamma}_k$, $k \in \mathcal{F}$, where, in dimension 2, each Γ_k is actually a linear segment, while, in dimension 3, Γ_k is a plane face (it is more convenient to assume that Γ_k is open!). If Ω has slits, we assume that each slit is split up into two faces.

In order to consider mixed boundary conditions, we fix a partition of \mathcal{F} into $\mathcal{D} \cup \mathcal{N}$, where \mathcal{D} will correspond to Dirichlet boundary conditions, while \mathcal{N} to Neumann boundary conditions.

Given a function w or a vector field μ defined on $\partial\Omega$, it will be convenient to denote by $w^{(k)}$, respectively $\mu^{(k)}$, its restriction to Γ_k , for all $k \in \mathcal{F}$. Moreover, for a vector \mathbf{u} of \mathbb{R}^n we denote by u_i its i -th component, for all $i \in \{1, \dots, n\}$, i.e. $\mathbf{u} = (u_i)_{i=1}^n$.

We associate with the displacement vector field \mathbf{u} the linearized strain tensor $\varepsilon(\mathbf{u})$ defined by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (D_j u_i + D_i u_j), \quad \forall i, j = 1, \dots, n,$$

and the linearized stress tensor $\sigma(\mathbf{u})$ given by Hooke's law, using the Lamé coefficients λ and μ (λ and μ are always assumed to be positive):

$$\sigma_{ij}(\mathbf{u}) = 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda \operatorname{tr} \varepsilon(\mathbf{u}) \delta_{ij}, \quad \forall i, j = 1, \dots, n.$$

We introduce the Lamé operator

$$L\mathbf{u} = \left(\sum_{j=1}^n D_j \sigma_{ij}(\mathbf{u}) \right)_{i=1}^n.$$

For all $k \in \mathcal{F}$, we also denote by γ_k the trace operator on the face Γ_k , $\nu^{(k)}$ the outward normal unit vector on Γ_k and

$$T^{(k)}\mathbf{u} = \left(\sum_{j=1}^n \gamma_k \sigma_{ij}(\mathbf{u}) \nu_j^{(k)} \right)_{i=1}^n .$$

Given a vector field $\mathbf{f} \in (L^2(\Omega))^n$ (which represents the force density applied to the body Ω) and $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma_k))^n$, for all $k \in \mathcal{N}$, we consider the weak solution $\mathbf{u} \in (H^1(\Omega))^n$ of the Lamé system

$$(1.1) \quad L\mathbf{u} = -\mathbf{f} \text{ in } \Omega,$$

with mixed boundary conditions

$$(1.2) \quad \gamma_k \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_k, \quad \forall k \in \mathcal{D},$$

$$(1.3) \quad T^{(k)}\mathbf{u} = \mathbf{g}^{(k)} \quad \text{on } \Gamma_k, \quad \forall k \in \mathcal{N}.$$

This problem admits the following variational formulation: we introduce the Hilbert space

$$V = \{u \in (H^1(\Omega))^n \text{ fulfilling (1.2)}\},$$

and the continuous sesquilinear form

$$a_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^n \sigma_{ij}(\mathbf{u}) \varepsilon_{ij}(\overline{\mathbf{v}}) dx, \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Therefore, we shall say that \mathbf{u} is a weak solution of problem (1.1)–(1.3), if $\mathbf{u} \in V$ is a solution of

$$(1.4) \quad a_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{f}, \mathbf{v}) dx + \sum_{k \in \mathcal{N}} \int_{\Gamma_k} (\mathbf{g}^{(k)}, \gamma_k \mathbf{v}) d\sigma, \quad \forall \mathbf{v} \in V,$$

where, from now on, (\cdot, \cdot) denotes the inner product in \mathbb{C}^n .

In all this paper, we shall use the Sobolev spaces $W^{s,p}(\Omega)$ defined, for instance, in Paragraph 1.3.2 of [6], when $s \in \mathbb{R}$ and $p > 1$. When $p = 2$, they are usually denoted by $H^s(\Omega)$. Moreover, we also use the weighted Sobolev spaces $H_\gamma^s(\Gamma)$ defined in (AA.2) of [3], when $s \geq 0$, $\gamma \in \mathbb{R}$ and Γ is a cone of \mathbb{R}^n .

2. - Regularity of the solution of the Lamé system in dimension 2

The behaviour of a solution \mathbf{u} of (1.4) near the vertices of Ω is wellknown; using Theorem I of [9], we obtain immediately the following (see also [21]):

THEOREM 2.1. *Let $\mathbf{u} \in V$ be a solution of (1.4) with data $\mathbf{f} \in (L^2(\Omega))^2$, $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma_k))^2, \forall k \in \mathcal{N}$. Let us fix $j, k \in \mathcal{N}$ such that $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$ and let us denote by S their common vertex and by ω the interior angle between Γ_j and Γ_k . Then there exist coefficients $c_{\alpha,\nu}$ such that*

$$(2.1) \quad \mathbf{u} - \sum_{\alpha} \sum_{\nu=1}^{N(\alpha)} c_{\alpha,\nu} \sigma^{\alpha,\nu} \in (W^{2,p}(W))^2, \quad .$$

where W is a neighbourhood of S , the sum extends to all roots $\alpha \in \mathbb{C}$ of

$$(2.2) \quad \sin^2 \alpha\omega = \alpha^2 \sin^2 \omega,$$

in the strip $\Re(\alpha) \in]0, 2 - 2/p[$. $N(\alpha)$ is the multiplicity of α in (2.2) ($N(\alpha) = 1$ or 2 , see [22]). Finally, $\sigma^{\alpha,\nu}$ are the so-called singular functions defined by (1.4) of [9]. This result holds for all $p < 2$ such that the equation (2.2) has no root on the line $\Re(\alpha) = 2 - 2/p$.

If $j, k \in \mathcal{D}$, this result remains true if we replace (2.2) by

$$(2.3) \quad \sin^2 \alpha\omega = \frac{(\lambda + \mu)^2}{(\lambda + 3\mu)^2} \alpha^2 \sin^2 \omega,$$

in that case, the $\sigma^{\alpha,\nu}$'s are defined in Paragraph 6.1 of [9].

If $j \in \mathcal{N}, k \in \mathcal{D}$ or $j \in \mathcal{D}, k \in \mathcal{N}$, then again this result still holds when (2.2) is replaced by

$$(2.4) \quad \sin^2 \alpha\omega = \frac{(\lambda + 2\mu)^2 - (\lambda + \mu)^2 \alpha^2 \sin^2 \omega}{(\lambda + \mu)(\lambda + 3\mu)},$$

the $\sigma^{\alpha,\nu}$'s being defined in Paragraph 6.2 of [9].

In view of this theorem, if we want to get a maximal regularity for \mathbf{u} , it is necessary to show that a strip $\Re(\alpha) \in]0, 2 - 2/p[$ has no root of (2.2), (2.3) or (2.4). This is the purpose of the

THEOREM 2.2. *If $\omega \in]0, 2\pi[$, then the equations (2.2) and (2.3) have no root in the strip $\Re(\alpha) \in \left]0, \frac{1}{2}\right]$. On the other hand, the equation (2.4) has no root in the same strip if $\omega \in]0, \pi[$.*

Before proving this result, let us give its consequence:

THEOREM 2.3. *If Ω satisfies the assumption*

(H₂) $\forall j, k \in \mathcal{F}$ such that $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$, the interior angle ω between Γ_j and Γ_k fulfils $\omega < 2\pi$ and moreover, if $j \in \mathcal{D}$ and $k \in \mathcal{N}$, $\omega < \pi$.

Then a solution $\mathbf{u} \in V$ of (1.4) with data $\mathbf{f} \in (L^2(\Omega))^2$ and $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma_k))^2, \forall k \in \mathcal{N}$ satisfies

$$(2.5) \quad \mathbf{u} \in (H^{3/2+\varepsilon}(\Omega))^2, \quad \text{for some } \varepsilon > 0.$$

PROOF. By Theorem 2.2 and the hypothesis (H₂), the strip $\mathfrak{R}(\alpha) \in \left]0, \frac{1}{2}\right]$ is free of root at each vertex of Ω . Moreover, it is wellknown that in a fixed strip $\mathfrak{R}(\alpha) \in [a, b]$, with $a, b \in \mathbb{R}$, the equations (2.2) to (2.4) have only a finite number of isolated roots; therefore there exists $p \in]4/3, 2[$ (sufficiently closed to $4/3$ if necessary) such that the strip $\mathfrak{R}(\alpha) \in]0, 2 - 2/p]$ is free of root at each vertex of Ω . Owing to Theorem 2.1, we deduce that $\mathbf{u} \in (W^{2,p}(\Omega))^2$, for such a p . Using the Sobolev embedding theorem (see Theorem 1.4.4.1 of [6]), we obtain (2.5) since the assumption (H₂) implies that Ω has a Lipschitz boundary. □

PROOF OF THEOREM 2.2. The equation (2.2) was studied by a lot of authors (see [22], [14], [7], [4] for instance). Actually, our result for (2.2) is a direct consequence of §5 of [4]. Let us now study the roots of

$$(2.6) \quad \sin^2 \alpha \omega = K^2 \alpha^2 \sin^2 \omega$$

with $K \in]0, 1]$. This equation (2.6) recovers (2.3) since $\frac{\lambda + \mu}{\lambda + 3\mu} \in]0, 1[$; it also recovers (2.2) by taking $K = 1$. It is obvious that α is a solution of (2.6) if and only if α fulfils (2.7) or (2.8) below:

$$(2.7) \quad \sin \alpha \omega = K \alpha \sin \omega,$$

$$(2.8) \quad \sin \alpha \omega = -K \alpha \sin \omega.$$

Let us show that if $\omega \in]0, 2\pi[$, then (2.7) has no root in the strip $\mathfrak{R}(\alpha) \in]0, 1/2]$. An analogous argument shows the same result for (2.8).

Writing $\alpha = \xi + i\eta$, with $\xi, \eta \in \mathbb{R}$, and taking the real part and imaginary part of (2.7), we arrive to the system

$$(2.9) \quad \sin(\xi\omega)\text{ch}(\eta\omega) = K \sin \omega \xi,$$

$$(2.10) \quad \cos(\xi\omega)\text{sh}(\eta\omega) = K \sin \omega \eta.$$

For a fixed $\eta \in \mathbb{R}$, consider the two functions:

$$f_1 : \mathbb{R} \rightarrow \mathbb{R} : \xi \rightarrow \sin(\xi\omega)\text{ch}(\eta\omega)$$

$$f_2 : \mathbb{R} \rightarrow \mathbb{R} : \xi \rightarrow K \sin \omega \xi$$

Then we easily check that

$$f_1(0) = f_2(0) = 0, \quad f_1(1/2) > f_2(1/2).$$

Since f_1 is concave in the interval $[0, \pi/\omega]$ (notice that $1/2 < \pi/\omega$), we get for all $\xi \in \left]0, \frac{1}{2}\right]$:

$$f_1(\xi) \geq 2\xi f_1(1/2) > 2\xi f_2(1/2) = f_2(\xi).$$

Therefore (2.9) has no solution ξ in the interval $\xi \in \left]0, \frac{1}{2}\right]$, so does (2.7).

Let us now pass to the equation (2.4) under the hypothesis $\omega \in [0, \pi[$. It can be written

$$(2.11) \quad \sin^2 \alpha \omega = K_1 - K \alpha^2 \sin^2 \omega,$$

where we set $K_1 = \frac{(\lambda + 2\mu)^2}{(\lambda + \mu)(\lambda + 3\mu)}$ and $K = \frac{\lambda + \mu}{\lambda + 3\mu}$. Writing $\alpha = \xi + i\eta$, with $\xi, \eta \in \mathbb{R}$, (2.11) is equivalent to:

$$(2.12) \quad \sin^2(\xi\omega) \operatorname{ch}^2(\eta\omega) - \cos^2(\xi\omega) \operatorname{sh}^2(\eta\omega) = K_1 - K \sin^2 \omega (\xi^2 - \eta^2),$$

$$(2.13) \quad \sin(2\xi\omega) \operatorname{sh}(2\eta\omega) = -4K \xi \eta \sin^2 \omega.$$

First case. If $\eta \neq 0$, then a solution $\xi > 0$ of (2.13) fulfils

$$(2.14) \quad \xi \geq \frac{\pi}{2\omega} > 1/2.$$

Indeed, (2.13) is then equivalent to

$$(2.15) \quad \frac{\sin(2\xi\omega)}{\xi} = -\frac{4K \sin^2 \omega \eta}{\operatorname{sh}(2\eta\omega)}.$$

We obtain (2.14) since on the interval $\xi \in \left]0, \frac{\pi}{2\omega}\right[$, the left-hand side of (2.15) is positive, while the right-hand side is always negative.

