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Classe di Scienze

KEWEI ZHANG

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1. - Introduction

In this paper we develop a method for constructing nontrivial quasiconvex functions with p-th growth at infinity from known quasiconvex functions. The main result is the following:

THEOREM 1.1. *Suppose that the continuous function* $f : M^{N\times n} \to \mathbb{R}$ *is quasiconvex in the sense of Morrey (cf.* [17], *also see Definition* 2.1*) and that for some real constant* α, *the level set*

$$K_\alpha := \{P \in M^{N\times n} : f(P) \leq \alpha\}$$

is compact. Then, for every $1 \leq q < +\infty$, *there is a continuous quasiconvex function* $g_q \geq 0$, *such that*

$$-C_1 + c|P|^q \leq g_q(P) \leq C_1 + C_2|P|^q \tag{1.1}$$

and

$$K_\alpha = \{P \in M^{N\times n} : g_q(P) = 0\} \tag{1.2}$$

where $C_1 \geq 0$, $c > 0$, $C_2 > 0$ *are constants.*

When the level set K_α of some quasiconvex function is unbounded, we establish the following result for a compact subset of K_α under an additional assumption.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.1 *without assuming that* K_α *is compact, for any compact subset* $H \subset K_\alpha$, *satisfying*

$$K_\alpha \cap (\text{conv } H \backslash H) = \emptyset,$$

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and $1 \leq q < \infty$, *there exists a non-negative quasiconvex function* g_q *satisfying* (1.1) *and with* H *as its zero set:*

$$H = \{P \in M^{N\times n} : g_q(P) = 0\}.$$

(For relevant notations and definitions, see Section 2 *below).*

With these results we can construct a rich class of quasiconvex functions with linear growth, for example, function with the two-point set $\{A, B\}$ being its zero set provided that rank $A - B \neq 1$. However for sets like $SO(n)$, we need much deeper result to cope with (see Theorem 4.1 below). These sets are important in the study of quasiconformal mappings (Reshetnyak [18], [19]) and phase transitions (Kinderlehrer [15], Ball and James [8]). Also, we can establish connection between these results and Tartar's conjecture on sets without rank-one connections (see Section 4). We prove that for any compact subset $K \subset R_+SO(n)$ we can construct quasiconvex functions with K as its zero set and with prescribed growth at infinity.

The basic idea for proving the main result is to apply maximal function method developed by Acerbi and Fusco [1] in the study of weak lower semicontinuity for the calculus of variations and an approximating result motivated by a work of V. Šverák on two-dimensional two-well problems with linear growth.

In Section 2, notation and preliminaries are given which will be used in the proof of the main result. In Section 3, we prove Theorem 1.1 while in Section 4 we study the relation between Tartar's conjecture and our basic constructions and give some examples to show how quasiconvex functions with linear growth and non-convex zero sets can be constructed.

2. - Notation and preliminaries

Throughout the rest of this paper Ω denotes a bounded open subset of $\mathbb{R}^n$. We denote by $M^{N\times n}$ the space of real $N \times n$ matrices, with norm $|P| = (\operatorname{tr} P^T P)^{1/2}$. We write $C_0(\Omega)$ for the space of continuous functions $\phi : \Omega \to \mathbb{R}$ having compact support in Ω, and define $C_0^1(\Omega) = C^1(\Omega) \cap C_0(\Omega)$. If $1 \leq p \leq \infty$ we denote by $L^p(\Omega; \mathbb{R}^N)$ the Banach space of mappings $u : \Omega \to \mathbb{R}^N$, $u = (u_1, \cdots, u_N)$, such that $u_i \in L^p(\Omega)$ for each i, with norm $||u||_{L^p(\Omega;\mathbb{R}^N)} = \sum_{i=1}^{N} ||u_i||_{L^p(\Omega)}$. Similarly, we denote by $W^{1,p}(\Omega; \mathbb{R}^N)$ the usual Sobolev space of mappings $u \in L^p(\Omega; \mathbb{R}^N)$ all of whose distributional derivatives $\dfrac{\partial u_i}{\partial x_j} = u_{i,j}$, $1 \leq i \leq N$, $1 \leq j \leq n$, belong to $L^p(\Omega)$. $W^{1,p}(\Omega; \mathbb{R}^N)$ is a Banach space under the norm

$$||u||_{W^{1,p}(\Omega;\mathbb{R}^N)} = ||u||_{L^p(\Omega;\mathbb{R}^N)} + ||Du||_{L^p(\Omega;M^{N\times n})},$$

where $\mathrm{D}u = (u_{i,j})$, and we define, as usual, $W_0^{1,p}(\Omega;\mathbb{R}^N)$ the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in the topology of $W^{1,p}(\Omega;\mathbb{R}^N)$.

