

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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FRANÇOIS DRESS

Sums of 19 biquadrates : on the representation of large integers

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 19,
n° 1 (1992), p. 113-153

http://www.numdam.org/item?id=ASNSP_1992_4_19_1_113_0

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Sums of 19 Biquadrates: On the Representation of Large Integers

JEAN-MARC DESHOUILLEERS* - FRANÇOIS DRESS

We give a detailed proof of the following result, announced in [4] by Balasubramanian, Deshouillers and Dress.

THEOREM. *Every integer larger than 10^{367} is a sum of 19 biquadrates.*

In the first edition of his *Meditationes Algebraicae*, in 1770, Waring¹ states that every natural integer is a sum of at most 4 squares, 9 cubes, 19 biquadrates. A few years later, he adds that a similar statement holds for higher powers. Those “statements” would be phrased nowadays as “conjectures”, and we refer to them as “Waring’s problem”.

The case of the squares was solved by Lagrange in 1770, and that of the cubes by Wieferich and Kempner in 1909-1912.

After Liouville proved around 1859 that every integer is the sum of at most 53 biquadrates, there have been different numerical improvements of his result, as well as extensions to some higher powers. Hilbert gave in 1909 the first proof of the fact that for any $k \geq 1$ there exists an integer s such that every integer is the sum of at most s perfect k^{th} powers. (We denote by $g(k)$ the least admissible such s). His method does not lead to the determination of $g(k)$.

Hardy and Ramanujan introduced in 1917 a powerful analytical method for dealing with additive problems: the circle method, which they originally applied to the study of partitions. Shortly afterwards, Hardy and Littlewood gave a new proof of Hilbert’s theorem: typically, their method leads to the existence of effectively computable constants $s(k)$ and $N(k)$ such that every integer larger than $N(k)$ is a sum of $s(k)$ perfect k^{th} powers. In the case of biquadrates, they showed in 1925 that $s(4) = 19$ is an admissible value [17],

* This research has been supported in part by the Institute for Advanced Study, Princeton, NJ 08540.

¹ References to the papers which appeared before 1920 are to be found in Dickson’s *History of the Theory Numbers*.

Pervenuto alla Redazione il 26 Luglio 1991.

opening the route towards the solution of Waring's problem for biquadrates. Auluck [1] proved in 1940 that every integer larger than $\exp \exp(204)$ is a sum of 19 biquadrates.

Let us shortly describe the status of higher powers. J.A. Euler observed around 1772 that $g(k)$ is at least $2^k + [(3/2)^k] - 2$, by considering the integer $[(3/2)^k] \cdot 2^k - 1$. In 1934, Vinogradov [26] succeeded in obtaining by the circle method, admissible values of $s(k)$ not exceeding $2^k + [(3/2)^k] - 2$, and he even proved [27] that one may take $s(k)$ as small as $O(k \log k)$. This means that the larger k is, the easier is the determination of $g(k)$, this number being controlled by the representation of small integers. Indeed, in 1936, Dickson [16] and Pillai [21] independently gave the expression of $g(k)$ for k between 7 and a few hundreds, and their work, completed by Rubugunday and Niven eventually led in 1944 to the complete determination of $g(k)$ for $k \geq 6$ (the complexity of the computation of $g(k)$ is discussed by Delmer and Deshouillers in [10]).

In 1938, Davenport [8] introduced a new idea which turned out to be specially efficient for small k 's. For biquadrates [9], his method relies on the direct consideration of numbers which are sums of 4 biquadrates, counted each with weight one, regardless of their number of representations under the given form. He proved that every sufficiently large integer which is not congruent to 0 or 15 modulo 16 is a sum of at most 14 biquadrates. The difficulty in using Davenport's method for actual computation is its dependence on the maximum of divisor-like functions; one can figure out the importance of such terms by noticing that the maximum of the usual divisor function $d(n)$ for n up to 10^{100} is larger than 10^{15} , whereas it is a well known fact that $d(n) = O_\varepsilon(n^\varepsilon)$ for any positive ε . This difficulty delayed the solution of Waring's problem for 5th powers to 1964, when Chen Jing-run [7] proved that $g(5) = 37$. For biquadrates, Davenport's method has led Thomas [23] to prove in 1974 that $g(4) \leq 22$, and Balasubramanian [2] and [3] to announce $g(4) \leq 21$ in 1979 and $g(4) \leq 20$ in 1985. (Those two latter papers rely on a numerical announcement by Thomas which has been subject to criticism [12], but is not used by Thomas in his proof of $g(4) \leq 22$). The second article [3] contains an innovative idea in the treatment of Weyl's trigonometrical sums which allows his author to show that every integer larger than 10^{700} is a sum of 19 biquadrates, whereas Thomas obtained the bound 10^{1409} .

That same year 1938, Hua [18] introduced another idea, which enabled him to give an asymptotic estimate for the number of representations of an integer as a sum of $2^k + 1$ perfect k -th powers. Apparently, Hua's method has the same drawback as that of Davenport, in that they both depend on estimates for the maximum of divisor-like functions; on the other hand, Hua's method can lead only to the existence of representations of large integers as sums of 17 biquadrates, whereas Davenport's method is efficient already with sums of 14 biquadrates. The first named author of this paper noticed however that the maximum of the implied divisor function may be replaced by an average of those values. The reader should remember that, the maximum of $d(n)$ of $n \leq 10^{100}$ is at least 10^{15} , whereas the mean-value of $d(n)$ over the same range

is less than 231. Although the averaging is performed over the values of a polynomial, this remark led to the proof that every integer less than 10^{625} is a sum of 19 biquadrates [11].

The two innovations introduced in [3] and [11] fortunately turned out to be compatible: when combined, they imply that every integer larger than 10^{530} is a sum of 19 biquadrates. The remaining gap would, still now, be hard to cover by numerical computations, if only possible.

The last main ingredient was a careful study of the divisor sum implied in the modified Hua's method. We succeeded [13] in that way to reduce the asymptotic bound beyond which every integer is a sum of 19 biquadrates, to a size, 10^{400} , which was within the range of possible computation. At the same time, the second named author noticed an improvement in Balasubramanian's method: the asymptotic bound fell below 10^{367} , well within the range of computation; we indeed showed at that time that every integer less than 10^{378} is a sum of 19 biquadrates: Waring was right!