Second case. If $\eta = 0$, then (2.12) has no solution $\xi \in \left[0, \frac{1}{2}\right]$. In that case, (2.12) becomes

$$(2.16) \quad \sin^2(\xi\omega) = K_1 - K \sin^2 \omega \xi^2.$$

By a direct computation, we check that the left-hand side is (strictly) lower than the right-hand side at $\xi = \frac{1}{2}$. We obtain the result since in the interval $\left[0, \frac{\pi}{2\omega}\right]$,

the left-hand side of (2.16) is increasing, while its right-hand side is decreasing (notice that $\omega < \pi$ implies that $\frac{1}{2} < \frac{\pi}{2\omega}$).

Joining together the two cases, we obtain the result for (2.4). This completes the proof of Theorem 2.2. □

REMARK 2.4. The geometrical assumptions made in Theorem 2.3 to get the regularity result (2.5) are exactly the same as for the Laplace operator with mixed boundary conditions made by P. Grisvard in [8]. Moreover, they are in accordance with some figures given in [21] for some particular examples. They are our motivations to establish Theorems 2.2 and 2.3. Moreover, they are necessary in the sense that if (H_2) fails then there exist singularities which do not belong to $H^{3/2+\varepsilon}(\Omega)^2$.

3. - Vertex and edge singularities of the Lamé system in dimension 3

In dimension 3, the behaviour of a solution \mathbf{u} of problem (1.4) along the edges was given by P. Grisvard in [9]. Moreover, it is possible to prove a general regularity result at the vertices and along the edges as Theorem 17.13 of [3] when the boundary conditions (1.2)–(1.3) are homogeneous (adapting Paragraphs 17, 22, 23 and 24 of [3] to this problem). A sketch of the proof was given in Paragraph 1 of [20]. Since we are interested in non-homogeneous boundary conditions and since we allow cracked domains, it is impossible to use a trace theorem to go back to homogeneous boundary conditions. Therefore, we shall show that Theorem 17.13 of [3] still holds for our system (1.1)–(1.3) but only with a regular part in $H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ (instead of H^2). Fortunately, it is sufficient for the applications to the exact controllability (see [8]).

Let us start with the singularities at the vertices. To do that, we fix a vertex S of Ω . In a sufficiently small neighbourhood of S , Ω coincides with a polyhedral cone Γ_S of \mathbb{R}^3 . We denote by Ω_S the intersection between Γ_S and the unit sphere centered at S . We shall also use spherical coordinates (r, ω) with origin at S ; in that way, we have

$$\Omega_S = \{\omega \in S_2 : (r, \omega) \in \Gamma_S\}.$$

Let us denote by \mathcal{F}_S , the set of faces of Γ_S , i.e. $\mathcal{F}_S = \{k \in \mathcal{F} : S \in \bar{\Gamma}_k\}$. For each $k \in \mathcal{F}_S$, we shall denote by Γ'_k the face of Γ_S containing the face Γ_k of Ω ; Γ''_k will be the corresponding arc of the boundary $\partial\Omega_S$ of Ω_S . Finally, we misuse the notation γ_k for the trace operator on Γ'_k or Γ''_k , for all $k \in \mathcal{F}_S$.

Obviously, the partition $\mathcal{D} \cup \mathcal{N}$ of \mathcal{F} induces a partition $\mathcal{D}_S \cup \mathcal{N}_S$ of \mathcal{F}_S . We are now able to set

$$\begin{aligned} V(\Gamma_S) &= \{\mathbf{u} \in (H^1(\Gamma_S))^3 \text{ fulfilling } \gamma_k \mathbf{u} = \mathbf{0} \text{ on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S\}, \\ V(\Omega_S) &= \{\mathbf{u} \in (H^1(\Omega_S))^3 \text{ fulfilling } \gamma_k \mathbf{u} = \mathbf{0} \text{ on } \Gamma''_k, \quad \forall k \in \mathcal{D}_S\}. \end{aligned}$$

Using a cut-off function, to study the behaviour of a solution \mathbf{u} of (1.4) near the vertex S , we may suppose that \mathbf{u} has a compact support, let us say

$$(3.1) \quad \text{supp } \mathbf{u} \subset B(S, 1),$$

and that it fulfils

$$(3.2) \quad \int_{\Gamma_S} \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij}(\bar{\mathbf{v}}) dx = \int_{\Gamma_S} (\mathbf{f}, \mathbf{v}) dx + \sum_{k \in \mathcal{N}_S} \int_{\Gamma'_k} (\mathbf{g}^{(k)}, \gamma_k \mathbf{v}) d\sigma,$$

for all $\mathbf{v} \in V(\Gamma_S)$; where $\mathbf{f} \in (L^2(\Gamma_S))^3$, $\mathbf{g}^{(k)} \in (H^{(1/2)}(\Gamma'_k))^3$, for all $k \in \mathcal{N}_S$ with a compact support.

As usual [10], [15], the asymptotic behaviour of \mathbf{u} near S depends on a family of operators $\mathcal{A}_S(\alpha)$, with complex parameter α , that we now introduce (in a variational way): we write the operator D_j in spherical coordinates and we set

$$D_j(\omega, r \partial_r, D_\omega) = r D_j, \quad \forall j \in \{1, 2, 3\}.$$

For $\alpha \in \mathbb{C}$ and a vector field \mathbf{v} , we set

$$\mathcal{E}_{ij}(\omega, \alpha, D_\omega) \mathbf{v} = \frac{1}{2} (D_i(\omega, \alpha, D_\omega) v_j + D_j(\omega, \alpha, D_\omega) v_i),$$

$$\mathcal{S}_{ij}(\omega, \alpha, D_\omega) \mathbf{v} = 2\mu \mathcal{E}_{ij}(\omega, \alpha, D_\omega) \mathbf{v} + \lambda \text{tr } \mathcal{E}(\omega, \alpha, D_\omega) \mathbf{v} \delta_{ij}, \quad \forall i, j \in \{1, 2, 3\}.$$

For all $\alpha \in \mathbb{C}$, we introduce the continuous sesquilinear form $a_S(\alpha)$ on $V(\Omega_S)$ defined by

$$a_S(\alpha) \{ \mathbf{u}, \mathbf{v} \} = \int_{\Omega_S} \sum_{i,j=1}^3 \mathcal{S}_{ij}(\omega, \alpha, D_\omega) \mathbf{u} \mathcal{E}_{ij}(\omega, -(\alpha + 1), D_\omega) \bar{\mathbf{v}} d\omega, \quad \forall \mathbf{u}, \mathbf{v} \in V(\Omega_S).$$

Finally, the operator $\mathcal{A}_S(\alpha) : V(\Omega_S) \rightarrow V(\Omega_S)'$ is defined by

$$\mathcal{A}_S(\alpha) \mathbf{u}(\mathbf{v}) = a_S(\alpha) \{ \mathbf{u}, \mathbf{v} \}.$$

We shall now give a result analogous to Lemma 17.4 of [3]; its proof is similar but using the variational formulation as in Proposition 24.1 of [3]. As M. Dauge, we shall use the Mellin transform. Let us recall that for $u \in \mathcal{D}(\mathbb{R}^+)$, the Mellin transform of u , $\mathcal{M}[u]$, is defined by

$$\mathcal{M}[u](\alpha) = \int_0^{+\infty} r^{-\alpha} u(r) \frac{dr}{r}, \quad \forall \alpha \in \mathbb{C}.$$

In the following, we shall use without comment the properties of the Mellin transform given in the Appendix AA of [3].

Since $\mathbf{u} \in (H^1(\Gamma_S))^3$ and has a compact support, we deduce that

$$\mathbf{U}(\alpha) = (\mathcal{M}[u_i](\alpha))_{i=1}^3$$

is analytic with values in $(H^1(\Omega_S))^3$ in the half-space $\Re(\alpha) < -1/2$. In the same way, if we set

$$\mathbf{F}(\alpha) = (\mathcal{M}[r^2 f_i](\alpha))_{i=1}^3,$$

$$\mathbf{G}^{(k)}(\alpha) = (\mathcal{M}[r g_i^{(k)}](\alpha))_{i=1}^3, \quad \forall k \in \mathcal{F}_S,$$

we know that \mathbf{F} (respectively $\mathbf{G}^{(k)}$) is analytic with values in $(L^2(\Omega_S))^3$ (respectively $(H^{1/2}(\Gamma_k''))^3$) for $\Re(\alpha) < 1/2$.

Using the change of variable $r = e^t$ and the Parseval identity, we see that (3.2) implies that (roughly speaking, it corresponds to apply the Mellin transform to (3.2))

$$(3.3) \quad \mathcal{A}_S(\alpha)\mathbf{U}(\alpha) = I(\mathbf{F}(\alpha), (\mathbf{G}^{(k)}(\alpha))_{k \in \mathcal{N}_S}),$$

for $\Re(\alpha) < -1/2$, when

$$I : (L^2(\Omega_S))^3 \times \prod_{k \in \mathcal{N}_S} (H^{1/2}(\Gamma_k''))^3 \rightarrow V(\Omega_S)'$$

is defined by

$$\begin{aligned} & I(\mathbf{F}, (\mathbf{G}^{(k)})_{k \in \mathcal{N}_S})(\mathbf{v}) \\ &= \int_{\Omega_S} (\mathbf{F}, \mathbf{v}) d\omega + \sum_{k \in \mathcal{N}_S} \int_{\Gamma_k''} (\mathbf{G}^{(k)}, \gamma_k \mathbf{v}) d\sigma, \quad \forall \mathbf{v} \in V(\Omega_S). \end{aligned}$$

But arguing as in Proposition 8.4 of [3] and using the fact that Korn's inequality holds on Γ_S , we can prove the

LEMMA 3.1. *For $\beta, \gamma \in \mathbb{R}$ fixed, there exist two constants $C_{\beta, \gamma}$ and $\Lambda_{\beta, \gamma}$ such that for all α fulfilling $\Re(\alpha) \in [\beta, \gamma]$ and $|\alpha| \geq \Lambda_{\beta, \gamma}$, we have*

$$(3.4) \quad \|\mathbf{u}\|_{(H^1(\Omega_S, |\alpha|))^3} \leq C_{\beta, \gamma} \Re(\alpha) \{\mathbf{u}, \mathbf{u}\},$$

for all $\mathbf{u} \in (H^1(\Omega_S))^3$ (see (AA.17) of [3] for the definition of the norm of $H^1(\Omega_S, |\alpha|)$).

Moreover, it is easy to see that for all $\alpha, \alpha' \in \mathbb{C}$, $\mathcal{A}_S(\alpha) - \mathcal{A}_S(\alpha')$ is a compact operator. Owing to the analytic Fredholm theorem, $\mathcal{A}_S(\alpha)^{-1}$ is meromorphic on \mathbb{C} . So there exists $\varepsilon > 0$ such that $\mathcal{A}_S(\alpha)$ is one-to-one on the line $\Re(\alpha) = \varepsilon$.

We now conclude as in Lemma 17.4 of [3]; we set

$$(3.5) \quad \mathbf{v}(\alpha) = \mathcal{A}_S(\alpha)^{-1} I(\mathbf{F}(\alpha), (\mathbf{G}^{(k)}(\alpha))_{k \in \mathcal{N}_S}),$$

on the line $\Re(\alpha) = \varepsilon$. Then the estimate (3.4) and the definition of $\mathcal{A}_S(\alpha)$ imply that there exists a constant $C > 0$ such that

$$\|\mathbf{v}(\alpha)\|_{(H^1(\Omega_S, |\alpha|))^3} \leq C \left\{ \|\mathbf{F}(\alpha)\|_{(L^2(\Omega_S))^3} + \sum_{k \in \mathcal{N}_S} \|\mathbf{G}^{(k)}(\alpha)\|_{(H^{1/2}(\Gamma_k^r, |\alpha|))^3} \right\}.$$

So by the Mellin inversion formula on the line $\Re(\alpha) = \varepsilon$, we get that the function \mathbf{u}_0 belongs to $(H^1_{-\frac{1}{2}-\varepsilon}(\Gamma_S))^3$, since the hypotheses made on \mathbf{f} and $(\mathbf{g}^{(k)})_{k \in \mathcal{N}_S}$ insure that

$$\int_{\Re(\alpha)=\varepsilon} \left\{ \|\mathbf{F}(\alpha)\|_{(L^2(\Omega_S))^3}^2 + \sum_{k \in \mathcal{N}_S} \|\mathbf{G}^{(k)}(\alpha)\|_{(H^{1/2}(\Gamma_k^r, |\alpha|))^3}^2 \right\} d\alpha < +\infty.$$

Since on the line $\Re(\alpha) = \varepsilon$, $\mathcal{M}[\mathbf{u}_0](\alpha) = \mathbf{v}(\alpha)$ and \mathbf{F} and $(\mathbf{G}^{(k)})_{k \in \mathcal{N}_S}$ are holomorphic in a neighbourhood of this line, the Parseval identity allows us to show that \mathbf{u}_0 fulfils

$$(3.6) \quad \int_{\Gamma_S} \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}_0) \varepsilon_{ij}(\overline{\mathbf{v}}) dx = \int_{\Gamma_S} (\mathbf{f}, \mathbf{v}) dx + \sum_{k \in \mathcal{N}_S} \int_{\Gamma_k^r} (\mathbf{g}^{(k)}, \gamma_k \mathbf{v}) d\sigma,$$

for all $\mathbf{v} \in V_0(\Gamma_S) := \{\mathbf{w} \in V(\Gamma_S) : \exists R > r > 0 \text{ such that } \text{supp } \mathbf{w} \subset B(S, R) \setminus \overline{B(S, r)}\}$.