Weak and weak $*$ convergence of sequences are written $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ respectively. The convex hull of a compact set K in $M^{N\times n}$ is denoted by $\operatorname{conv} K$. If $H \subset M^{N\times n}$, $P \in M^{N\times n}$, we write $H+P$ the set $\{P+Q : Q \in H\}$. We define the distance function for a set $K \subset M^{N\times n}$ by

$$f(P) = \operatorname{dist}(P,K) := \inf_{Q\in K} |P-Q|.$$

DEFINITION 2.1 (see Morrey [17], Ball [3, 4], Ball, Currie and Olver [7]). *A continuous function $f : M^{N\times n} \to \mathbb{R}$ is* quasiconvex *if*

$$\int_U f(P+\mathrm{D}\phi(x))\,\mathrm{d}x \geq f(P)\,\mathrm{meas}(U)$$

for every $P \in M^{N\times n}$, $\phi \in C_0^1(U;R^N)$, and every open bounded subset $U \subset \mathbb{R}^n$.

To construct quasiconvex functions, we need the following

DEFINITION 2.2 (see Dacorogna [11]). *Suppose $f : M^{N\times n} \to \mathbb{R}$ is a continuous function. The quasiconvexification of f is defined by*

$$\sup\{g \leq f; g \text{ quasiconvex}\}$$

and will be denoted by Qf.

PROPOSITION 2.2 (see Dacorogna [11]). *Suppose $f : M^{N\times n} \to \mathbb{R}$ is continuous, then*

$$Qf(P) = \inf_{\phi\in C_0^\infty(\Omega;\mathbb{R}^N)} \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega f(P+\mathrm{D}\phi(x))\,\mathrm{d}x, \tag{2.1}$$

where $\Omega \subset R^n$ is a bounded domain. In particular the infimum in (2.1) *is independent of the choice of Ω.*

We use the following theorem concerning the existence and properties of Young measures from Tartar [22]. For results in a more general context and proofs the reader is referred to Berliocchi and Lasry [10], Balder [2] and Ball [6].

THEOREM 2.4. *Let $z^{(j)}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^s)$. Then there exist a subsequence $z^{(\nu)}$ of $z^{(j)}$ and a family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^s$, depending measurably on $x \in \Omega$, such that*

$$f(z^{(\nu)}) \overset{*}{\rightharpoonup} \langle \nu_x, f(\cdot)\rangle \qquad \text{in } L^\infty(\Omega)$$

for every continuous function $f : \mathbb{R}^s \to \mathbb{R}$.

Let $r > 0$ and $x \in \mathbb{R}^n$, set $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and

$$\text{meas}\,(B(x,r)) = \omega_x \times r^n.$$

DEFINITION 2.5 (The Maximal Function). *Let* $\in C_0^\infty(\mathbb{R}^n)$, *we define*

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^{n} (Mu_{,\alpha})(x)$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B(x,r)} |f(y)| \mathrm{d}y$$

for every locally summable f, *where* ω_n *is the volume of the* n *dimensional unit ball.*

LEMMA 2.6 (cf. [20, p. 5, Th. 1.(b)]). *If* $f \in L^1(\mathbb{R}^n)$, *then for every* $\lambda > 0$

$$\text{meas}(\{x \in \mathbb{R}^n : (Mf)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f|\, \mathrm{d}x.$$

LEMMA 2.7. *If* $u \in C_0^\infty(\mathbb{R}^n)$, *then* $M^*u \in C^0(\mathbb{R}^n)$ *and*

$$|u(x)| + \sum_{\alpha=1}^{n} |u_{,\alpha}| \leq (M^*u)(x)$$

for all $x \in \mathbb{R}^n$. *Moreover (see* [20]*) if* $p > 1$, *then*

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leq c(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

and if $p \geq 1$, *then*

$$\text{meas}(\{x \in R^n : (M^*u)(x) \geq \lambda\}) \leq \frac{c(n,p)}{\lambda^p} \|u\|^p_{W^{1,p}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

LEMMA 2.8 (*see* [1]). *Let* $u \in C_0^\infty(\mathbb{R}^n)$ *and* $\lambda > 0$, *and set*

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}.$$

Then for every x, $y \in H^\lambda$ *we have*

$$\frac{|u(x) - u(y)|}{|x - y|} \leq C(n)\lambda.$$

LEMMA 2.9. *Let X be a metric space, E a subspace of X, and k a positive real number. Then any k-Lipschitz mapping from E into $\mathbb{R}$ can be extended to a k-Lipschitz mapping from X into $\mathbb{R}$.*

For the proof see [13, p. 298].