Although we have now extended our computation [15] to 10^{448} , making it possible to relax some of the arguments in the asymptotic part of the proof, we think that it may have some historical interest to provide a complete, almost self-contained, and accessible proof of our main theorem, following the lines presented in [4] and [5]. This is made possible through this paper and the articles [13] and [14].

Davenport has shown that only finitely many integers cannot be expressed as sum of 18 biquadrates. Under a strong numerical evidence, one conjectures that only 79, 159, 239, 319, 399, 479 and 559 require as many as 19 biquadrates, and it seems desirable to substantiate this statement. Although our method leads to an effective bound beyond which every integer is a sum of 18 biquadrates, we doubt that it may soon lead to the determination of the integers that cannot be written as sums of 18 biquadrates. Landreau [19] could reduce the contribution of the divisor function implied in Davenport's method, thanks to an averaging over a sufficiently large set, and this could lead to a "revenge" of Davenport's approach.

Acknowledgement. When working on this problem we (either or both) received encouragement and suggestions from E. Bombieri, H. Halberstram, H. Iwaniec, H. Maier, C. Stewart, G. Tenenbaum and W. Schmidt: it is our pleasure to thank them for their help.

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Formulae have been extensively numbered in order to help the reader in finding his/her way through the paper.

§1. - General Outline of the Proof

1.1 Hua's version of the circle method

Let N be a positive integer. We define t as the integer in the interval $[4, 19]$ which is congruent to $N \pmod{16}$. By P we denote an integer in the interval $[0.14N^{1/4}, 0.17N^{1/4}]$ which will be chosen later on. Our aim is to show that the number of ways, let us call it $r(N)$, to represent N as a sum of the fourth powers of t odd integers in $]2P, 4P]$ and $s = 19 - t$ even integers in the same interval is strictly positive. For that purpose, we follow the circle method, based on the integral representation

$$(1.1.1) \quad r(N) = \int_0^1 S_0^s(\alpha) S_1^t(\alpha) \exp(-2\pi i \alpha N) d\alpha$$

where

$$(1.1.2) \quad S_\varepsilon(\alpha) := \sum_{P-\varepsilon/2 < x \leq 2P-\varepsilon/2} \exp(2\pi i \alpha (2x + \varepsilon)^4).$$

The interval $[0, 1]$ is decomposed as a union of two distinct sets \mathcal{M} and m , respectively called the major and minor arcs. A precise definition is given in section 1.2; roughly speaking, \mathcal{M} is composed of those α 's which are close to a rational with a small denominator. Because of this arithmetic property, a lower bound of the contribution of \mathcal{M} to the integral 1.1.1 can be obtained; this is the aim of the second chapter.

On the minor arcs, we use Hua's method, that is to say that we rely on upper bounds for $\int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha$. The first step, which strictly follows [11], is to reduce the problem to finding an upper bound for the mean value of the number of ways to represent the values of a certain polynomial as a product of 3 integers. This is done in the third chapter. We dealt with this divisor sum in an independent publication [13], and we simply exploit it in chapter 4.

The next step is to get an upper bound for the maximum of the trigonometrical sums $|S_\varepsilon(\alpha)|$ over the minor arcs; we again invoke an independent publication, namely [14] which presents an improvement on Balasubramanian's method [3].

In the last chapter, we put together the different estimates, in order to show that the contribution of the major arcs is larger than that of the minor arcs, concluding that $r(N) > 0$, and thus that N is the sum of 19 biquadrates when $N \geq 10^{364}$, which is a bit stronger than stated in the Theorem.

1.2 Notation. The Farey dissection

By ν_0 we denote a real number in the interval [85, 151], the existence of which is asserted in section 2.1.

By N , we denote an integer $\geq 10^{10}$; the integers s and t are defined by the relations

$$(1.2.1) \quad 4 \leq t \leq 19, \quad N \equiv t \pmod{16}, \quad s + t = 19.$$

We denote by P_0 and P the quantities

$$(1.2.2) \quad P_0 := (N/16\nu_0)^{1/4} \text{ and } P := [P_0].$$

We call *major arcs* the union \mathcal{M} of the pairwise distinct intervals

$$(1.2.3) \quad \mathcal{M}_{a,q} := \left[\frac{a}{q} - \frac{975}{qP^3}, \frac{a}{q} + \frac{975}{qP^3} \right],$$

for $q \leq P^{1/2}$, $0 \leq a < q$, $(a, q) = 1$.

The *minor arcs* m is the complementary set of \mathcal{M} in the interval $\left[-\frac{975}{P^3}, 1 - \frac{975}{P^3} \right]$.

Finally, for $\varepsilon \in \{0, 1\}$, we write

$$(1.2.4) \quad S_\varepsilon(\alpha) := \sum_{P - \frac{\varepsilon}{2} < x \leq 2P - \frac{\varepsilon}{2}} e(\alpha(2x + \varepsilon)^4),$$

where $e(u) = \exp(2\pi i u)$. For an integral q , we write $e_q(u) = e(u/q)$.

§2. - Contribution of the major arcs

With the notation introduced in section 1.2, we have

THEOREM 2. *Let N be an integer larger than 10^{320} ; we have*

$$\int_{\mathcal{M}} S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha \geq 0.0079 P^{15}$$

2.1 The singular integral

PROPOSITION 2.1. *There exists a real number $\nu_0 \in [85, 151]$ such that on has*

$$(2.1.1) \quad K(\nu_0) \geq 0.01,$$

where

$$(2.1.2) \quad K(\nu) := \int_{-\infty}^{\infty} \left(\int_1^2 e(\beta t^4) dt \right)^{19} e(-\beta \nu) d\beta$$

PROOF OF PROPOSITION 2.1. We consider a family (X_1, \dots, X_{19}) of independent random variables, equidistributed on the interval $[1, 2]$, and we let

$$(2.1.3) \quad Z = \sum_{i=1}^{19} X_i^4.$$

We have

$$(2.1.4) \quad \mu := E(Z) = 19E(X_1^4) = \frac{589}{5},$$

$$(2.1.5) \quad \sigma^2 := \sigma^2(Z) = 19\sigma^2(X_1^4) = \frac{78394}{225}.$$