By a standard argument (see [10]) based upon the Cauchy formula for $r^\alpha \mathbf{v}(\alpha)$ on rectangular paths tending to the infinite path $\Re(\alpha) = -1/2$, $\Re(\alpha) = \varepsilon$, we get that (it is easy to show that $\mathcal{A}_S(\alpha)$ is one-to-one on the line $\Re(\alpha) = -1/2$)

$$(3.7) \quad \mathbf{u} - \mathbf{u}_0 = \sum_{\alpha} \sigma^\alpha,$$

where the sum extends to all α in the strip $\Re(\alpha) \in]-\frac{1}{2}, \varepsilon[$ such that $\mathcal{A}_S(\alpha)^{-1}$ does not exist and

$$\sigma^\alpha = \text{Res}_{\beta=\alpha} \{ r^\beta \mathcal{A}_S(\beta)^{-1} I(\mathbf{F}(\beta), (\mathbf{G}^{(k)}(\beta))_{k \in \mathcal{N}_S}) \}.$$

Therefore, it is clear that there exist $Q(\alpha) \in \mathbf{N}$ and functions $\varphi^{\alpha,q}$, $q \in \{1, \dots, Q(\alpha)\}$ defined on Ω_S such that

$$(3.8) \quad \sigma^\alpha(r, \omega) = r^\alpha \sum_{q=0}^{Q(\alpha)} (\log r)^{Q(\alpha)-q} \varphi^{\alpha,q}(\omega).$$

Let us now show that σ^α and $r^\alpha \varphi^{\alpha,0}$ satisfy (3.9) below:

$$(3.9) \quad \begin{cases} L\mathbf{v} = \mathbf{0} & \text{in } \Gamma_S, \\ \gamma_k \mathbf{v} = \mathbf{0} & \text{on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S, \\ T^{(k)} \mathbf{v} = \mathbf{0} & \text{on } \Gamma'_k, \quad \forall k \in \mathcal{N}_S. \end{cases}$$

Firstly, if we compare (3.2) with (3.6), $\mathbf{u} - \mathbf{u}_0$ fulfils

$$\int_{\Gamma_S} \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u} - \mathbf{u}_0) \varepsilon_{ij}(\bar{\mathbf{v}}) dx = 0, \quad \forall \mathbf{v} \in V_0(\Gamma_S).$$

Hence, $\mathbf{u} - \mathbf{u}_0$ is regular (i.e. in H^2) far from the vertex S and the edges of Γ_S . Taking test-function $\mathbf{v} \in V_0(\Gamma_S) \cap (\mathcal{D}(\bar{\Gamma}_S))^3$ such that $\mathbf{v} = \mathbf{0}$ in a neighbourhood of the edges of Γ_S , by Green's formula, we deduce that $\mathbf{u} - \mathbf{u}_0$ fulfils (3.9). Writing L and $T^{(k)}$ in spherical coordinates and using the fact that functions of the form $r^{\alpha_i}(\log r)^{q_i}$, with different α_i and $q_i \in \mathbb{N}$, $i = 1, \dots, N$, are linearly independent, we conclude that all σ^α 's and $r^\alpha \varphi^{\alpha,0}$'s satisfy (3.9).

In summary, we have proven the

THEOREM 3.2. *Let $\mathbf{f} \in (L^2(\Gamma_S))^3$, $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma'_k))^3$, $\forall k \in \mathcal{N}_S$ and let $\mathbf{u} \in V(\Gamma_S)$ be a solution of (3.2) with a compact support. Then there exists $\varepsilon > 0$ and a function $\mathbf{u}_0 \in (H^{1-\frac{1}{2}-\varepsilon}(\Gamma_S))^3$, which is a solution of (3.6), such that*

$$(3.10) \quad \mathbf{u} = \mathbf{u}_0 + \sum_{\alpha} \sigma^\alpha,$$

where the sum extends to all α in the strip $\Re(\alpha) \in \left] -\frac{1}{2}, \varepsilon \right[$ such that $\mathcal{A}_S(\alpha)^{-1}$ does not exist; the function σ^α admits the expansion (3.8) and finally, σ^α and $r^\alpha \varphi^{\alpha,0}$ satisfy (3.9).

Now, in order to get edge singularities up to the vertex S , let us introduce some notations (see §17.B of [3]):

- $A(\Omega_S)$ denotes the set of vertices of Ω_S (it is the set of edges of Γ_S).
- If $x \in A(\Omega_S)$, there exists a local chart χ_x sending a neighbourhood of x in Ω_S onto a neighbourhood of 0 in a cone C_x of \mathbb{R}^2 with opening ω_x . Since x corresponds to the common edge between Γ_k and Γ_j , for some $j, k \in \mathcal{F}_S$, ω_x is the interior dihedral angle between Γ_j and Γ_k . We shall denote by z_x the cartesian coordinates in C_x . $\sigma_x^{\alpha,\nu}$ will be the singularities introduced in Paragraph 2 associated with the Lamé system in the cone C_x with boundary conditions induced by the boundary conditions imposed on Γ_j and Γ_k , when α is a solution of (2.2), (2.3) or (2.4) with $\omega = \omega_x$. In the same way, we introduce the singularities $\sigma_x^{\alpha'}$ of the Laplace operator in

the cone C_x with boundary conditions induced by the boundary conditions imposed on Γ_j and Γ_k . More precisely, using polar coordinates (r_x, θ_x) in the cone C_x such that $\theta_x = 0$ on Γ_k and $\theta_x = \omega_x$ on Γ_j , we set

i) If $j, k \in \mathcal{D}$, then $\alpha' = m\pi/\omega_x$, for all $m \in \mathbf{N}^*$ and

$$(3.11) \quad \sigma_x^{\alpha'}(r_x, \theta_x) = r_x^{\alpha'} \sin(\alpha' \theta_x).$$

ii) If $j \in \mathcal{D}, k \in \mathcal{N}$, then $\alpha' = \left(m - \frac{1}{2}\right) \pi/\omega_x$ for all $m \in \mathbf{N}^*$ and

$$(3.12) \quad \sigma_x^{\alpha'}(r_x, \theta_x) = r_x^{\alpha'} \cos(\alpha' \theta_x).$$

iii) If $j \in \mathcal{N}, k \in \mathcal{D}$, then $\alpha' = \left(m - \frac{1}{2}\right) \pi/\omega_x$, for all $m \in \mathbf{N}^*$ and $\sigma_x^{\alpha'}$ is defined by (3.11).

iv) If $j, k \in \mathcal{N}$, then $\alpha' = m\pi/\omega_x$, for all $m \in \mathbf{N}^*$ and $\sigma_x^{\alpha'}$ is defined by (3.12).

For $\varepsilon > 0$, we set

$$\Lambda_x(\varepsilon) = \left\{ (\alpha, \nu) : \alpha \text{ is a solution of (2.2), (2.3) or (2.4) such that } \Re(\alpha) \in \left] 0, \frac{1}{2} + \varepsilon \right[\text{ and } \nu \in \{1, \dots, N(\alpha)\} \right\},$$

$$\Lambda'_x(\varepsilon) = \left\{ \alpha' \text{ equal to } m\pi/\omega_x \text{ or } \left(m - \frac{1}{2}\right) \pi/\omega_x \text{ such that } \alpha' \in \left] 0, \frac{1}{2} + \varepsilon \right[\right\}.$$

Finally, we introduce the smoothing operator \mathcal{R}_x defined by

$$\mathcal{R}_x(c)(r, \omega) = \varphi_x(\omega) r^\varepsilon [c \star_t \phi](\ln r, z_x),$$

where φ_x is a cut-off function defined on Ω_S such that $\varphi_x = 1$ in a neighbourhood of x and $\varphi_x = 0$ in a neighbourhood of the other vertices. ϕ is the function introduced by M. Dauge in (16.6) of [3] i.e.

$$\phi(t, z_x) = (-2\pi)^{-1} \int_{\mathbb{R}} e^{it\xi} r(\xi, z_x) d\xi,$$

when $r(\xi, z_x) = \varphi(|z_x| \chi(\xi))$, χ is a continuous function on \mathbb{R} such that $\chi \geq 1$ on \mathbb{R} and $\chi(t) = |t|$ if $t \geq t_0 > 0$ and φ is a rapidly decreasing function on \mathbb{R}^+

such that $\varphi = 1$ in a neighbourhood of 0. By $c \star_t \phi$, we mean

$$(c \star_t \phi)(t, z_x) = \int_{\mathbb{R}} c(s)\phi(t - s, z_x)ds.$$

- We also introduce the matrix

$$A = \begin{pmatrix} \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi \\ \sin \theta \sin \varphi & \cos \theta \sin \varphi & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix}.$$

This matrix allows us to pass from \mathbf{u} to $(u_r, u_\theta, u_\varphi)$, the projections of the vector \mathbf{u} in cartesian coordinates onto the spherical basis i.e.

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = A \begin{pmatrix} u_r \\ u_\theta \\ u_\varphi \end{pmatrix}.$$

Now, we are ready to give the analogue of Theorem 17.13 of [3] to our system.

THEOREM 3.3. *Let $\mathbf{f} \in (L^2(\Gamma_S))^3$, $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma_k'))^3$, $\forall k \in \mathcal{N}_S$, and let $\mathbf{u} \in V(\Gamma_S)$ be a solution of (3.2) with a compact support. Then there exists $\varepsilon > 0$ such that*

$$(3.13) \quad \begin{aligned} \mathbf{u} = \mathbf{u}_r + \sum_{\alpha} \sigma^{\alpha} + \sum_{x \in A(\Omega_S)} \left\{ \sum_{(\alpha, \nu) \in \Lambda_x(\varepsilon)} \mathcal{R}_x(c_x^{\alpha, \nu}) A \begin{pmatrix} 0 \\ \sigma_x^{\alpha, \nu}(z_x) \end{pmatrix} \right. \\ \left. + \sum_{\alpha' \in \Lambda'_x(\varepsilon)} \mathcal{R}_x(c_x^{\alpha'}) A \begin{pmatrix} \sigma_x^{\alpha'}(z_x) \\ 0 \\ 0 \end{pmatrix} \right\}, \end{aligned}$$

where $\mathbf{u}_r \in (H^{3/2+\varepsilon}(\Omega))^3$, $c_x^{\alpha, \nu} \in H^{1/2+\varepsilon-\Re(\alpha)}(\mathbb{R})$, $c_x^{\alpha'} \in H^{1/2+\varepsilon-\alpha'}(\mathbb{R})$.