3. - Construction of quasiconvex functions

In this section we prove Theorem 1.1. The following lemma is crucial in the prove of the theorem.

LEMMA 3.1. *Suppose $u_j \rightharpoonup 0$ in $W_0^{1,1}(\Omega;\mathbb{R}^N)$ and there is $K > 0$ such that*

$$\int_{\Omega\cap\{|\mathrm{D}u_j|\geq K\}} |\mathrm{D}u_j|\,\mathrm{d}x \to 0 \qquad \text{as } j \to \infty. \tag{3.1}$$

Then there exists a bounded sequence g_j in $W^{1,\infty}(\Omega;\mathbb{R}^N)$ such that

$$\int_{\Omega} |\mathrm{D}u_j - \mathrm{D}g_j|\,\mathrm{d}x \to 0 \qquad \text{as } j \to \infty. \tag{3.2}$$

PROOF. For each fixed j, extend u_j by zero outside Ω so that it is defined on $\mathbb{R}^n$. Since $C_0^\infty(\mathbb{R}^n,\mathbb{R}^N)$ is dense in $W_0^{1,1}(\mathbb{R}^n;\mathbb{R}^N)$, there exists a sequence w_j in $C_0^\infty(\mathbb{R}^n;\mathbb{R}^N)$ such that

$$\|u_j - w_j\|_{W_0^{1,1}(\mathbb{R}^n;\mathbb{R}^N)} < \frac{1}{j},$$

and

$$\int_{\{x\in\mathbb{R}^n:|\mathrm{D}w_j|\geq 2K\}} |\mathrm{D}w_j(x)|\,\mathrm{d}x \to 0$$

as $j \to \infty$, so that we can assume that $u_j \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^N)$.

For each fixed j, i, define

$$H_{i,j}^\lambda = \{x \in \mathbb{R}^n : (M^* u_j^i)(x) < \lambda\}, \qquad H_j^\lambda = \bigcap_{i=1}^N H_{ij}^\lambda, \qquad \lambda \geq 4nK.$$

Lemma 2.8 ensures that for all x, $y \in H_j^\lambda$,

$$\frac{|u_j^i(x) - u_j^i(y)|}{|y - x|} \leq C(n)\lambda.$$

Let g_j^i be a Lipschitz function extending u_j^i outside H_j^λ with Lipschitz constant not greater than $C(n)\lambda$ (Lemma 2.9). Since H_j^λ is an open set, we have

$$g_j^i(x) = u_j^i(x), \qquad \mathrm{D}g_j^i(x) = \mathrm{D}u_j^i(x)$$

for all $x \in H_j^\lambda$ and

$$\|\mathrm{D}g_j^i\|_{L^\infty(\mathbb{R}^n)} \leq C(n)\lambda.$$

We may also assume

$$\|g_j^i\|_{L^\infty} \leq \|z_j^i\|_{L^\infty(H_j^\lambda)} \leq C(n)\lambda$$

and set $g_j^i = (g_j^1, \dots, g_j^N)$.

In order to prove that $u_j - g_j \to 0$ strongly in $W^{1,1}(\Omega; \mathbb{R}^N)$, we have

$$\int_\Omega |\mathrm{D}u_j - \mathrm{D}g_j| \, \mathrm{d}x \leq \int_{\Omega \setminus H_j^\lambda} (|\mathrm{D}u_j| + |\mathrm{D}g_j|) \, \mathrm{d}x. \tag{3.3}$$