The integral $\int_1^2 e(\beta t^4) dt$ can be interpreted as the characteristic function (i.e. Fourier transform) of X_i ; thus, $\left(\int_1^2 e(\beta t^4) dt \right)^{19}$ represents the characteristic function of Z , and by inverse Fourier transform, $K(\nu)$ represents the density of Z at ν . By the Bienaymé-Čebyšev theorem, we have

$$(2.1.6) \quad \begin{aligned} & 2\sqrt{3}\sigma \quad \text{Sup} \quad K(\nu) \geq \\ & \quad \quad \quad [\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma] \\ & \geq \int_{\mu - \sqrt{3}\sigma}^{\mu + \sqrt{3}\sigma} K(\nu) d\nu = P \left\{ |Z - \mu| \leq \sqrt{3}\sigma \right\} \geq 1 - (\sqrt{3})^{-2} = \frac{2}{3} \end{aligned}$$

whence the result. □

2.2. Gauß sums

For $\varepsilon \in \{0, 1\}$, we get an upper bound for the sums

$$(2.2.1) \quad G_\varepsilon(a, q; \nu) := \sum_{h=0}^{q-1} e_q(a(2h + \varepsilon)^4 + \nu h)$$

A general result for biquadratic trigonometrical sum is given by Nečaev and Topunov. However, using directly this result would lead to a constant in

Proposition 2.2, too large for our purpose. We shall derive the main result of this section from an upperbound for Gauß sums of the shape

$$(2.2.2) \quad S(a, q; b) := \sum_{h=0}^{q-1} e_q(ah^4 + bh)$$

given in the thesis of Thomas [23], and that we shall prove again, for the sake of the completeness.

PROPOSITION 2.2. *For ν integer, a and q coprime, and $\varepsilon \in \{0, 1\}$, we have*

$$(2.2.3) \quad |G_\varepsilon(a, q; \nu)| \leq 18q^{3/4}$$

LEMMA 2.2.1 (NEČAEV AND TOPUNOV [20]). *Let $F(X) = a_4X^4 + a_3X^3 + a_2X^2 + a_1X$ be a polynomial with integral coefficients, and let p be a prime number and s a positive integer such that*

$$\gcd(a_1, a_2, a_3, a_4, p) = 1.$$

We have

$$(2.2.4) \quad \left| \sum_{h=0}^{p^s-1} \exp(2\pi i F(h)/p^s) \right| \leq p^{3s/4}$$

in the following cases:

$$\begin{aligned} p = 2 \text{ and } s \geq 8 \\ p = 3 \text{ or } 5 \text{ and } s \geq 3 \\ p \geq 7 \text{ and } s \geq 2 \\ 43 \leq p \leq 53 \text{ or } p \geq 61, \text{ and } s \geq 1. \end{aligned}$$

LEMMA 2.2.2 (THOMAS [23]). *Let a , b and q be three integers such that*

$$\gcd(a, b, q) = 1.$$

We have

$$(2.2.5) \quad \left| \sum_{h=0}^{q-1} e_q(ah^2 + bh) \right| \leq 4.5 q^{3/4}.$$

PROOF OF LEMMA 2.2.2. By the multiplicativity of the trigonometrical sums, we have

$$(2.2.6) \quad |S(a, q; b)| q^{-3/4} \leq \prod_p \max_{s \geq 0} \max_{(a', b', p)=1} |S(a', p^s; b')| p^{-3s/4}.$$

If we denote by $c(p)$ the factors in the *RHS* of 2.2.6, Lemma 2.2.2 and a direct computation imply that one has

$$(2.2.7) \quad c(p) \leq 1,$$

except when p is 2, 5, 13, 17 or 41, in which cases one has

$$(2.2.8) \quad \begin{aligned} c(2) &\leq 1.962, \\ c(5) &\leq 1.320, \\ c(13) &\leq 1.138, \\ c(17) &\leq 1.269, \\ c(41) &\leq 1.142 \end{aligned}$$

from which Lemma 2.2.2 follows. □

PROOF OF PROPOSITION 2.2. In a first step, we reduce the majorization of $G_1(a, q; \nu)$ to that of $G_0(a, q; \nu)$, since we have

$$\begin{aligned} G_0(a, q; \nu) + e_{2q}(\nu)G_1(a, q; \nu) &= \sum_{h=0}^{2q-1} e_{2q}(2ah^4 + \nu h) \\ &= \sum_{h=0}^{q-1} (e_{2q}(2ah^4 + \nu h) + e_{2q}(2ah^4 + \nu(h+q))) \\ &= \left(1 + e\left(\frac{\nu}{2}\right)\right) \sum_{h=0}^{q-1} e_q(ah^4 + \nu h/2), \end{aligned}$$

so that, for odd ν we get

$$(2.2.9) \quad |G_1(a, q; \nu)| = |G_0(a, q; \nu)|,$$

and for even ν , Lemma 2.2.1 leads to

$$(2.2.10) \quad |G_1(a, q; \nu)| \leq 9q^{3/4} + |G_0(a, q; \nu)|.$$

Since a and q are coprime, the g.c.d. of $16a$, ν and q is a divisor of 16; let us call it 2^k (with $k \leq 4$). We have

$$(2.2.11) \quad |G_0(a, q; \nu)| \leq 2^k |S(16a/2^k, q/2^k; \nu/2^k)|,$$

and by Lemma 2.2.2, we have

$$(2.2.12) \quad |G_0(a, q; \nu)| \leq 2^k (q/2^k)^{3/4} \leq 2^{k/4} q^{3/4} \leq 2q^{3/4},$$

since k is at most 4.

Proposition 2.2 now follows from 2.2.9, 2.2.10 and 2.2.12.