PROOF. In view of Theorem 3.2, it suffices to study the regularity of the function $\mathbf{u}_0 \in (H^1_{-1/2-\varepsilon}(\Gamma_S))^3$, solution of (3.6). Let us perform the change of variable $r = e^t$ and set

$$(3.14) \quad \begin{cases} \mathbf{w}(t, \omega) = e^{-\varepsilon t} \mathbf{u}_0(e^t, \omega), \\ \mathbf{F}(t, \omega) = e^{(2-\varepsilon)t} \mathbf{f}(e^t, \omega), \\ \mathbf{G}^{(k)}(t, \omega') = e^{(1-\varepsilon)t} \mathbf{g}^{(k)}(e^t, \omega'), \quad \forall k \in \mathcal{N}_S. \end{cases}$$

By Theorem AA.3 of [3], we can show that (recall that \mathbf{f} and $\mathbf{g}^{(k)}$ have compact

supports in Γ_S):

$$\begin{aligned} \mathbf{w} &\in (H^1(\mathbb{R} \times \Omega_S))^3, \\ \mathbf{F} &\in (L^2(\mathbb{R} \times \Omega_S))^3, \\ \mathbf{G}^{(k)} &\in (H^{1/2}(\mathbb{R} \times \Gamma_k''))^3, \quad \forall k \in \mathcal{N}_S. \end{aligned}$$

Moreover, the Dirichlet boundary conditions fulfilled by \mathbf{u}_0 on Γ_S induced analogous boundary conditions on $\mathbb{R} \times \Omega_S$, namely $\mathbf{w} \in V(\mathbb{R} \times \Omega_S)$, when we set

$$V(\mathbb{R} \times \Omega_S) = \{ \mathbf{u} \in (H^1(\mathbb{R} \times \Omega_S))^3 \text{ fulfilling } \gamma_k \mathbf{u} = \mathbf{0} \text{ on } \mathbb{R} \times \Gamma_k'', \quad \forall k \in \mathcal{D}_S \},$$

where γ_k denotes improperly the trace operator on $\mathbb{R} \times \Gamma_k''$. Finally, by this change of variable and (3.14), the identity (3.6) becomes

$$\begin{aligned} (3.15) \quad a_\varepsilon(\mathbf{w}, \mathbf{v}) &= \int_{\mathbb{R} \times \Omega_S} (\mathbf{F}(t, \omega), \mathbf{v}(t, \omega)) e^t dt d\omega \\ &+ \sum_{k \in \mathcal{N}_S} \int_{\mathbb{R} \times \Gamma_k''} (\mathbf{G}^{(k)}(t, \omega'), \gamma_k \mathbf{v}(t, \omega')) e^t dt d\sigma', \quad \forall \mathbf{v} \in V_\infty(\mathbb{R} \times \Omega_S), \end{aligned}$$

where we have set

$$\begin{aligned} a_\varepsilon(\mathbf{w}, \mathbf{v}) &= \int_{\mathbb{R} \times \Omega_S} \sum_{i,j=1}^3 S_{ij} \left(\omega, \frac{\partial}{\partial t} + \varepsilon, D_\omega \right) \mathbf{w} \mathcal{E}_{ij} \left(\omega, \frac{\partial}{\partial t} - \varepsilon, D_\omega \right) \bar{\mathbf{v}} e^t dt d\omega, \\ V_\infty(\mathbb{R} \times \Omega_S) &= \{ \mathbf{v} \in V(\mathbb{R} \times \Omega_S) \text{ with a compact support} \}. \end{aligned}$$

This shows that \mathbf{w} is solution of a boundary value problem on the dihedral cone $\mathbb{R} \times \Omega_S$. So to obtain (3.13), it suffices to give the singularities of \mathbf{w} along the edges of $\mathbb{R} \times \Omega_S$. This is proven as Theorem 16.9 of [3], using Theorem 2.1 on Ω_S for the component (w_θ, w_φ) and Theorem 4.4.3.7 of [6] for w_r , instead of Theorem 5.11 of [3] because we notice that, using the local chart χ_x , the principal part of the operator \mathcal{A}_ε , induced by the sesquilinear form a_ε frozen at 0, is the system of elasticity in C_x for (w_θ, w_φ) and the Laplace operator for w_r . Finally, the assumption on coerciveness for this principal part holds here since Korn's inequality is true on the cone C_x . Actually, if Ω has a crack at x , then $C_x = \mathbb{R}^2 \setminus \mathbb{R}^+$, with the convention $\mathbb{R}^+ = \{(x_1, 0) : x_1 \geq 0\}$; then to get Korn's inequality on C_x , it suffices to split up C_x into two half-spaces where Korn's inequality holds. Therefore, as in Theorem 16.9 of [3], we deduce that

\mathbf{w} admits the following expansion

$$(3.16) \quad \mathbf{w} = \mathbf{w}_r + \sum_{x \in A(\Omega_S)} \varphi_x(\omega) \left\{ \sum_{(\alpha, \nu) \in \Lambda_x(\varepsilon)} (c_x^{\alpha, \nu} \star_t \phi)(t, z_x) A \begin{pmatrix} 0 \\ \sigma_x^{\alpha, \nu}(z_x) \end{pmatrix} + \sum_{\alpha' \in \Lambda'_x(\varepsilon)} (c_x^{\alpha'} \star_t \phi)(t, z_x) A \begin{pmatrix} \sigma_x^{\alpha'}(z_x) \\ 0 \\ 0 \end{pmatrix} \right\},$$

where $\mathbf{w}_r \in (H^{3/2+\varepsilon}(\mathbb{R} \times \Omega_S))^3$, $c_x^{\alpha, \nu} \in H^{1/2+\varepsilon-\Re(\alpha)}(\mathbb{R})$, $c_x^{\alpha'} \in H^{1/2+\varepsilon-\alpha'}(\mathbb{R})$.

Let us notice that this result holds if the equation (2.2), (2.3) or (2.4) with $\omega = \omega_x$ has no solution on the line $\Re(\alpha) = 1/2 + \varepsilon$, for all $x \in A(\Omega_S)$. Since the set of solutions of these equations is finite in a fixed strip and the set of α , such that $\mathcal{A}_S(\alpha)^{-1}$ does not exist, is also finite in a fixed strip, it is always possible to find a $\varepsilon > 0$ such that the line $\Re(\alpha) = 1/2 + \varepsilon$ has no solution of the equation (2.2), (2.3) or (2.4) at each vertex of Ω_S and such that, on the line $\Re(\alpha) = \varepsilon$, $\mathcal{A}_S(\alpha)^{-1}$ exists (at the beginning of this proof, we take the corresponding \mathbf{u}_0).

Using (3.16), the change of variable (3.14) and the expression (3.10) of \mathbf{u} , we obtain the expansion (3.13) for \mathbf{u} , when

$$\mathbf{u}_r(r, \omega) = r^\varepsilon \mathbf{w}_r(\ln r, \omega).$$

This completes the proof since Theorem AA.3 of [3] allows us to conclude that

$$\mathbf{u}_r \in \left(H_0^{\frac{3}{2}+\varepsilon}(\Gamma_S) \right)^3$$

and this last space is obviously embedded into $\left(H^{\frac{3}{2}+\varepsilon}(\Gamma_S) \right)^3$. □

REMARK 3.4. The proof of the edge behaviour of \mathbf{u}_0 is different from the proof of Proposition 17.12 of [3]. Our idea consists in setting (3.14) and therefore to go back to the boundary value problem (3.15) in the dihedral cone $\mathbb{R} \times \Omega_S$. This allows us to avoid the localization arguments of [3].

4. - Maximal regularity for the Lamé system in dimension 3

Theorem 3.3 shows that if we want to give a maximal regularity for a solution \mathbf{u} of problem (1.4), we need to control the edge and vertex singularities. The edge singularities do not pose any problem since they correspond to vertex singularities in dimension 2. It remains the vertex singularities. When the boundary conditions on all the faces are of Dirichlet type, an estimate of a strip free of pole for $\mathcal{A}_S(\alpha)^{-1}$ can be deduced from [16]:

LEMMA 4.1. *Let S be a fixed vertex of Ω . If $\overline{\Omega}_S$ is different from the unit sphere S_2 and $\mathcal{N}_S = \emptyset$, then $\mathcal{A}_S(\alpha)^{-1}$ exists for all α in the strip $\Re(\alpha) \in [-1, 0]$.*

PROOF. Let us suppose that there exists α in the strip $\Re(\alpha) \in [-1, 0]$ such that $\mathcal{A}_S(\alpha)^{-1}$ does not exist. In that case, Theorem 3.2 shows that there exists a function $\varphi^{\alpha,0}$ on Ω_S such that $r^\alpha \varphi^{\alpha,0}$ fulfils (3.9). But Theorem 1 of [16] excludes the existence of such a solution in the strip

$$\left| \Re(\alpha) + \frac{1}{2} \right| \leq \alpha_0 + \frac{1}{2},$$

for some $\alpha_0 > 0$. This is a contradiction since this strip is larger than ours. \square

Unfortunately, Theorem 1 of [16] uses in a basic way the Dirichlet boundary conditions and it seems to be impossible to extend it to mixed boundary conditions. For purely Neumann boundary conditions, using a different method, Maz'ya and Kozlov prove in Theorem 3 of [11] that, if Ω_S has no crack, then the conclusion of Lemma 4.1 still holds. Under a geometrical assumption, we now prove that their method can be adapted to mixed boundary conditions. For a fixed vertex S of Ω , let us set

$$C_{S,D} = \left\{ \sum_{k \in \mathcal{D}_S} \lambda_k \nu^{(k)} : \lambda_k \geq 0, \text{ not all zero} \right\},$$

$$C_{S,N} = \left\{ \sum_{k \in \mathcal{N}_S} \lambda_k \nu^{(k)} : \lambda_k \geq 0, \text{ not all zero} \right\},$$

with the agreement that $C_{S,D}$ (respectively $C_{S,N}$) is empty if \mathcal{D}_S (respectively \mathcal{N}_S) is empty.

THEOREM 4.2. *Let S be a fixed vertex of Ω . If Ω_S has no crack and $C_{S,D} \cap C_{S,N} = \emptyset$, then*

- i) *If $\mathcal{D}_S \neq \emptyset$, $\mathcal{A}_S(\alpha)^{-1}$ exists for all α in the strip $\Re(\alpha) \in [-1, 0]$.*
- ii) *If $\mathcal{D}_S = \emptyset$, $\mathcal{A}_S(\alpha)^{-1}$ exists for all α in the strip $\Re(\alpha) \in [-1, 0]$ except for $\alpha = 0$ and -1 , where $\ker \mathcal{A}_S(\alpha) = \mathbb{C}^3$.*

PROOF. The case $\mathcal{D}_S = \emptyset$ is precisely Theorem 3 of [11] (see Remark 2 of [11]). So from now on, we can suppose that $\mathcal{D}_S \neq \emptyset$. We firstly establish that $\mathcal{A}_S(\alpha)^{-1}$ exists for all α on the line $\Re(\alpha) = 0$. To do that, as in Lemma 4.1, we show that a function $\mathbf{u} \neq \mathbf{0}$ in the form

$$(4.1) \quad \mathbf{u}(r, \omega) = r^\alpha \varphi(\omega),$$

with some function φ defined on Ω_S and $\Re(\alpha) = 0$, cannot be a solution of (3.9).

Let us suppose the contrary. Then the assumption that Ω_S has no crack insure that the following Green formula has a meaning for \mathbf{u} :

$$(4.2) \quad \int_{\Gamma_{S\varepsilon}} \left(L\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_m} \right) dx = - \int_{\Gamma_{S\varepsilon}} \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij} \left(\frac{\partial \bar{\mathbf{u}}}{\partial x_m} \right) dx + \sum_{k \in \mathcal{J}_S} \int_{\Gamma_{k\varepsilon}} \left(T^{(k)}\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_m} \right) d\sigma,$$

for all $m \in \{1, 2, 3\}$, all $\varepsilon \in]0, 1[$, when we set

$$\Gamma_{S\varepsilon} = \{(r, \omega) \in \Gamma_S : \varepsilon < r < 1\},$$

$$\Gamma_{k\varepsilon} = \{(r, \omega') \in \Gamma'_k : \varepsilon < r < 1\}.$$

Let us remark that the boundary terms corresponding to $r = \varepsilon$ and $r = 1$ cancel since $\Re(\alpha) = 0$. Let us fix $m \in \{1, 2, 3\}$ and $\varepsilon \in]0, 1[$. Taking the real part of (4.2) and since we assume that \mathbf{u} fulfils (3.9), we get

$$(4.3) \quad 0 = - \int_{\Gamma_{S\varepsilon}} \Re \left(\sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij} \left(\frac{\partial \bar{\mathbf{u}}}{\partial x_m} \right) \right) dx + \sum_{k \in \mathcal{D}_S} \int_{\Gamma_{k\varepsilon}} \Re \left(\left(T^{(k)}\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_m} \right) \right) d\sigma.$$

Now, using the easily checked identity

$$\Re \left(\sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij} \left(\frac{\partial \bar{\mathbf{u}}}{\partial x_m} \right) \right) = \frac{1}{2} \frac{\partial}{\partial x_m} \left\{ \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij}(\bar{\mathbf{u}}) \right\}$$

and integrating by parts, (4.3) becomes

$$(4.4) \quad 0 = - \frac{1}{2} \sum_{k \in \mathcal{J}_S} \int_{\Gamma_{k\varepsilon}} \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij}(\bar{\mathbf{u}}) \nu_m^{(k)} d\sigma + \sum_{k \in \mathcal{D}_S} \int_{\Gamma_{k\varepsilon}} \Re \left(\left(T^{(k)}\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_m} \right) \right) d\sigma.$$

It is obvious that

$$(4.5) \quad \sum_{i,j=1}^3 \sigma_{ij}(\mathbf{u}) \varepsilon_{ij}(\bar{\mathbf{u}}) = 2\mu \sum_{i,j=1}^3 |\varepsilon_{ij}(\mathbf{u})|^2 + \lambda |\text{tr } \varepsilon(\mathbf{u})|^2.$$