Hence the left hand side of (3.2) tends to zero provided that

$$\operatorname{meas}(\Omega \setminus H_j^\lambda) \to 0.$$

From the definition of H_{ij}^λ, we have

$$\Omega \setminus H_{ij}^\lambda \subset \{x \in \Omega : (Mu_j^i)(x) \geq \lambda/2\} \cup \left\{ x \in \Omega : \sum_{\alpha=1}^n \left(M \frac{\partial u_j^i}{\partial x_\alpha} \right)(x) \geq \lambda/2 \right\},$$

and

$$\left\{ x \in \mathbb{R}^n : \sum_{\alpha=1}^n (Mu_{j,\alpha}^i)(x) \geq \lambda/2 \right\} \subset \bigcup_{\alpha=1}^n \left\{ x \in \mathbb{R}^n : (Mu_{j,\alpha}^i)(x) \geq \frac{\lambda}{2n} \right\}.$$

Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(s) = \begin{cases} 0 & \text{as } |s| < K, \\ |s| - K & \text{as } |s| \geq K, \end{cases}$$

so that we can prove that

$$\left\{ x \in \mathbb{R}^n : (Mu_{j,\alpha}^i)(x) \geq \frac{\lambda}{2n} \right\} \subset \left\{ x \in \mathbb{R}^n : (Mh(\mathrm{D}u_j^i))(x) \geq \frac{\lambda}{2n} - K \right\}. \tag{3.4}$$

In fact, when $Mu_{j,\alpha}^i(x) \geq \dfrac{\lambda}{2n}$, we have a sequence of $\epsilon_k > 0$, $\epsilon \to 0$ and a sequence of balls $B_k = B(x, \mathbb{R}_k)$ such that

$$\frac{1}{\mathrm{meas}(B_k)} \int_{B_k} |u^i_{j,\alpha}|\,\mathrm{d}x \geq \frac{\lambda}{2n} - \epsilon_k$$

which implies

$$\begin{aligned} Mh(\mathrm{D}u^i_j) &\geq \frac{1}{\mathrm{meas}(B_k)} \int\limits_{B_k \cap \{x : |\mathrm{D}u^i_j(x)| \geq K\}} (|\mathrm{D}u^i_j| - K)\,\mathrm{d}x \\ &\geq \frac{\lambda}{2n} - \frac{1}{\mathrm{meas}(B_k)} \int\limits_{B_k \cap \{x : |\mathrm{D}u^i_j| \leq K\}} |u^i_{j,\alpha}|\,\mathrm{d}x \\ &\quad - \frac{1}{\mathrm{meas}(B_k)} \int\limits_{B_k \cap \{x : |\mathrm{D}u^i_j(x)| \geq K\}} K\,\mathrm{d}x - \epsilon_k \geq \frac{\lambda}{2n} - K - \epsilon_k. \end{aligned} \tag{3.5}$$

Passing to the limit $k \to \infty$ in (3.5), we obtain (3.4) (here we choose $\frac{\lambda}{2n} > K$).

From Lemma 2.6, we have

$$\begin{aligned} &\mathrm{meas}\left(\left\{x \in \mathbb{R}^n : (Mh(\mathrm{D}u^i_j))(x) \geq \frac{\lambda}{2n} - K\right\}\right) \\ &\quad \leq \frac{1}{\frac{\lambda}{2n} - K} \int_{\mathbb{R}^n} |h(\mathrm{D}u^i_j)|\,\mathrm{d}x \leq \frac{1}{\frac{\lambda}{2n} - K} \int\limits_{\{x \in \Omega : |\mathrm{D}u^i_j| \geq K\}} |\mathrm{D}u^i_j|\,\mathrm{d}x \\ &\quad \leq \frac{1}{\frac{\lambda}{2n} - K} \int\limits_{\{x \in \Omega : |\mathrm{D}u_j| \geq K\}} |\mathrm{D}u_j|\,\mathrm{d}x \to 0 \end{aligned}$$

as $j \to \infty$. Also, from Lemma 2.6, together with the embedding theorem, we have

$$\mathrm{meas}(\{x \in \mathbb{R}^n : (Mu^i_j)(x) \geq \lambda/2\}) \leq \frac{1}{\lambda/2} \int_\Omega |u^i_j|\,\mathrm{d}x \to 0,$$

as $j \to \infty$, so that we conclude that

$$\mathrm{meas}(\Omega \backslash H^\lambda_j) \to 0 \qquad \text{as } j \to \infty.$$