2.3 The singular series

PROPOSITION 2.3. *Let us define*

$$(2.3.1) \quad \mathcal{S}(N) := \sum_{q=1}^{\infty} A(q, N)$$

where

$$(2.3.2) \quad A(q, N) := \sum_{(a,q)=1} q^{-19} G_0^8(a, q; 0) G_1^4(a, q; 0) e_q(-aN).$$

For any integer N , the series $\mathcal{S}(N)$ converges absolutely, and one has

$$(2.3.3) \quad \mathcal{S}(N) \geq 12.68$$

PROOF OF PROPOSITION 2.3. The absolute convergence of 2.3.1 comes from Proposition 2.2. Let us denote by $M(q, N)$ the number of solutions of the congruence

$$(2.3.4) \quad (2x_1)^4 + \dots + (2x_s)^4 + (2y_1 + 1)^4 + \dots + (2y_t + 1)^4 \equiv N \pmod{q}.$$

By the Chinese remainder theorem, the arithmetic function $q \mapsto M(q, N)$ is multiplicative. Moreover, by detecting the congruence mod q with an additive character one readily gets

$$M(q, N) = q^{18} \sum_{d|q} A(d, N),$$

which implies two facts:

(2.3.5) the function $d \mapsto A(d, N)$ is multiplicative so that \mathcal{S} can be written as an Euler product.

(2.3.6) each factor of this product can be written as $\lim_{n \rightarrow \infty} p^{-18n} M(p^n, N)$.

Our problem is thus reduced to getting a lower bound for $M(p^n, N)$.

LEMMA 2.3. *For prime p , a lower bound for $M(p^n, N)$ is given by the following relations*

$$(2.3.7) \quad M(2^n, N) = 16 \cdot 2^{18n} \text{ for } n \geq 4$$

$$(2.3.8) \quad M(3^n, N) \geq 387400806 \cdot 3^{18(n-1)} \text{ for } n \geq 1$$

$$(2.3.9) \quad M(5^n, N) \geq 3.131421 \cdot 10^{12} \cdot 5^{18(n-1)} \text{ for } n \geq 1$$

$$(2.3.10) \quad M(p^n, N) \geq (p^{18} - 3^{19}p^{8.5} - 1)p^{18(n-1)} \text{ for } p \equiv 1 \pmod{4} \text{ and for } n \geq 1$$

$$(2.3.11) \quad M(p^n, N) \geq (p^{18} - p^{8.5} - 1)p^{18(n-1)} \text{ for } p \equiv 3 \pmod{4} \text{ and for } n \geq 1$$

PROOF OF LEMMA 2.3. A common ingredient in the proof of the different inequalities is that above any non-singular solution of $f(x_0, \dots, x_s) \equiv 0 \pmod{p}$, one can find $p^{s(n-1)}$ solutions of

$$f(x_0, \dots, x_s) \equiv 0 \pmod{p^n}.$$

For 3 and 5, a direct computation can be performed by hand, since the only biquadrates are 0 and 1. For example, one has

$$(2.3.12) \quad M(5, 3) = \binom{19}{3}4^3 + \binom{19}{8}4^8 + \binom{19}{13}4^{13} + \binom{19}{18}4^{18};$$

one computes in a similar way $M(3, 0)$, $M(3, 1)$, $M(3, 2)$, $M(5, 0)$, $M(5, 1)$, $M(5, 2)$, $M(5, 4)$, and gets 2.3.8 and 2.3.9.

For odd p , it is sufficient to consider a lower bound for $M(p, N) - 1$, which is itself a lower bound for the number of non-singular solutions of 2.3.4. Following [6] (p. 15), we get

$$(2.3.13) \quad M(p, N) - 1 \geq p^{18} - (\gcd(4, p - 1) - 1)p^{8.5} - 1,$$

which readily leads to 2.3.10 and 2.3.11.

Finally, when $p = 2$, our choice of s and t leads to

$$(2.3.14) \quad 2^{-4 \times 18} M(2^4, N) = 16.$$

Let now n be ≥ 4 , and consider integers $x_1, \dots, x_s, y_1, \dots, y_t$ solutions of

$$(2.3.15) \quad (2x_1)^4 + \dots + (2x_s)^4 + (2y_1 + 1)^4 + \dots + (2y_t + 1)^4 \equiv N \pmod{2^n};$$

the integers $x_1 + 2^{n-3}, \dots, x_s + 2^{n-3}, y_1 + 2^{n-3}, \dots, y_t + 2^{n-3}$ are also solutions of 2.3.15. If we denote by $L(2^n, N)$ the number of arrays $(x_1, \dots, x_s, y_1, \dots, y_t)$ of integers satisfying

$$(2.3.16) \quad 0 \leq x_i < 2^{n-3}, \quad 0 \leq y_j < 2^{n-3}$$

and 2.3.15, we have

$$(2.3.17) \quad M(2^n, N) = 2^{3 \times 19} L(2^n, N).$$

We now compute $L(2^n, N)$ by induction on n . Let $(\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_t) \in \{0, 1\}^{19}$; we have, modulo 2^{n+1} :

$$(2.3.18) \quad \begin{aligned} & (2(x_1 + \lambda_1 2^{n-3}))^4 + \dots + (2(y_t + \mu_t 2^{n-3}) + 1)^4 \\ & \equiv (2x_1)^4 + \dots + (2y_t + 1)^4 + 2^n \mu_1 (2y_1 + 1)^3 + \dots + 2^n \mu_t (2y_t + 1)^4 \end{aligned}$$

Since $t > 0$ and $(2y_t + 1)$ is invertible mod 2, to each $(\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_{t-1})$ can be associated a unique μ_t in $\{0, 1\}$ such that one has

$$(2.3.19) \quad (2(x_1 + \lambda_1 2^{n-3}))^4 + \dots + (2(y_t + \mu_t 2^{n-3}) + 1)^4 \equiv N \pmod{2^{n+1}}.$$