Moreover using the fact that

$$\frac{\partial u_i}{\partial x_m} = \nu_m^{(k)} \frac{\partial u_i}{\partial \nu^{(k)}} \text{ on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S,$$

we can prove the following identity

$$(4.6) \quad \left(T^{(k)} \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_m} \right) = \nu_m^{(k)} \left\{ 2\mu \sum_{i,j=1}^3 |\varepsilon_{ij}(\mathbf{u})|^2 + \lambda |\text{tr } \varepsilon(\mathbf{u})|^2 \right\} \text{ on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S.$$

For all $k \in \mathcal{F}_S$, let us set

$$(4.7) \quad \lambda_{k\varepsilon} = \int_{\Gamma_{k\varepsilon}} \left\{ 2\mu \sum_{i,j=1}^3 |\varepsilon_{ij}(\mathbf{u})|^2 + \lambda |\text{tr } \varepsilon(\mathbf{u})|^2 \right\} d\sigma.$$

Then using (4.5) to (4.7) into (4.4), we arrive to

$$(4.8) \quad - \sum_{k \in \mathcal{N}_S} \lambda_{k\varepsilon} \nu_m^{(k)} + \sum_{k \in \mathcal{D}_S} \lambda_{k\varepsilon} \nu_m^{(k)} = 0.$$

Letting m vary into $\{1, 2, 3\}$, (4.8) is equivalent to

$$\sum_{k \in \mathcal{N}_S} \lambda_{k\varepsilon} \nu^{(k)} = \sum_{k \in \mathcal{D}_S} \lambda_{k\varepsilon} \nu^{(k)}.$$

Therefore the assumption $C_{S,D} \cap C_{S,N} = \emptyset$ imply that

$$\lambda_{k\varepsilon} = 0, \quad \forall k \in \mathcal{F}_S.$$

Since ε is arbitrary in $]0, 1[$, we finally obtain

$$(4.9) \quad 2\mu \sum_{i,j=1}^3 |\varepsilon_{ij}(\mathbf{u})|^2 + \lambda |\text{tr } \varepsilon(\mathbf{u})|^2 = 0 \quad \text{on } \Gamma'_k, \quad \forall k \in \mathcal{F}_S.$$

At this step, we follow Theorem 3 of [11]. The function

$$v = \text{tr } \varepsilon(\mathbf{u})$$

fulfils (owing to (4.9) and (3.10))

$$(4.10) \quad \begin{cases} \Delta v = 0 & \text{in } \Gamma_S, \\ \gamma_k v = 0 & \text{on } \Gamma'_k, \end{cases} \quad \forall k \in \mathcal{F}_S.$$

Since v has the form

$$(4.11) \quad v(r, \omega) = r^{\alpha-1} \psi(\omega),$$

with some function ψ defined on Ω_S , using Theorem 4.3 hereafter, we deduce that

$$v = 0.$$

Therefore, for all $i, j \in \{1, 2, 3\}$ the function $\varepsilon_{ij}(\mathbf{u})$ fulfils (4.10) (owing to (4.9) and (3.9)) and admits the expansion (4.11). So again Theorem 4.3 implies that

$$\varepsilon_{ij}(\mathbf{u}) = 0.$$

This shows that the displacement field \mathbf{u} is a rigid body motion. But this is incompatible with (4.1) except if $\alpha = 0$. In that last case, we deduce that there exists a vector $\mathbf{a} \in \mathbb{C}^3$ such that

$$\mathbf{u}(r, \omega) = \mathbf{a}.$$

Since $\mathcal{D}_S \neq \emptyset$, we arrive to

$$\mathbf{u} = \mathbf{0}.$$

This proves that $\mathcal{A}_S(\alpha)^{-1}$ exists for all α on the line $\Re(\alpha) = 0$. But using the definition of $\mathcal{A}_S(\alpha)$, we see that

$$\mathcal{A}_S(\alpha)^* = \mathcal{A}_S(-(\bar{\alpha} + 1)).$$

Therefore, $\mathcal{A}_S(\alpha)^{-1}$ exists also for all α on the line $\Re(\alpha) = -1$.

For all $t \in [0, 1]$, let us set $\mathcal{A}_S^t(\alpha)$ the family of operators (defined on Ω_S analogously to $\mathcal{A}_S(\alpha)$) associated with the following boundary value problem:

$$\begin{cases} (1-t)L\mathbf{u} + t\Delta\mathbf{u} = \mathbf{f} & \text{in } \Gamma_S, \\ \gamma_k\mathbf{u} = \mathbf{0} & \text{on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S, \\ (1-t)T^{(k)}\mathbf{u} + t\gamma_k \frac{\partial\mathbf{u}}{\partial\nu^{(k)}} = \mathbf{g}^{(k)} & \text{on } \Gamma''_k, \quad \forall k \in \mathcal{N}_S. \end{cases}$$

Using the previous argument, we can show that $\mathcal{A}_S^t(\alpha)^{-1}$ exists for all α on the lines $\Re(\alpha) = 0$ and $\Re(\alpha) = -1$, for any $t \in [0, 1]$. Moreover, Theorem 4.3 hereafter shows that $\mathcal{A}_S^1(\alpha)^{-1}$ exists for all α in the strip $\Re(\alpha) \in [-1, 0]$. Arguing as at the end of the proof of Theorem 1 of [11], we conclude that $\mathcal{A}_S^t(\alpha)^{-1}$ exists for all α in the same strip. \square

As we remark in the proof of Theorem 4.2, we need to study the family of operators $\mathcal{B}_S(\alpha)$ with complex parameter α associated with the following boundary value problem, when S is a fixed vertex of Ω :

$$(4.12) \quad \begin{cases} \Delta u = f & \text{in } \Gamma_S, \\ \gamma_k u = 0 & \text{on } \Gamma'_k, \quad \forall k \in \mathcal{D}_S, \\ \gamma_k \frac{\partial u}{\partial \nu^{(k)}} = 0 & \text{on } \Gamma''_k, \quad \forall k \in \mathcal{N}_S. \end{cases}$$

It is defined variationally as follows: we set

$$W(\Omega_S) = \{u \in H^1(\Omega_S) : \gamma_k u = 0 \text{ on } \Gamma'_k, \quad \forall k \in D_S\},$$

$$b_S(\alpha)\{u, v\} = \int_{\Omega_S} \{\nabla_\omega u \cdot \nabla_\omega \bar{v} - \alpha(\alpha + 1)u\bar{v}\} d\omega, \quad \forall u, v \in W(\Omega_S),$$

when $\nabla_\omega u = \left(\frac{\partial u}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial u}{\partial \varphi} \right)$. Then $B_S(\alpha)$ is the operator from $W(\Omega_S)$ into its dual $W(\Omega_S)'$ defined by

$$(B_S(\alpha)u)(v) = b_S(\alpha)\{u, v\}, \quad \forall u, v \in W(\Omega_S).$$

THEOREM 4.3. *Let S be a fixed vertex of Ω . If $D_S \neq \emptyset$, then $B_S(\alpha)^{-1}$ exists for all α in the strip $\mathfrak{R}(\alpha) \in [-1, 0]$; while if $D_S = \emptyset$, the same holds except for $\alpha = 0$ and $\alpha = -1$, where $\ker B_S(\alpha) = \mathbb{C}$.*

PROOF. Let us denote by $\{\lambda_k\}_{k \in \mathbb{N}}$ the sequence (in increasing order) of the Laplace-Beltrami operator $B_S(0)$ (let us recall that it is a nonnegative selfadjoint operator with a compact resolvent). Since $B_S(\alpha) = B_S(0) - \alpha(\alpha + 1)I$, we see that $B_S(\alpha)$ is one-to-one if and only if

$$\alpha(\alpha + 1) \neq \lambda_k, \quad \forall k \in \mathbb{N}.$$

This proves the result since $\lambda_0 > 0$ if $D_S \neq \emptyset$ and $\lambda = 0$ is of multiplicity 1 when $D_S = \emptyset$. □

REMARK 4.4. Theorem 4.3 is implicitly proven in Paragraph 5.1 of [8] and it precisely proves the fact that no vertex singularity for the Laplace operator appear in the strip $\mathfrak{R}(\alpha) \in [-1, 0]$ without any geometrical assumption on Ω_S . So we conjecture that the conclusions of Theorem 4.2 still hold without the geometrical assumptions made in Theorem 4.2.

With our notations, we have

$$V(\Omega_S) = (W(\Omega_S))^3 \text{ and } \mathcal{A}_S^1(\alpha)\mathbf{u} = (B_S(\alpha)u_i)_{i=1}^3, \quad \forall \mathbf{u} \in V(\Omega_S).$$

Let us mention two particular situations where the assumption $C_{S,D} \cap C_{S,N} = \emptyset$ is fulfilled:

- i) if Γ_S is nondegenerate trihedral cone, then for every choice of D_S and \mathcal{N}_S , this assumption is true.
- ii) If $\text{card } D_S \leq 1$ or $\text{card } \mathcal{N}_S \leq 1$, and if Γ_S is convex, then it holds; while if $\text{card } D_S \geq 2$ and $\text{card } \mathcal{N}_S \geq 2$, it may fail. □

Collecting the previous results, we arrive to the

THEOREM 4.5. *Let $\mathbf{u} \in V$ be a solution of problem (1.4) with data $\mathbf{f} \in (L^2(\Omega))^3$, $\mathbf{g}^{(k)} \in (H^{1/2}(\Gamma_k))^3$, $\forall k \in \mathcal{N}$. If the assumptions (H₃E) and (H₃V) hereafter are fulfilled, then*

$$(4.15) \quad \mathbf{u} \in (H^{3/2+\varepsilon}(\Omega))^3, \text{ for some } \varepsilon > 0.$$

(H₃E) $\forall j, k \in \mathcal{F}$ such that $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$, the interior dihedral angle ω_{jk} between Γ_j and Γ_k belongs to $]0, 2\pi[$ and if moreover $j \in \mathcal{D}$ and $k \in \mathcal{N}$, then $\omega_{jk} < \pi$.

(H₃V) For all vertex S of Ω , either $\mathcal{N}_S = \emptyset$ and $\bar{\Omega}_S \neq S_2$ or Ω_S has no crack and $C_{S,N} \cap C_{S,D} \neq \emptyset$.

5. - Setting of the coupled problem

Let us recall some notations introduced in §1 of [19] (when $n = 3$):

$$\begin{aligned} \Gamma &= \{x \in]-1, 1[^3 : x_2 = 0, 0 < x_1 < 1\}, \\ \omega &= \{x \in \mathbb{R}^3 : x_2 = 0, 0 < x_1 < 2 \text{ and } -1 < x_3 < 1\}, \\ \Omega &=]-1, 1[^3 \setminus \bar{\Gamma}. \end{aligned}$$

We sometimes identify Γ and ω with the open sets $]0, 1[\times]-1, 1[$ and $]0, 2[\times]-1, 1[$ of \mathbb{R}^2 , respectively. We notice that Ω is the unit cube with a slit along the half-plane $x_2 = 0, x_1 \geq 0$ (see Figure 1).

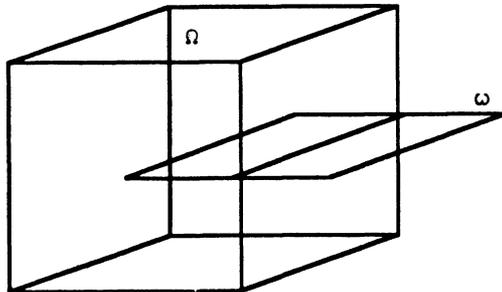


Figure 1

According to the convention of §1, the slit Γ of Ω will be splitted up into Γ_+ and Γ_- , so we denote by γ_+u (respectively γ_-u) the trace of a function u

on Γ from above (respectively from below) in Ω . The boundary $\partial\Omega$ of Ω will be decomposed as follows:

$$\Gamma_1 = \{x \in \partial\Omega : |x_3| = 1\},$$

$$\Gamma_2 = \partial\Omega \setminus \{\bar{\Gamma} \cup \Gamma_1\}.$$

It is also convenient to split up Γ_1 and Γ_2 into their plane faces i.e.

$$\Gamma_1 = \bigcup_{k=1,2} \bar{\Gamma}_{1k},$$

$$\Gamma_2 = \bigcup_{k=1}^5 \bar{\Gamma}_{2k}.$$

We denote by γ_{ik} , the trace operator on the face Γ_{ik} in Ω .