□

PROOF OF THEOREM 1.1. It is easy to see that the function

$$F(P) = \max\{0, f(P) - \alpha\}$$

is quasiconvex and satisfies assumption (1), with zero set

$$\{x \in M^{n\times N} : F(P) = 0\} = K_\alpha.$$

Define

$$\tilde{f}_\alpha = \operatorname{dist}(P; K_\alpha)$$

and

$$G(P) = Q\tilde{f}_\alpha.$$

We seek to prove that $G(P) = 0$ if and only if $P \in K_\alpha$. By definition of quasiconvexification of $\tilde{f}_\alpha$, G is zero on K_α. Conversely, suppose $G(P) = 0$, i.e.,

$$0 = G(P) = \inf_{\phi \in C_0^\infty(B;\mathbb{R}^N)} \frac{1}{\operatorname{meas}(B)} \int_B \tilde{f}_\alpha(P + \mathrm{D}\phi)\,\mathrm{d}x$$

for a ball $B \subset \mathbb{R}^n$, we have a sequence $\phi_j \in C_0^\infty(\Omega;\mathbb{R}^N)$ such that for $K \geq 2$ dist$(P; K_\alpha)$,

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_B \operatorname{dist}(P + \mathrm{D}\phi_j, K_\alpha)\,\mathrm{d}x \\ &\geq \lim_{j\to\infty} \int_{B\cap\{x\in\Omega : |\mathrm{D}\phi_j(x)|\geq K\}} [|\mathrm{D}\phi_j| - \operatorname{dist}(P; K_\alpha)]\,\mathrm{d}x \\ &\geq \lim_{j\to\infty} K/2 \operatorname{meas}(\{x \in \Omega : |\mathrm{D}\phi_j(x)| \geq K\}), \end{aligned}$$

hence

$$\int_{\{x\in\Omega : |\mathrm{D}\phi_j|\geq K\}} |\mathrm{D}\phi_j|\,\mathrm{d}x \to 0$$

as $j \to \infty$ and $(|\mathrm{D}\phi_j|)$ are equi-integrable on Ω with respect to j. Then, by a vector-valued version of the Dunford-Pettis theorem (A & C. Ionescu Tulcea [14, p. 117], Diestel and Uhl [12, p. 101, 76]) there exists a subsequence (still denoted by ϕ_j) which converges weakly in $W_0^{1,1}(B;\mathbb{R}^N)$ to a function ϕ. Moreover, by an argument of Tartar [22], and the embedding theorems, $\mathrm{D}\phi(x) \in \operatorname{conv} K_\alpha$ for a.e. $x \in B$, so that $\phi \in W_0^{1,\infty}(B;\mathbb{R}^N)$.

Define $\psi_j = \phi_j - \phi$. Then ψ_j satisfies all assumptions of Lemma 3.1. Hence there exists a bounded sequence $g_j \in W^{1,\infty}(B;\mathbb{R}^N)$, such that

$$\int_B |\mathrm{D}\psi_j - \mathrm{D}g_j|\,\mathrm{d}x \to 0, \qquad g_j \overset{*}{\rightharpoonup} 0 \qquad \text{in } W^{1,\infty}(B;\mathbb{R}^N),$$

as $j \to \infty$. Let $\{\nu_x\}_{x \in B}$ be the family of Young measures corresponding to the sequence $\mathrm{D}g_j$ (up to a subsequence), we have

$$\begin{aligned}
&\limsup_{j\to\infty} \int_B \tilde{f}_\alpha(P + \phi + \mathrm{D}g_j)\,\mathrm{d}x \\
(3.6)\qquad &\leq \lim_{j\to\infty} \int_B |\mathrm{D}\psi_j - \mathrm{D}g_j|\,\mathrm{d}x + \lim_{j\to\infty}\int_B \tilde{f}_\alpha(P + \mathrm{D}\phi + \mathrm{D}\psi_j)\,\mathrm{d}x = 0
\end{aligned}$$

which implies

$$\int_B \langle \nu_x, \tilde{f}_\alpha(P + \mathrm{D}\phi(x) + \lambda)\rangle\,\mathrm{d}x = 0$$

which further implies

$$\operatorname{supp}\nu_x \subset K_\alpha - P - \mathrm{D}\phi(x) \text{ for a.e. } x \in B. \tag{3.7}$$

Since $g_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(B;\mathbb{R}^N)$, by Ball and Zhang [9, Th. 2.1], and (3.7), up to a subsequence, we have

$$0 = F(P + \mathrm{D}\phi + \mathrm{D}g_j) \overset{*}{\rightharpoonup} \langle \nu_x, F(P + \mathrm{D}\phi(x) + \lambda)\rangle \geq F(P + \mathrm{D}\phi(x))$$

for a.e. $x \in B$, as $j \to \infty$. By the definition of quasiconvex functions, we have

$$0 = \int_B F(P + \mathrm{D}\phi(x))\,\mathrm{d}x \geq F(P)\operatorname{meas}(B)$$

which implies $F(P) = 0$, $P \in K_\alpha$.