Thus, for $n \geq 4$, one has

$$(2.3.20) \quad L(2^{n+1}, M) = 2^{18} L(2^n, M).$$

Now, 2.3.7 follows from 2.3.14, 2.3.17 and 2.3.20. \square

We return to the proof of the Proposition 3.3. By the Lemma 2.3 and the facts 2.3.5 and 2.3.6, we have

$$(2.3.21) \quad \begin{aligned} S(N) & \geq 16 \times \frac{387400806}{3^{18}} \\ & \times \frac{3.131421 \times 10^{12}}{5^{18}} \prod_{p \geq 7} (1 - ((4, p - 1) - 1)^{19} p^{-9.5} - p^{-18}) \end{aligned}$$

and a hand computation leads to 2.3.3. \square

2.4 Approximation of $S_\varepsilon(\alpha)$

On the interval $\left[\frac{a}{q} - \frac{975}{qP^3}, \frac{a}{q} + \frac{975}{qP^3}\right]$, we give an approximation of $S_\varepsilon(\alpha)$ in terms of the Gauß sums $G_\varepsilon(a, q; 0)$ introduced in 2.2.1, and of the integral

$$(2.4.1) \quad I(\beta, N) := \int_{2P_0}^{4P_0} e(\beta t^4) dt.$$

PROPOSITION 2.4. For $\alpha = \frac{a}{q} + \beta$, with $0 \leq a < q$, $(a, q) = 1$ and $|\beta| \leq 975/(qP^3)$, we have

$$(2.4.2) \quad \left| S_\varepsilon(\alpha) - \frac{1}{2q} G_\varepsilon(a, q; 0) I(\beta, N) \right| \leq 2.7 \cdot 10^6 q^{1/4} P^{1/2} + 61 q^{3/4} (\text{Log } q + 1).$$

In particular, for $q \leq P^{1/2}$, we have

$$(2.4.3) \quad \left| S_\varepsilon(\alpha) - \frac{1}{2q} G_\varepsilon(a, q; 0) I(\beta, N) \right| \leq 3 \cdot 10^6 q^{1/4} P^{1/2}.$$

Before embarking on the proof of Proposition 2.4, we state and prove the following lemma

LEMMA 2.4. *Let $P > 0$, $|\beta| \leq 975/(qP^3)$ and $\nu \neq 0$. We have*

$$(2.4.4) \quad \int_{2P}^{4P} e\left(\beta t^4 - \frac{\nu t}{2q}\right) dt = -\frac{q}{\pi i \nu} (e(256\beta P^4 - 2\nu P q^{-1}) - e(16\beta P^4 - \nu P q^{-1})) + E(\lambda, q, P),$$

where

$$(2.4.5) \quad |E(\nu, q, P)| \leq 10500 \frac{q^{1/2} P^{1/2}}{|\nu|}.$$

Furthermore, if $|\beta| \leq (1024qP^3)^{-1}$ or $|\nu| \geq 10^6$, we have

$$(2.4.6) \quad |E(\nu, q, P)| \leq 1.3 \cdot 10^6 q |\nu|^{-2}.$$

PROOF OF LEMMA 2.4. As in Lemma 9.3 from [23], we follow Titchmarsh [25], except that we use his Lemma 4.5 (stationary phase method). Integrating by parts leads us to

$$(2.4.7) \quad E(\nu, q, P) = \int_{2P}^{4P} \frac{8\beta q t^3}{\nu} e\left(\beta t^4 - \frac{\nu t}{2q}\right) dt;$$

we then apply Titchmarsh's Lemma with

$$G(t) := \frac{8\beta q t^3}{\nu} \text{ and } F(t) := 2\pi \left(\beta t^4 - \frac{\nu t}{2q}\right);$$

this leads to 2.4.4; furthermore, if $|\beta| \leq (1024qP^3)^{-1}$ or $|\nu| \geq 10^6$, we note that $\frac{G}{F}$ is monotonic, continuous and that $\left|\frac{G}{F'}(t)\right|$ is at most $\frac{320000q}{|\nu|^2}$ over $[2P, 4P]$. □

PROOF OF PROPOSITION 2.4. Let $J(h)$ be the interval

$$\left] \frac{P - h - \varepsilon/2}{q}, \frac{2P - h - \varepsilon/2}{q} \right];$$

we note that there are at most 2 values of $h \pmod q$ such that either endpoint

of $J(h)$ is an integer. We have

$$\begin{aligned}
 (2.4.8) \quad S_\varepsilon(\alpha) &= \sum_{h=0}^{q-1} \sum_{m \in J(h)} e \left(\left(\frac{a}{q} + \beta \right) (2mq + 2h + \varepsilon)^4 \right) \\
 &= \sum_{h=0}^{q-1} e_q(a(2h + \varepsilon)^4) \sum_{m \in J(h)}^* e(\beta(2mq_2h + \varepsilon)^4) + \theta,
 \end{aligned}$$

where $|\theta| \leq 1$, and the asterisk means that a weight $1/2$ is attached to any term corresponding to an endpoint of $J(h)$.

By Poisson summation formula, we have

$$\begin{aligned}
 (2.4.9) \quad S_\varepsilon(\alpha) &= \frac{1}{2q} G_\varepsilon(a, q; 0) \int_{2P}^{4P} e(\beta t^4) dt + \\
 &+ \frac{1}{2q} \sum_{\nu \neq 0} e \left(\frac{\nu \varepsilon}{2q} \right) G_\varepsilon(a, q; \nu) \int_{2P}^{4P} e \left(\beta t^4 - \frac{\nu t}{2q} \right) dt + \theta,
 \end{aligned}$$

which easily leads to

$$\begin{aligned}
 (2.4.10) \quad & \left| S_\varepsilon(\alpha) - \frac{1}{2q} G_\varepsilon(a, q; 0) I(\beta, N) \right| \leq \\
 & \frac{1}{2q} \left| \sum_{\nu \neq 0} G_\varepsilon(a, q; \nu) \int_{2P}^{4P} e \left(\beta t^4 - \frac{\nu t}{2q} \right) dt \right| + 4.
 \end{aligned}$$