Inspired from [2], we consider the following boundary value problem: given $\mathbf{f} \in (L^2(\Omega))^3$ and $g \in L^2(\omega)$, find weak solutions $\mathbf{u} \in (H^1(\Omega))^3$ and $\xi \in H^2(\omega)$ of problem (5.1)–(5.7) hereafter:

$$(5.1) \quad L\mathbf{u} = -\mathbf{f} \quad \text{in } \Omega,$$

$$(5.2) \quad \gamma_{1k}\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{1k}, \quad k = 1, 2,$$

$$(5.3) \quad T^{(2k)}\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{2k}, \quad k = 1, \dots, 5,$$

$$(5.4) \quad \rho\Delta^2\xi + \{\gamma_{-\sigma_{22}}(\mathbf{u}) - \gamma_{+\sigma_{22}}(\mathbf{u})\}\chi_\Gamma = g \quad \text{in } \omega,$$

where $\rho = \frac{8}{3} \frac{\mu(\lambda + \mu)}{\lambda + 2\mu}$ and $\chi_\Gamma(x) = 1$ if $x \in \Gamma$ and 0 elsewhere

$$(5.5) \quad \xi = \frac{\partial\xi}{\partial\nu} = 0 \quad \text{on } \partial\omega,$$

$$(5.6) \quad \gamma_+u_2 = \gamma_-u_2 = \xi \quad \text{on } \Gamma,$$

$$(5.7) \quad \gamma_+u_\alpha = \gamma_-u_\alpha = 0 \quad \text{on } \Gamma, \quad \alpha = 1, 3.$$

In order to give the variational formulation of this problem, we introduce the two Hilbert spaces:

$$H = (L^2(\Omega))^3 \times L^2(\omega),$$

$$V = \{\mathbf{U} = (\mathbf{u}, \xi) \in (H^1(\Omega))^3 \times H_0^2(\omega) \text{ fulfilling (5.2), (5.6) and (5.7)}\},$$

this last one being equipped with the norm of $(H^1(\Omega))^3 \times H^2(\omega)$. We define the continuous sesquilinear form on V as follows:

$$(5.8) \quad a(\mathbf{U}, \mathbf{V}) = a_\Omega(\mathbf{u}, \mathbf{v}) + b_\omega(\xi, \eta),$$

for all $\mathbf{U} = (\mathbf{u}, \xi), \mathbf{V} = (\mathbf{v}, \eta) \in V$, where a_Ω is defined in §1 and

$$b_\omega(\xi, \eta) = \frac{4\mu}{3} \int_\omega \sum_{\alpha, \beta=1,3} \left(\frac{\partial^2 \xi}{\partial x_\alpha \partial x_\beta} + \frac{\lambda}{\lambda + 2\mu} \Delta \xi \delta_{\alpha\beta} \right) \frac{\partial^2 \eta}{\partial x_\alpha \partial x_\beta} dx'.$$

Let us notice that this form b_ω is a multiple of the form (2.13) of [18], using the classical convention of linear elasticity.

LEMMA 5.1. *For all $(\mathbf{f}, g) \in H$, there exists a unique solution $\mathbf{U} \in V$ of*

$$(5.9) \quad a(\mathbf{U}, \mathbf{V}) = \int_\Omega (\mathbf{f}, \mathbf{v}) dx + \int_\omega g \bar{\eta} dx', \quad \forall \mathbf{V} = (\mathbf{v}, \eta) \in V.$$

PROOF. Let us set

$$\begin{aligned} \Omega^+ &= \{x \in \Omega : x_2 > 0\}, \\ \Omega^- &= \{x \in \Omega : x_2 < 0\}. \end{aligned}$$

Using Korn's inequality in Ω^+ and Ω^- and adding the results, we deduce the existence of a constant $\alpha_1 > 0$ such that

$$(5.10) \quad a_\Omega(\mathbf{u}, \mathbf{u}) \geq \alpha_1 \|\mathbf{u}\|_{(H^1(\Omega))^3}^2, \quad \forall \mathbf{U} = (\mathbf{u}, \xi) \in V.$$

Moreover, the inequality (2.14) of [18] may be written

$$(5.11) \quad a_\omega(\xi, \xi) \geq \alpha_2 \|\xi\|_{H^2(\omega)}^2, \quad \forall \xi \in H_0^2(\omega),$$

for some constant $\alpha_2 > 0$. The addition of (5.10) and (5.11) shows that a is V -coercive, i.e. there exists $\alpha > 0$ such that

$$(5.12) \quad a(\mathbf{U}, \mathbf{U}) \geq \alpha \|\mathbf{U}\|_V^2, \quad \forall \mathbf{U} \in V.$$

The conclusion follows from the Lax-Milgram lemma. □

In order to show that \mathbf{U} , solution of (5.9), is actually a solution of the boundary value problem (5.1)–(5.7), we need the extension of Theorems 1.5.3.10 and 1.5.3.11 of [6] in dimension 3 for the Lamé system instead of the Laplace operator. This is the purpose of

THEOREM 5.2. *Let θ be a bounded open set of \mathbb{R}^3 , with a polyhedral boundary $\partial\theta$, θ lying on only one side of its boundary and set $\partial\theta = \bigcup_{j=1}^N \bar{\Gamma}_j$, where Γ_j are disjoint plane open sets. For all $k = 1, \dots, N$, let us denote by γ_k the trace operator on the face Γ_k . For $p > 1$, let us set*

$$E(L, L^p(\theta)) = \{\mathbf{u} \in (H^1(\theta))^3 : L\mathbf{u} \in (L^p(\theta))^3\}.$$

Then the mapping

$$\mathbf{u} \rightarrow T^{(k)}\mathbf{u}$$

which is defined on $(D(\bar{\theta}))^3$, has a unique continuous extension as an operator from $E(L, L^p(\theta))$ into $((\tilde{H}^{1/2}(\Gamma_k))')^3$. Moreover, the following half Green identity still holds:

$$(5.13) \quad \int_{\theta} (L\mathbf{u}, \mathbf{v})dx = -a_{\theta}(\mathbf{u}, \mathbf{v}) + \sum_{k=1}^n \langle T^{(k)}\mathbf{u}; \gamma_k \bar{\mathbf{v}} \rangle,$$

for all $\mathbf{u} \in E(L, L^p(\theta))$ and all $\mathbf{v} \in (D(\bar{\theta}))^3$ such that $\gamma_k \mathbf{v} \in (D(\Gamma_k))^3$, for every $k \in \{1, \dots, N\}$.

PROOF. It suffices to follow the proof of Theorem 1.5.3.10 and 1.5.3.11 of [6] using Theorem 2.2 of [19] instead of Theorem 1.5.2.3 of [6] and since the Green identity (5.13) holds for $\mathbf{u} \in (H^2(\theta))^3$ and $\mathbf{v} \in (H^1(\theta))^3$. \square

THEOREM 5.3. Let $\mathbf{U} = (\mathbf{u}, \xi) \in V$ be a solution of (5.9) with data $(\mathbf{f}, g) \in H$. Then it fulfils (5.1), (5.3) and (5.4).

PROOF. (5.1) is a direct consequence of (5.9) applied with $\mathbf{v} \in (D(\Omega))^3$ and $\eta = 0$. Therefore, we deduce that $\mathbf{u} \in E(L, L^2(\Omega))$. Since Ω does not fulfil the assumption of Theorem 5.2, as in Theorem 2.9 of [19], we use some tricky splitting of Ω . Let us firstly consider $\varphi \in (D(\bar{\Omega}))^3$ fulfilling

$$\varphi = \mathbf{0} \text{ on } x_2 = 0, \quad \text{and } \varphi = \mathbf{0} \text{ on } \Gamma_1.$$

We moreover suppose that the restriction of φ to each face of Γ_2 has a compact support. So the restriction φ^+ (respectively φ^-) to Ω^+ (respectively Ω^-) fulfils the assumption of Theorem 5.2. Since $\mathbf{u} \in E(L, L^2(\Omega^+)) \cap E(L, L^2(\Omega^-))$, applying the identity (5.13) to the pair (\mathbf{u}, φ^+) (respectively (\mathbf{u}, φ^-)) in Ω^+ (respectively Ω^-) and adding the results, we get

$$(5.14) \quad \int_{\Omega} (L\mathbf{u}, \varphi)dx = -a_{\Omega}(\mathbf{u}, \varphi) + \sum_{k=1}^5 \langle T^{(2k)}\mathbf{u}, \gamma_{2k} \bar{\varphi} \rangle.$$

Comparing (5.14) with (5.9) applied with $\mathbf{V} = (\varphi, \mathbf{0}) \in V$, we arrive to (5.3).

To prove (5.4), we argue as in Theorem 2.9 of [19]; we set

$$\begin{aligned} \hat{\Omega} &= \{x \in]-1, 1[{}^n : x_1 > 0\}, \\ \hat{\Omega}^+ &= \{x \in \hat{\Omega} : x_2 > 0\}, \\ \hat{\Omega}^- &= \{x \in \hat{\Omega} : x_2 < 0\}. \end{aligned}$$

Since $\mathbf{u} \in E(L, L^2(\hat{\Omega}^+)) \cap E(L, L^2(\hat{\Omega}^-))$, we know that $\gamma_+ T(\mathbf{u}) = (-\gamma_+ \sigma_{i2}$

$(\mathbf{u})_{i=1}^3$ and $\gamma_-T(\mathbf{u}) = (\gamma_- \sigma_{i2}(\mathbf{u}))_{i=1}^3$ belong to $(\tilde{H}^{1/2}(\Gamma))^3$. Let us fix $\eta \in \mathcal{D}(\omega)$ such that $\text{supp } \eta \subset \Gamma \cup \omega_1$, where $\omega_1 = \omega \setminus \bar{\Gamma}$. It is obvious that there exists $v_2 \in \mathcal{D}(\hat{\Omega})$ such that

$$v_2 = \eta \text{ on } \Gamma.$$

If we set $\mathbf{v} = (0, v_2, 0)$, then $\mathbf{V}_1 = (\mathbf{v}, \eta) \in V$. But \mathbf{v}^+ (respectively \mathbf{v}^-), the restriction of \mathbf{v} to $\hat{\Omega}^+$ (respectively $\hat{\Omega}^-$), fulfils the assumptions of Theorem 5.2; applying the Green identity (5.13) to the pairs $(\mathbf{u}, \mathbf{v}^+)$ in $\hat{\Omega}^+$ and $(\mathbf{u}, \mathbf{v}^-)$ in $\hat{\Omega}^-$, we get by addition

$$\int_{\Omega} (L\mathbf{u}, \dot{\mathbf{v}}) dx = -a_{\Omega}(\mathbf{u}, \mathbf{v}) + \langle \gamma_- \sigma_{22}(\mathbf{u}) - \gamma_+ \sigma_{22}(\mathbf{u}), \gamma_+ \bar{v}_2 \rangle.$$

Applying (5.9) with the test-function \mathbf{V}_1 and using the previous identity, we obtain

$$(5.15) \quad b_{\omega}(\xi, \eta) + \langle \gamma_- \sigma_{22}(\mathbf{u}) - \gamma_+ \sigma_{22}(\mathbf{u}), \chi_{\Gamma} \bar{\eta} \rangle = \int_{\omega} g \bar{\eta} dx'.$$

This proves (5.4) because (5.15) holds for all $\eta \in \mathcal{D}(\omega)$ such that $\text{supp } \eta \subset \Gamma \cup \omega_1$. □

Before going on, let us remark that (5.15) does not imply that $\xi \in H_0^2(\omega)$ is the variational solution of

$$(5.16) \quad b_{\omega}(\xi, \eta) = \int_{\omega} g \bar{\eta} dx' + \langle \gamma_+ \sigma_{22}(\mathbf{u}) - \gamma_- \sigma_{22}(\mathbf{u}), \chi_{\Gamma} \bar{\eta} \rangle, \quad \forall \eta \in H_0^2(\omega).$$

Actually to show this, we shall need the regularity of \mathbf{u} in Ω established in §6 hereafter. From (2.14), we only obtain (5.16) for all $\eta \in H_0^2(\Gamma) \cap H_0^2(\omega_1)$.

6. - Regularity of the solution of the coupled problem

Firstly, we establish that the 3D-part \mathbf{u} of \mathbf{U} , solution of (5.9), may be seen as a solution of the Lamé system in Ω with non-homogeneous boundary conditions.