Now, for $q > 1$, define

$$g_q(P) = \max\{[\operatorname{dist}(P, \operatorname{conv} K_\alpha)]^q, Q\operatorname{dist}(P, K_\alpha)\}.$$

It is easy to see that g_q satisfies (1.1) and (1.2). □

PROOF OF COROLLARY 1.2. As in the proof of Theorem 1.1, firstly we have $P \in K$, $\operatorname{supp}\nu_x \subset H - P - \mathrm{D}\phi(x)$ for a.e. $x \in B$, so that $P + \mathrm{D}\phi(x) \in \operatorname{conv} H \cap K$. Hence,

$$\frac{1}{\operatorname{meas} B}\int_B (P + \mathrm{D}\phi(x))\,\mathrm{d}x = P \in \operatorname{conv} H \cap K = K.$$

For $q > 1$, similar argument as above works. □

4. - Tartar's conjecture and some examples

With Theorem 1.1 in hand, we can study the connection between out constructions and Tartar's conjecture on oscillations of gradients (cf. Tartar [23], Ball [5]).

TARTAR'S CONJECTURE. *Let $K \subset M^{N\times n}$ be closed and have no rank-one connections, i.e. for every A, $B \in K$,* $\operatorname{rank}(A-B) \neq 1$. *Let z_j be a bounded sequence in $W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N)$ and the Young measures (ν_x) associated with $\mathrm{D}z_j$ satisfies $\nu_x \subset K$, and such that $f(\mathrm{D}z_j)$ is weak-$*$ convergent in $L^\infty(\mathbb{R}^n)$ for every continuous $f : M^{N\times n} \to \mathbb{R}$. Then (ν_x) is a Dirac mass.*

The answer of this conjecture is, in general, negative (cf. Ball [5]). However, there are a number of cases when Tartar's conjecture is known to be true for gradients under supplementary hypotheses on the set K.

(i) $K_1 = \{A, B\}$ with $\operatorname{rank}(A-B) > 1$ (Ball and James [8]),

(ii) $n = N > 1$, $K = SO(n)$ (Kinderlehrer [15]). In fact, more generally, (see Ball [5]), for $n > 1$ and

$$K_2 = \{tR : t \geq 0, R \in SO(n)\} := R_+SO(n).$$

Based on these examples, we can use similar argument as in the proof of Theorem 1.1 to prove the following

THEOREM 4.1. *Suppose $K \subset M^{N\times n}$ has no rank-one connections and Tartar's conjecture is known to be true for K. Moreover, for any bounded $Q^{1,\infty}$ sequence with Young measures $\nu_x \subset K$ has the property that $\nu_x = \delta_T$ with T a constant matrix in K ($T = \langle \nu_x, \lambda \rangle$). Then for any non-empty compact subset $H \subset K$, any $1 \leq p < \infty$, there exist a continuous quasiconvex function $f \geq 0$, such that*

(i) $c(p)|P|^p - C(P) \leq f(P) \leq C_1(p)(1+|P|)^p$, *with* $c(p)$, $C_1(p) > 0$, $C(p) \geq 0$;

(ii) $\{P \in M^{N\times n} : f(P) = 0\} = H$.

REMARK 4.2. In the case $K = K_1$, Kohn [16] constructs a quasiconvex function with the above properties when $p = 2$ and n, $N > 1$ arbitrary; Šverák [21] does the same in the case $p \geq 1$, $n = N = 2$.

PROOF OF THEOREM 4.1. We employ a similar argument as that of Theorem 1.1.