By Lemma 2.4 and Proposition 2.2, the *RHS* of 2.4.10 is at most

$$\begin{aligned}
 (2.4.11) \quad & 4 + \frac{1}{2q} \sum_{0 < |\nu| \leq 10^6} 18q^{3/4} 10500 \frac{q^{1/2} P^{1/2}}{|\nu|} \\
 & + \frac{1}{2q} \sum_{|\nu| > 10^6} 18q^{3/4} 1.3 \cdot 10^6 \frac{q}{|\nu|^2} + \frac{1}{2q} \sum_{0 < |\nu| \leq 4q^2} 18q^{3/4} \frac{2q}{\pi |\nu|} \\
 & + \frac{1}{2q} \left| \sum_{|\nu| > 4q^2} e \left(\frac{\nu \varepsilon}{2q} \right) G_\varepsilon(a, q; \nu) \frac{q}{\pi \nu} \right. \\
 & \quad \left. \left(e \left(256\beta P^4 - \frac{2\nu P}{q} \right) e \left(16\beta P^4 - \frac{\nu P}{q} \right) \right) \right| \\
 & \leq 2.7 \cdot 10^6 q^{1/4} P^{1/2} + 60.3(\text{Log } q + 1)q^{3/4} + \frac{1}{2q} \left| \sum \dots \right|;
 \end{aligned}$$

we get an upper bound for the last sum by using a technique already used by Thomas [23], which rely on the inequality

$$(2.4.12) \quad \left| \sum_{\nu \geq q^2+1} \nu^{-1} \sin \frac{2\pi\nu j}{q} \right| \leq \frac{1}{2q},$$

which is valid for any integral j .

By the definition of G_ε , we get

$$(2.4.13) \quad \left| \sum_{\nu > 4q^2} \frac{1}{\nu} e\left(\frac{\nu\varepsilon}{2q}\right) G_\varepsilon(a, q; \nu) e(256\beta P^4 - 2\nu P q^{-1}) \right| \\ \leq \sum_{h=0}^{q-1} \left| \sum_{\nu > 4q^2} \nu^{-1} \sin \left(\frac{2\pi\nu((2h + \varepsilon) - 4P)}{2q} \right) \right| \leq \frac{1}{4}.$$

Applying three more times this technique, we get

$$(2.4.14) \quad \frac{1}{2q} \left| \sum_{|\nu| > 4q^2} e\left(\frac{\nu\varepsilon}{2q}\right) G_\varepsilon(a, q; \nu) \frac{q}{\pi\nu} \right. \\ \left. \left(e\left(256\beta P^4 - \frac{2\nu P}{q}\right) - e\left(16\beta P^4 - \frac{\nu P}{q}\right) \right) \right| \leq 4 \times \frac{1}{2q} \times \frac{q}{\pi} \times \frac{1}{4} = \frac{1}{2\pi}.$$

The relation 2.4.2 follows then from 2.4.10, 2.4.11 and 2.4.14, and relation 2.4.3 is a direct consequence of 2.4.2. □

2.5 Contribution of the major arcs

The key result of this section is the following

PROPOSITION 2.5. For $P \geq 10^{60}$ we have

$$(2.5.1) \quad \left| \sum_{q \leq P^{1/2}} \sum_{(a,q)=1} \int_{\mathcal{M}_{a,q}} S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha \right. \\ \left. - \frac{P^{15}}{16} K \left(\frac{N}{(2P_0)^4} \right) \sum_{q \leq P^{1/2}} q^{-19} \sum_{(a,q)=1} e_q(-aN) G_0^s(a, q; 0) G_1^t(a, q; 0) \right| \\ \leq 10^{34} P^{29/2},$$

where K is defined by 2.1.2.

Under the conditions of Proposition 2.4, we have

$$(2.5.2) \quad |S_\varepsilon(\alpha) - T_\varepsilon(\alpha)| \leq R,$$

where

$$(2.5.3) \quad T_\varepsilon(\alpha) := \frac{1}{2q} G_\varepsilon(a, q; 0) I(\beta, N),$$

and

$$(2.5.4) \quad R := 3 \cdot 10^6 q^{1/4} P^{1/2}.$$

We first get an upperbound for $I(\beta, N)$.

LEMMA 2.5.1. *We have*

$$(2.5.5) \quad |I(\beta, N)| \leq \min(2P_0, (16|\beta|P^3)^{-1}).$$

PROOF OF LEMMA 2.5.1. The first upperbound is trivial. Let us assume now that $\beta \neq 0$; we change the variable and integrate by parts, getting

$$\begin{aligned} I(\beta, N) &= \int_{2P_0}^{4P_0} e(\beta t^4) dt = \frac{1}{4} \int_{(2P_0)^4}^{(4P_0)^4} \tau^{-3/4} e(\beta \tau) d\tau \\ &= \frac{1}{4} \left[\frac{e(\beta \tau)}{2\pi i \beta \tau^{3/4}} \right]_{(2P_0)^4}^{(4P_0)^4} + \frac{1}{4} \int_{(2P_0)^4}^{(4P_0)^4} \frac{3e(\beta \tau)}{8\pi i \beta \tau^{7/4}} d\tau, \end{aligned}$$

whence

$$(2.5.6) \quad |I(\beta, N)| \leq \frac{9/8}{8\pi|\beta|(2P_0)^3} + \frac{45}{32\pi|\beta|(2P_0)^3} \leq \frac{1}{16|\beta|P^3}$$

and so 2.5.5 is proven. □

LEMMA 2.5.2. *We have*

$$(2.5.7) \quad \left| \int_{\mathcal{M}_{a,q}} S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha - \int_{-\infty}^{\infty} T_0^s(\alpha) T_0^t(\alpha) e(-\alpha N) d\alpha \right| \leq 8 \cdot 10^{128} P^{95/8} + 8 \cdot 10^{33} q^{-17/4} P^{29/2}.$$

PROOF OF LEMMA 2.5.2.