PROPOSITION 6.1. *Let $\mathbf{U} = (\mathbf{u}, \xi) \in V$ be the solution of problem (5.9) with data $(\mathbf{f}, g) \in H$. Then there exists $\mathbf{w} \in (H^2(\Omega))^3$ such that $\mathbf{u}_0 = \mathbf{u} - \mathbf{w}$ is the variational solution of*

$$(6.1) \quad L\mathbf{u}_0 = -(\mathbf{f} + L\mathbf{w}) \quad \text{in } \Omega,$$

$$(6.2) \quad \gamma_+ \mathbf{u}_0 = \gamma_- \mathbf{u}_0 = 0 \quad \text{on } \Gamma,$$

$$(6.3) \quad \gamma_{1k} \mathbf{u}_0 = \mathbf{0} \quad \text{on } \Gamma_{1k}, \quad k = 1, 2,$$

$$(6.4) \quad T^{(2k)} \mathbf{u}_0 = -T^{(2k)} \mathbf{w} \quad \text{on } \Gamma_{2k}, \quad k = 1, \dots, 5,$$

In other words, if we set

$$W = \{ \mathbf{v} \in (H^1(\Omega))^3 \text{ fulfilling (6.2) and (6.3)} \},$$

then $\mathbf{u}_0 \in W$ fulfils

$$(6.5) \quad a_\Omega(\mathbf{u}_0, \mathbf{v}) = \int_\Omega (\mathbf{f} + L\mathbf{w}, \mathbf{v}) dx - \sum_{k=1}^5 \int_{\Gamma_{2k}} (T^{(2k)} \mathbf{w}, \gamma_{2k} \mathbf{v}) d\sigma, \quad \forall \mathbf{v} \in W.$$

PROOF. Applying (5.9) with $\mathbf{V} = (\mathbf{v}, \eta) \in V$ such that $\eta = 0$, we see that $\mathbf{u} \in (H^1(\Omega))^3$ fulfils (5.2), (5.6), (5.7) and

$$(6.6) \quad a_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{f}, \mathbf{v}) dx, \quad \forall \mathbf{v} \in W.$$

In order to prove (6.5) from (6.6), we need a trace result, which follows from [5]. Let us denote by $\tilde{\Gamma}$, the common part of the boundaries of Ω^+ and Ω^- i.e.

$$\tilde{\Gamma} = \{x \in]-1, 1[^3 : x_2 = 0\}.$$

It is clear that the extension $\tilde{\xi}$ of ξ to $\tilde{\Gamma}$ by zero outside Γ belongs to $H^2(\tilde{\Gamma})$. Owing to the results of [5], there exist $w^+ \in H^{2+\delta}(\Omega^+)$ and $w^- \in H^{2+\delta}(\Omega^-)$ for some fixed $\delta \in]0, \frac{1}{2}[$ such that

$$\begin{cases} w^+ = \tilde{\xi} & \text{on } \tilde{\Gamma}, \\ D_2 w^+ = 0 & \text{on } \tilde{\Gamma}, \\ w^+ = 0 & \text{on } \tilde{\Gamma}_1^+ := \{x \in \Gamma_1 : x_2 > 0\}. \end{cases}$$

$$\begin{cases} w^- = \tilde{\xi} & \text{on } \tilde{\Gamma}, \\ D_2 w^- = 0 & \text{on } \tilde{\Gamma}, \\ w^- = 0 & \text{on } \tilde{\Gamma}_1^- := \{x \in \Gamma_1 : x_2 < 0\}. \end{cases}$$

So the function w_2 defined in Ω by

$$w_2 = \begin{cases} w^+ & \text{in } \Omega^+, \\ w^- & \text{in } \Omega^-, \end{cases}$$

belongs to $H^2(]-1, 1[^3)$ and fulfils (5.6). The conclusion follows by setting $\mathbf{w} = (0, w_2, 0)$, because the Green identity (5.13) can be applied in Ω^+ and Ω^- to the pair (\mathbf{w}, \mathbf{v}) , for $\mathbf{v} \in W$. □

Since \mathbf{u}_0 is a solution of the Lamé system in Ω with non-homogeneous mixed boundary conditions, we can apply Theorem 3.3 to \mathbf{u}_0 . This will allow us to give the behaviour of \mathbf{u} near the vertices and the edges of Ω . To do that, we introduce some notations: let us denote by S_1 and S_{-1} , the two vertices of Ω belonging to the bottom of the crack, i.e. $S_1 = (0, 0, 1)$ and $S_{-1} = (0, 0, -1)$. For $i = 1, -1$, we denote by $(r_i, \theta_i, \varphi_i)$ the spherical coordinates with origin S_i such that $\theta_i = \pi/2$ on Γ and $\theta_i = \pi/2, \varphi_i = \pi/2$ on the bottom of the crack. So Ω_i , the intersection between Ω and the unit sphere of center S_i will be (denoted previously Ω_{S_i} , see Figure 2)

$$\Omega_i = \{(\theta_i, \varphi_i) \in]0, \pi[\times]0, \pi[\setminus \{(\theta_i, \varphi_i) \text{ such that } \varphi_i \in]0, \pi/2] \text{ and } \theta_i = \pi/2\}.$$

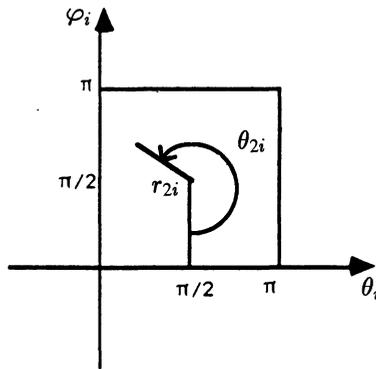


Figure 2

Contrary to §3, we do not need a local chart χ_i , it suffices to take $z_i = (\theta_i - \frac{\pi}{2}, \varphi_i - \frac{\pi}{2})$. Therefore denote (r_{2i}, θ_{2i}) the polar coordinates with origin $(\pi/2, \pi/2)$ such that $\theta_{2i} = 0$ on the crack of Ω_i .

We finally introduce two cut-off functions $\eta_1, \eta_{-1} \in \mathcal{D}(\mathbb{R}^3)$ such that $\eta_i = 1$ in a neighbourhood of S_i , $\text{supp } \eta_i$ included in a neighbourhood of S_i , let us say $\text{supp } \eta_i \subset B(S_i, 3/2)$, and $\eta_1 + \eta_{-1} = 1$ in a neighbourhood of the bottom of the crack of Ω .

THEOREM 6.2. *Let $\mathbf{U} = (\mathbf{u}, \xi) \in V$ be the solution of (5.9) with data $(\mathbf{f}, \mathbf{g}) \in H$. Then there exists $\varepsilon > 0$ such that*

$$(6.7) \quad \mathbf{u} = \mathbf{u}_r + \sum_{i=1,-1} \eta_i \left\{ \sum_{\nu=1,2} \mathcal{R}_i(c_i^\nu) A_i \begin{pmatrix} 0 \\ \sigma_i^\nu(z_i) \end{pmatrix} + \mathcal{R}_i(c_i) A_i \begin{pmatrix} \sigma_i(z_i) \\ 0 \\ 0 \end{pmatrix} \right\},$$

where $\mathbf{u}_r \in (H^{3/2+\varepsilon}(\Omega))^3$, c_i^ν , $c_i \in H^\varepsilon(\mathbb{R})$, σ_i^ν , $\nu = 1, 2$ (respectively σ_i) are the singular functions of the Lamé system (respectively the Laplace operator) with Dirichlet boundary conditions along the crack of Ω_i associated with the bottom of the crack with exponent $\alpha = 1/2$ (respectively $\alpha' = 1/2$), \mathcal{R}_i is the smoothing operator \mathcal{R}_x , when x is the bottom of the crack of Ω_i , and finally due to the particular choice of the spherical coordinates $(r_i, \theta_i, \varphi_i)$, we have

$$A_i = \begin{pmatrix} \sin \theta_i \cos \varphi_i & \cos \theta_i \cos \varphi_i & -\sin \varphi_i \\ \cos \theta_i & -\sin \theta_i & 0 \\ -i \sin \theta_i \sin \varphi_i & -i \cos \theta_i \sin \varphi_i & -i \cos \varphi_i \end{pmatrix}.$$

PROOF. In view of Proposition 6.1, it suffices to study the behaviour of the solution \mathbf{u}_0 of (6.5). Applying Theorem 3.3 to \mathbf{u}_0 , we see that \mathbf{u}_0 admits the expansion (3.13) in a neighbourhood of each vertex of Ω . Let us firstly show that no vertex singularity appears: for the vertices S_1 and S_{-1} , Ω presents a crack, but fortunately the boundary conditions on the adjacent faces are of Dirichlet type. Owing to Lemma 4.1, we deduce that the strip $\mathfrak{R}(\alpha) \in [-1, 0]$ is free of pole at these vertices. At another vertex S , we use Theorem 4.2 and Remark 4.4 to show the same result since the cone Γ_S has three faces without crack. Except at the bottom of the crack, the dihedral angle along the edges of Ω is equal to $\pi/2$; therefore, owing to Theorem 2.2, we know that no edge singularity occurs in the strip $\mathfrak{R}(\alpha) \in \left] 0, \frac{1}{2} \right]$. It remains the edge singularity of \mathbf{u}_0 near S_1 and S_{-1} corresponding to the exponent $\alpha = \alpha' = \frac{1}{2}$. This proves the expansion (6.7). \square

Actually, we shall now show that the regular part \mathbf{u}_r of the decomposition (6.7) of \mathbf{u} belongs to H^2 in a neighbourhood of the bottom of the crack. This will be useful in part II of this paper.

LEMMA 6.3. *Let \mathcal{V} be a neighbourhood of the bottom of the crack in Ω such that $\overline{\mathcal{V}} \cap \Gamma_2 = \emptyset$. If $\mathbf{U} = (\mathbf{u}, \xi)$ is a solution of (5.9) with data $(\mathbf{f}, g) \in H$, then the regular part \mathbf{u}_r of the decomposition (6.7) of \mathbf{u} fulfils*

$$\mathbf{u}_r \in (H^2(\mathcal{V}))^3.$$

PROOF. It suffices to look at the regularity of \mathbf{u}_0 defined in Proposition 6.1. But, in the neighbourhood \mathcal{V} , \mathbf{u}_0 is solution of a homogeneous Dirichlet problem (for the elasticity system) with data in L^2 . Therefore, owing to Theorem 1.4 of [20] (see also Theorem 17.13 of [3]), \mathbf{u}_0 admits, in this neighbourhood \mathcal{V} , the decomposition (6.7) with \mathbf{u}_r in $(H^2(\mathcal{V}))^3$. Indeed, no new singularity appears:

- a) *Vertex singularities.* The corollary of Theorem 3.2 of [12] implies that for all α in the strip $\mathfrak{R}(\alpha) \in]-2, 1[$, $\mathcal{A}_S(\alpha)^{-1}$ exists, when $S = S_1$ or S_{-1} . So,

there is no vertex singularity in the strip $\Re(\alpha) \in \left] -\frac{1}{2}, \frac{1}{2} \right]$.

- b) *Edge singularities.* Owing to Theorems 2.1.1 and 4.2.1 of [7], edge singularities (in \mathcal{V}) in the strip $]0, 1[$ are only those due to the bottom of the crack and correspond to $\alpha = \alpha' = \frac{1}{2}$.

This proves Lemma 6.3. □

Since ξ is solution of (5.4), if we want to study the behaviour of ξ in ω , it is necessary to know the regularity of $\gamma_+\sigma_{22}(\mathbf{u})$ and $\gamma_-\sigma_{22}(\mathbf{u})$.

PROPOSITION 6.4. *Under the assumption of Theorem 6.2, we have*

$$(6.8) \quad \gamma_+\sigma_{22}(\mathbf{u}) - \gamma_-\sigma_{22}(\mathbf{u}) \in L^p(\Gamma), \quad \forall p \in]1, 2[.$$

PROOF. In view of the expansion (6.7) of \mathbf{u} , it suffices to show that the singular part fulfils (6.8) and by symmetry, only on the support of η_1 . For simplicity, when we shall use spherical coordinates centered at S_1 , we shall drop the index 1. Let us denote

$$(6.9) \quad \mathbf{z} = \sum_{\nu=1,2} \mathcal{R}_1(c_1^\nu) A_1 \begin{pmatrix} 0 \\ \sigma_1^\nu(z_1) \end{pmatrix} + \mathcal{R}_1(c_1) A_1 \begin{pmatrix} \sigma_1(z_1) \\ 0 \\ 0 \end{pmatrix}.$$

Since $\gamma_+\mathbf{z} = \gamma_-\mathbf{z} = \mathbf{0}$, it suffices to check that \mathbf{z} fulfils (6.8) on $\text{supp } \eta_1 \cap \Gamma$. Using the spherical coordinates and the boundary conditions $\gamma_+\mathbf{z} = \gamma_-\mathbf{z} = \mathbf{0}$, we can show that

$$\gamma_+\sigma_{22}(\mathbf{z}) = (\lambda + 2\mu) \frac{1}{r} \frac{\partial z_\theta}{\partial \theta} (\theta = \pi/2^+).$$

In view of the definition of z_θ and of (6.9), we see that

$$z_\theta = \sum_{\nu=1,2} \mathcal{R}_1(c_1^\nu) \sigma_{11}^\nu(z_1),$$

where $\sigma_{11}^\nu(z_1)$ is the first component of the (2-dimensional) vector $\sigma_1^\nu(z_1)$. This leads to

$$(6.10) \quad \gamma_+\sigma_{22}(\mathbf{z}) = \frac{(\lambda + 2\mu)}{r} \sum_{\nu=1,2} \mathcal{R}_1(c_1^\nu) (\theta = \pi/2^+) \frac{\partial \sigma_{11}^\nu}{\partial \theta} (\theta = \pi/2^+).$$

Finally, using the explicit definition of σ_{11}^ν and computing $\partial \sigma_{11}^\nu / \partial \theta$ in the polar coordinates (r_{21}, θ_{21}) , we deduce that there exist two constants k_ν , $\nu = 1, 2$, such that

$$(6.11) \quad \gamma_+\sigma_{22}(\mathbf{z}) = (\lambda + 2\mu) |\varphi - \pi/2|^{-1/2} r^{-1} \sum_{\nu=1,2} k_\nu \mathcal{R}_1(c_1^\nu) (\theta = \pi/2^+).$$

In order to conclude, we shall use the following lemma.