Firstly, we construct a quasiconvex function with linear growth. Define as before

$$G(P) = \operatorname{dist}(P, H) \qquad \text{and } f(P) = QG(P)$$

and assume that

$$f(P) = \inf_{\phi \in C_0^\infty(B;\mathbb{R}^N)} \int_B G(P + \mathrm{D}\phi)\,\mathrm{d}x = 0$$

to derive a sequence $\phi_j \rightharpoonup \phi$ in $W_0^{1,1}(B;\mathbb{R}^N)$ with $\phi \in W_0^{1,\infty}(B;\mathbb{R}^N)$. In fact, we can assume $\phi_j \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^N)$ and $\phi \in W_0^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N)$ supported in B. It is easy to see that $\mathrm{D}\phi_j$ converges in measure to the set $H - P$. Let g_j be the approximate sequence in $W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N)$, we have the Young measures $(\nu_x)_{x\in\mathbb{R}^n}$ associated with $\mathrm{D}g_j$ satisfy $\operatorname{supp}\nu_x \subset H - P - \mathrm{D}\phi(x)$ for a.e. $x \in \mathbb{R}^n$. Therefore, the Young measures associated with $P + \mathrm{D}\phi(x) + \mathrm{D}g_j(x)$ will be supported in $H \subset K$, so that from the assumption, they are the same Dirac measure. Since $\langle \nu_x, \lambda\rangle = \mathrm{D}g(x) = 0$, $P + \mathrm{D}\phi(x) = \text{constant} \in H$. Therefore $\phi = 0$ a.e. and $P \in H$.

EXAMPLE 4.3. Let $K_1 = \{A, B\}$ with A, $B \in M^{N\times n}$ and assume that rank $(A-B) > 1$. It is known (Kohn [16]) that there exists a non-negative quasiconvex function f with quadric growth, such that

$$\{P \in M^{N\times n} : f(P) = 0\} = \{A, B\}.$$

From Theorem 1.1, the zero set of the quasiconvex function with linear growth $Q\,\mathrm{dist}(P; K_1)$ should be K_1.

EXAMPLE 4.4. Let $K_2 = \{P = tQ : t \geq 0, Q \in SO(n)\} = R_+SO(n)$ and let H be any non-empty compact subset of K_2. Then, we can apply Theorem 4.1 and a result due to Reshetnyak [18], [19] to show that

$$\{P \in M^{n\times n} : Q\,\mathrm{dist}(P, H) = 0\} = H.$$

Here we employ the approach based on an argument of Ball [5]. Following the proof of Theorem 1.1, the Young measures $\{\nu_x\}_{x\in B}$ associated with $\mathrm{D}g_j$ are supported in $H - P - \mathrm{D}\phi(x)$ for a.e. $x \in B$. Let us consider the quasiconvex function (see, e.g. Ball [5])

$$F(P) = |P|^n - n^{n/2}\det P$$

which is non-negative and has K_2 as its zero set. We have

$$\begin{aligned}
0 &= \liminf_{j\to\infty} \int_B F(P + \mathrm{D}\phi + \mathrm{D}g_j)\,\mathrm{d}x \\
&= \int_B \langle \nu_x, F(P + \mathrm{D}\phi(x) + \lambda)\rangle\,\mathrm{d}x \geq \int_B F(P + \mathrm{D}\phi(x))\,\mathrm{d}x \geq F(P)\,\mathrm{meas}(B).
\end{aligned}$$

Since the function $|\cdot|^n$ is strictly convex and

$$\int_B \langle \nu_x, \det(P + D\phi(x) + \lambda)\rangle \, dx = \int_B \det(P + D\phi(x))\, dx = \det P \operatorname{meas}(B),$$

we have

$$\int_B \langle \nu_x, |P + D\phi(x) + \lambda|^n\rangle \, dx = \int_B |P + D\phi(x)|^n \, dx = |P|^n \operatorname{meas}(B)$$

which implies

$$\nu_x = \delta_0, \qquad \text{and} \qquad D\phi(x) = 0 \qquad \text{a.e.}$$

so that $P \in H$.

REMARK 4.5. Since any non-empty compact subset of $R_+SO(n)$ can be the zero set of some non-negative quasiconvex function, the topology of zero sets for quasiconvex functions can be very complicated. For example, let K be any compact subset of $\mathbb{R}^2$, define

$$K_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : (a, b) \in K \right\},$$

then $K_1 \subset R_+SO(2)$ and has the same topology as K.

REMARK 4.6. The method used in the proof of Theorem 1.1 depends heavily on the compactness of the level set K_α. I do not know, for example, whether the function $Q\,\mathrm{dist}(P, R_+SO(n))$ has $R_+SO(n)$ as its zero set or not.

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School of Mathematics, Physics,
Computing and Electronics Macquarie University
North Ryde, NSW 2109
Australia