1st step. From Lemma 2.5.1 and Proposition 2.2, we get

$$\begin{aligned}
 (2.5.8) \quad & \left| \int_{-\infty}^{\infty} T_0^s(\alpha) T_0^t(\alpha) e(-\alpha N) d\alpha - \int_{\mathcal{M}_{a,q}} T_0^s(\alpha) T_0^t(\alpha) e(-\alpha N) d\alpha \right| \\
 & \leq 2 \int_{975(qP^3)^{-1}}^{\infty} (9q^{-1/4})^{19} \frac{d\beta}{16^{19} \beta^{19} P^{57}} \\
 & \leq \frac{2}{18} \left(\frac{9}{16} \right)^{19} q^{-19/4} P^{-57} \left(\frac{qP^3}{975} \right)^{18} \\
 & \leq q^{53/4} P^{-3} \leq P^4
 \end{aligned}$$

2nd step. Let $T(\alpha) := \text{Sup}(|T_0(\alpha)|, |T_1(\alpha)|)$. For $\varepsilon \in \{0, 1\}$, u in Z_+ and α in $\mathcal{M}_{a,q}$, we have

$$(2.5.9) \quad |S_0^s(\alpha) S_1^t(\alpha) - T_0^s(\alpha) T_1^t(\alpha)| \leq 2^{19} R T^{18}(\alpha) + 2^{19} R^{19},$$

whence

$$\begin{aligned}
 (2.5.10) \quad & \int_{\mathcal{M}_{a,q}} |S_0^s(\alpha) S_1^t(\alpha) - T_0^s(\alpha) T_1^t(\alpha)| d\alpha \\
 & \leq 2^{19} R^{19} + 2^{20} R (9q^{-1/4})^{18} \int_0^{(32P^4)^{-1}} (2P)^{18} d\beta \\
 & \quad + 2^{20} R (9q^{-1/4})^{18} \int_{(32P^4)^{-1}}^{\infty} \frac{d\beta}{(16\beta P^3)^{18}} \\
 & \leq 8 \cdot 10^{128} P^{95/8} + 8 \cdot 10^{33} q^{-17/4} P^{29/2}. \quad \square
 \end{aligned}$$

PROOF OF PROPOSITION 2.5. We perform the summation over the major arcs, and use 2.5.7 in conjunction with the relations

$$(2.5.11) \quad \sum_{q \leq P^{1/2}} \sum_{(a,q)=1} 1 \leq \sum_{q \leq P^{1/2}} q \leq P,$$

$$(2.5.12) \quad \sum_{q \leq P^{1/2}} \sum_{(a,q)=1} q^{-17/4} \leq \sum_{q \leq P^{1/2}} q^{-13/4} \leq \sum_q \frac{1}{q^2} \leq 2$$

$$(2.5.13) \quad 10^{129} P^{103/8} + 8 \cdot 10^{33} P^{29/2} \leq 10^{34} P^{29/2} \text{ for } P \geq 10^{60}.$$

By changing the variable, the quantity

$$\frac{1}{2^{19}} \int_{-\infty}^{\infty} \left(\int_{2P_0}^{4P_0} e(\beta t^4) dt \right)^{19} e(-\beta N) d\beta \text{ becomes}$$

$$\frac{P_0^{15}}{16} \int_{-\infty}^{\infty} \left(\int_1^2 e(\beta t^4) dt \right)^{19} e\left(-\beta \frac{N}{(2P_0)^4}\right) d\beta = \frac{P_0^{15}}{16} K\left(\frac{N}{(2P_0)^4}\right)$$

and this leads to 2.5.1.

PROOF OF THEOREM 2. Because of the definition of $\mathcal{S}(N)$ (cf. 2.3.1 and 2.3.2), and the Proposition 2.2, we have

$$(2.5.14) \quad \left| \mathcal{S}(N) - \sum_{q \leq P^{1/2}} \sum_{(a,q)=1} q^{-19} G_0^s(a, q; 0) G_1^t(a, q; 0) e_q(-aN) \right|$$

$$\leq \sum_{q > P^{1/2}} q \cdot q^{-19} (18)^{19} q^{57/4} \leq 10^{24} \sum_{q > P^{1/2}} q^{-3} \leq 10^{24} \cdot P_0^{-1}.$$

We notice that both the double sums in the *LHS* of Proposition 2.5 are real. Combining 2.5.1 and 2.5.14 we get

$$\int_{\mathcal{M}} S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha \geq$$

$$\frac{P_0^{15}}{16} K\left(\frac{N}{(2P_0)^4}\right) (\mathcal{S}(N) - 10^{24} P_0^{-1}) - 10^{34} P_0^{29/2}.$$

We now recall the definition 2.0.2 of P_0 , that of ν_0 in Proposition 2.1 and the lower bounds 2.1.1 and 2.3.3 for $K(\nu_0)$ and $\mathcal{S}(N)$ respectively. We get

$$(2.5.15) \quad \int_{\mathcal{M}} S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha$$

$$\geq \frac{P_0^{15}}{16} \cdot 0.01 \cdot 12.68 - 10^{24} P_0^{14} - 10^{34} P_0^{29/2}$$

$$\geq 0.0079 P_0^{15} + (2 \cdot 10^{-5} - 10^{24} P_0^{-1} 10^{34} P_0^{-1/2}) P_0^{15},$$

and Theorem 2 follows from the fact that for $N \geq 10^{320}$, P_0 is larger than 10^{78} and so the second term in the right hand side of 2.5.15 is positive. \square

§3. - Hua's inequality: reduction to a divisor sum

In section 3.3, we shall prove the following result

THEOREM 3. *For $\varepsilon = 0$ or 1 , we assume that there exists a real number $B_\varepsilon(P)$ satisfying the relations*

$$(3.0.1) \quad 0.1 P^3 (\text{Log } P)^8 \leq B_\varepsilon(P) \leq P^{3.6}$$

$$(3.0.2) \quad 2 \sum_{h,k,y}^{(3.0.3)} d_3(|3(2y + \varepsilon + h + k)^2 + h^2 + k^2| h k) \leq B_\varepsilon(P)$$

where the summation condition is

$$(3.0.3) \quad \begin{cases} h k \neq 0, & 0 < |h| + |k| < P, & P - \varepsilon/2 < y \leq 2P - \varepsilon/2 \\ & P - \varepsilon/2 < y + h + k \leq 2P - \varepsilon/2. \end{cases}$$

For $P \geq 10^{80}$ and $\varepsilon = 0$ or 1 , one has

$$(3.0.4) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 60000 \frac{(4.5)^{2\lambda}}{1 + \lambda} P^{12} (\text{Log } P)^{7+4\lambda} + 66 \cdot 2^{-\lambda} B_\varepsilon(P) P^9 (\text{Log } P)^{1-\lambda},$$

where λ is any real number with $0 \leq \lambda \leq 0.5$.