LEMMA 6.5. *Let $c \in H^\varepsilon(\mathbb{R})$, then for all $p \in]1, 2]$, there exists a constant C_p such that*

$$(6.12) \quad \int_0^1 \left| \mathcal{R}_1(c)(r, \theta = \pi/2, \varphi) r^{\frac{1}{p}-1} \right|^p dr \leq C_p.$$

Indeed if (6.12) is true, then

$$\int_0^{\pi/2} \int_0^1 \left| |\varphi - \pi/2|^{-1/2} r^{-1} \mathcal{R}_1(c'_1)(r, \theta = \pi/2, \varphi) \right|^p r dr d\varphi \leq C_p \int_0^{\pi/2} |\varphi - \pi/2|^{-p/2} d\varphi.$$

Since $p < 2$, this last integral is finite and we conclude that

$$\gamma_{+\sigma 22}(\mathbf{z}) \in L^p(\Gamma), \quad \forall p \in]1, 2[.$$

This completes the proof of Proposition 6.4 since clearly

$$\gamma_{-\sigma 22}(\mathbf{z}) = -\gamma_{+\sigma 22}(\mathbf{z}). \quad \square$$

PROOF OF LEMMA 6.5. Let us recall that

$$\mathcal{R}_1(c)(r, \theta, \varphi) = \chi_1(\theta, \varphi) r^\varepsilon (c \star_t \phi)(\ln r, z_1),$$

where χ_1 is a cut-off function with a compact support in a neighbourhood of the bottom of the crack of Ω_1 . Owing to Remark (16.7) of [3], there exist $r_0 > 0$ and $C_1 > 0$ such that if $|z_1| \leq r_0$, then

$$\int_{\mathbb{R}} |\phi(t, z_1)| dt \leq C_1.$$

Moreover, the Sobolev embedding Theorem (see Theorem 1.4.4.1 of [3]) shows that

$$c \in L^q(\mathbb{R}),$$

for q fulfilling $\varepsilon - 1/2 = -1/q$ (notice that $q > 2$). Therefore, Young's Theorem allows us to conclude that for $|z_1| \leq r_0$:

$$(6.13) \quad \|(c \star_t \phi)(\cdot, z_1)\|_{L^q(\mathbb{R})} \leq C_2,$$

for some $q > 2$, where C_2 is independent of z_1 . By the change of variable $r = e^t$, this inequality implies that

$$(6.14) \quad \int_0^1 |r^{-1/q} (c \star_t \phi)(\ln r, \pi/2, \varphi)|^q dr \leq C_2^q,$$

for all $\varphi : |\varphi - \pi/2| \leq r_0$, for some $q > 2$. But it is easily checked that

$$(6.15) \quad r^{\varepsilon + \frac{1}{p} - 1 + \frac{1}{q}} \in L^s((0, 1)),$$

for all $p \in]1, 2]$, when $s > 1$ is defined by $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$. Therefore, (6.14), (6.15) and Hölder's inequality imply that

$$(6.16) \quad \int_0^1 \left| (c \star_t \phi)(\ln r, \pi/2, \varphi) r^{\frac{1}{p} - 1 + \varepsilon} \right|^p dr \leq C_p,$$

for all $p \in]1, 2]$ and all $\varphi : |\varphi - \pi/2| \leq r_0$, where C_p does not depend on φ . Choosing the cut-off function χ_1 such that

$$\chi_1(\theta = \pi/2, \varphi) = 0 \text{ if } |\varphi - \pi/2| > r_0,$$

the inequality (6.16) proves (6.12). □

Now, we are ready to give the regularity of the 2D-part ξ of \mathbf{U} , solution of (5.9).

THEOREM 6.6. *Under the assumption of Theorem 6.2, then ξ is the variational solution of (5.16) and there exists $\varepsilon' > 0$ such that*

$$(6.17) \quad \xi \in H^{7/2+\varepsilon'}(\omega).$$

PROOF. We firstly show that $\xi \in H_0^2(\omega)$ is the solution of (5.16). Indeed, let us fix $\eta \in \mathcal{D}(\omega)$ and define $\mathbf{v} \in (\mathcal{D}(\overline{\Omega}))^3$ by

$$\begin{aligned} v_1 &= v_3 = 0, \\ v_2(x_1, x_2, x_3) &= \eta(x_1, x_3). \end{aligned}$$

Then clearly $\mathbf{V} = (\mathbf{v}, \eta)$ belongs to V . Moreover, there exists $\delta > 0$ such that $\text{supp } \mathbf{v} \subset \Omega \setminus R_\delta$, where we set

$$R_\delta = \{(x_1, x_2, x_3) : -\delta \leq x_1, x_2 \leq \delta, 1 \leq x_3 \leq 1\}.$$

The decomposition (6.7) of \mathbf{u} shows that $\mathbf{u} \in (H^{3/2+\varepsilon}(\Omega \setminus R_\delta))^3$. Therefore, applying Theorem 6.7 hereafter in $\Omega^+ \setminus R_\delta$ and $\Omega^- \setminus R_\delta$ to the pair (\mathbf{u}, \mathbf{v}) and adding the results, we obtain

$$\int_{\Omega} (\mathbf{f}, \mathbf{v}) dx = -a_{\Omega}(\mathbf{u}, \mathbf{v}) + \int_{\Gamma} (\gamma_{-\sigma 22}(\mathbf{u}) - \gamma_{+\sigma 22}(\mathbf{u})) \gamma_+ \overline{v_2} d\sigma.$$

Comparing with (5.9), we see that ξ fulfils (5.16) for all $\eta \in \mathcal{D}(\omega)$. The conclusion follows by density.

Now, (5.16) and Proposition 6.4 show that $\xi \in H_0^2(\omega)$ is the solution of

$$b_\omega(\xi, \eta) = \int_\omega h \bar{\eta} \, dx', \quad \forall \eta \in H_0^2(\omega),$$

for some $h \in L^p(\omega)$, $\forall p \in]1, 2[$. This is precisely a boundary value problem studied in [18] (see also [1] and Paragraph 7 of [6]). Theorem 4.2 of [18] proves that ξ admits a decomposition into a regular part $\xi_0 \in W^{4,p}(\omega)$ and a singular one, which is a linear combination of some functions $\sigma^{\alpha,\nu}$, when $\alpha \in \mathbb{C}$ is solution of

$$(6.18) \quad \sin^2 \left((\alpha - 1) \frac{\pi}{2} \right) = (\alpha - 1)^2,$$

in the strip $\Re(\alpha) \in]1, 4 - 2/p[$ at each vertex of ω , when the line $\Re(\alpha) = 4 - 2/p$ has no solution of (6.18). By Proposition 4 of [17], there is no solution of (6.18) in the strip $\Re(\alpha) \in]1, 5/2 + \varepsilon'[$ for some $\varepsilon' > 0$. Taking $p \in]4/3, 2[$ such that $4 - 2/p = 5/2 + \varepsilon'$, we deduce that

$$\xi \in W^{4,p}(\omega).$$

The Sobolev embedding theorem leads to (6.17). □

THEOREM 6.7. *Under the assumptions of Theorem 5.2, let $\mathbf{u} \in E(L, L^2(\theta)) \cap (H^{3/2+\varepsilon}(\theta))^3$, for some $\varepsilon > 0$. Then for all $\mathbf{v} \in (H^1(\theta))^3$, we have*

$$(6.19) \quad \int_\theta (L\mathbf{u}, \mathbf{v}) \, dx = -a_\theta(\mathbf{u}, \mathbf{v}) + \sum_{k=1}^N \int_{\Gamma_k} (T^{(k)}\mathbf{u}, \gamma_k \mathbf{v}) \, d\sigma.$$

PROOF. Since $\mathcal{D}(\bar{\theta})$ is dense in $H^{3/2+\varepsilon}(\theta)$ (see Theorem 1.4.2.1 of [6]), using the trace theorem of [5] and the Green identity, we obtain (6.19), where the left-hand side is written as a duality bracket between $(H^{-\frac{1}{2}+\varepsilon}(\theta))^3$ and $(H_0^{\frac{1}{2}-\varepsilon}(\theta))^3$, because

$$L\mathbf{u} \in (H^{-1/2+\varepsilon}(\theta))^3 \text{ and } v_i \in H^1(\theta) \hookrightarrow H^{1/2-\varepsilon}(\theta) = H_0^{1/2-\varepsilon}(\theta), \quad i = 1, 2, 3,$$

owing to Theorem 1.4.4.6 and 1.4.2.4 of [6]. The assumption $\mathbf{u} \in E(L, L^2(\theta))$ allows us to replace this duality bracket into an inner product between $L\mathbf{u}$ and \mathbf{v} . □

To finish this paper, let us give some comments about the coupled problem set by [2]. Let us denote

$$\gamma_\omega = \{(x_1, 0, x_3) \in \partial\omega : x_3 = 1\}.$$

The problem of [2] is the following: given $(\mathbf{f}, g) \in H$, let $\mathbf{U} = (\mathbf{u}, \xi)$ be the weak solution of

$$\left\{ \begin{array}{ll} L\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ T\mathbf{u} = \mathbf{0} & \text{on } \Gamma_1 \cup \Gamma_2, \\ \gamma_+ u_2 = \gamma_- u_2 = \xi & \text{on } \Gamma, \\ \gamma_+ u_\alpha = \gamma_- u_\alpha = 0 & \text{on } \Gamma, \alpha = 1, 3, \\ \rho\Delta^2 \xi + \{\gamma_- \sigma_{22}(\mathbf{u}) - \gamma_+ \sigma_{22}(\mathbf{u})\} \chi_\Gamma = g & \text{in } \omega, \\ \xi = \frac{\partial \xi}{\partial \nu} & \text{on } \gamma_0, \\ M(\xi) = N(\xi) = 0 & \text{on } \partial\omega \setminus \gamma_0, \end{array} \right.$$

where the operators M and N are defined in [1] or in [18] for instance. In other words, setting

$$\begin{aligned} \tilde{V} = \{(\mathbf{u}, \xi) \in (H^1(\Omega))^3 \times H^2(\omega) \text{ fulfilling (5.6),} \\ (5.7) \text{ and } \xi = \partial\xi/\partial\nu = 0 \text{ on } \gamma_0\}; \end{aligned}$$

$\mathbf{U} = (\mathbf{u}, \xi)$ belongs to \tilde{V} and fulfils (5.9) for all $\mathbf{V} \in \tilde{V}$.

For this problem, Theorem 5.3 remains almost true, i.e. we can show that (6.20) holds except the last boundary condition! Nevertheless, we can establish a result analogous to Proposition 6.1 and therefore we can give a decomposition into a regular and a singular part for \mathbf{u} . Unfortunately, in that case, we are not able to prove that there is no vertex singularity near the vertices S_1 and S_{-1} . In view of the results of [8] and [19], it seems to be impossible to adapt the Hilbert Uniqueness Method of J.-L. Lions [13] to prove the exact controllability of the associate dynamical problem. Moreover, since we cannot prove that ξ fulfils

$$(6.21) \quad M(\xi) = N(\xi) = 0 \text{ on } \partial\omega \setminus \gamma_0,$$

it is impossible to study the regularity of ξ on ω . In any case, we can show that (6.21) holds in a neighbourhood of the vertices S_3 and S_4 of ω , when S_3 and S_4 are the extremities of γ_0 . By numerical computations (see [1], for instance), we can show that ξ has vertex singularities near S_3 and S_4 for some reasonable Lamé coefficients (remark that ξ fulfils a mixed boundary condition near S_3 and S_4). So we have the same problem for the exact controllability.

Let us notice that other boundary conditions on Γ_1 , Γ_2 or $\partial\omega$ lead to analogous problems.

On the contrary with the coupled problem studied in [19], we do not impose Dirichlet boundary conditions on all of $\partial\Omega \setminus \bar{\Gamma}$ in order to avoid the splitting phenomenon of [19].

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