3.1 On the congruence $x^4 \equiv y^4 \pmod{k}$

PROPOSITION 3.1. *Let $X \geq 10^{80}$ and*

$$A(X) := \#\{(k, x, y) | 0 < k < X, x \in]X, 2X], y \in]X, 2X], x^4 \equiv y^4 \pmod{k}\}$$

we have

$$(3.1.1) \quad A(X) \leq 13 X^2 (\text{Log } X)^4 - X^2.$$

We should notice that an effective result on prime in the arithmetic progressions mod 4 would lead to an upper bound $A(X) \ll X^2 (\text{Log } X)^3$.

Let $N(k)$ denote the number of solutions of the congruence $\xi^4 \equiv \eta^4 \pmod{k}$. We have

$$(3.1.2) \quad \begin{aligned} A(X) &= \sum_{0 < k < X} \sum_{\substack{\xi, \eta \pmod{k} \\ \xi^4 \equiv \eta^4}} \sum_{\substack{x \in]X, 2X] \\ x \equiv \xi}} \sum_{\substack{y \in]X, 2X] \\ y \equiv \eta}} 1 \\ &\leq 4X^2 \sum_{0 < k < X} \frac{N(k)}{k^2} \end{aligned}$$

where all the congruences are taken modulo k .

The function N is multiplicative, and we simply show an upper bound of $N(p^\ell)$ which is good enough for our purpose. We denote

$$(3.1.3) \quad N_0(p^\ell) = \#\{(\xi, \eta) \mid 0 \leq \xi < p^\ell, 0 \leq \eta < p^\ell, p \mid \xi\eta \text{ and } \xi^4 \equiv \eta^4 \pmod{p^\ell}\}$$

$$(3.1.4) \quad N_1(p^\ell) = \#\{(\xi, \eta) \mid 0 \leq \xi < p^\ell, 0 \leq \eta < p^\ell, p \nmid \xi\eta \text{ and } \xi^4 \equiv \eta^4 \pmod{p^\ell}\},$$

so that

$$(3.1.5) \quad N(p^\ell) = N_0(p^\ell) + N_1(p^\ell).$$

LEMMA 3.1.1. *For any prime number p , we have*

$$(3.1.6) \quad N_0(p^\ell) = p^{2(\ell-1)} \text{ for } 1 \leq \ell \leq 4$$

$$(3.1.7) \quad N_0(p^\ell) = p^6 N(p^{\ell-4}) \text{ for } \ell \geq 5$$

$$(3.1.8) \quad \sum_{\ell \geq 2} \frac{N_0(p^\ell)}{p^{2\ell}} \leq \frac{4}{p^2 - 1}.$$

PROOF OF LEMMA 3.1.1. As soon as p divides ξ and $\xi^4 - \eta^4$, it divides η ; but, if p divides ξ and η , then p^4 divides $\xi^4 - \eta^4$; this leads to 3.1.6.

Let now $\ell \geq 5$, and consider the solutions of

$$(3.1.9) \quad 0 \leq \xi < p^\ell, 0 \leq \eta < p^\ell, p \mid \xi\eta, \xi^4 - \eta^4 \equiv 0 \pmod{p^\ell}.$$

We let $\xi = px$, $\eta = py$, so that 3.1.9 is equivalent to

$$(3.1.10) \quad 0 \leq x < p^{\ell-1}, 0 \leq y < p^{\ell-1}, x^4 - y^4 \equiv 0 \pmod{p^{\ell-4}},$$

and each of the $N(p^{\ell-4})$ solutions of

$$0 \leq x < p^{\ell-4}, 0 \leq y < p^{\ell-4}, x^4 - y^4 \equiv 0 \pmod{p^{\ell-4}}$$

leads to p^6 solutions of 3.1.10. This shows 3.1.7.

From 3.1.6. and 3.1.7, we have

$$\sum_{\ell \geq 2} \frac{N_0(p^\ell)}{p^{2\ell}} = \frac{p^2}{p^4} + \frac{p^4}{p^6} + \frac{p^6}{p^8} + \frac{p^6}{p^{10}} + \frac{1}{p^2} \sum_{m \geq 2} \frac{N_0(p^m)}{p^{2m}},$$

whence we get

$$\left(1 - \frac{1}{p^2}\right) \sum_{\ell \geq 2} \frac{N_0(p^\ell)}{p^{2\ell}} \leq \frac{4}{p^2}$$

which is equivalent to 3.1.8.

LEMMA 3.1.2. *For any odd prime numbers p , we have*

$$(3.1.11) \quad N_1(p) \leq 4(p-1) \text{ and } N_1(3) = 4$$

$$(3.1.12) \quad N_1(p^\ell) = pN_1(p^{\ell-1}) \text{ for } \ell \geq 2.$$

$$(3.1.13) \quad \sum_{\ell \geq 2} \frac{N_1(p^\ell)}{p^{2\ell}} = \frac{N_1(p)}{p^2(p-1)}.$$

REMARK. We have indeed $N_1(p) = 4(p-1)$ or $2(p-1)$ according as p is congruent to 1 or 3 modulo 4.

PROOF OF LEMMA 3.1.2. Let $\alpha \in [0, p[$ and let $r(\alpha)$ denote the number of ξ in $[0, p[$ such that $\xi^4 \equiv \alpha [p]$. Since $p \nmid \xi$ if and only if $\alpha = 0$, we have

$$(3.1.14) \quad N_1(p) = \sum_{\alpha=1}^{p-1} r^2(\alpha) \text{ and } \sum_{\alpha=1}^{p-1} r(\alpha) = p-1;$$

since p is prime, the congruence $\xi^4 \equiv \alpha [p]$ has at most 4 solutions, and so

$$(3.1.15) \quad N_1(p) = \sum_{\alpha=1}^{p-1} r^2(\alpha) \leq 4 \sum_{\alpha=1}^{p-1} r(\alpha) = 4(p-1),$$

which proves the first part of 3.1.11; we compute directly $N_1(3)$.

Let now $\ell \geq 2$ and (x, y) be a solution of

$$(3.1.16) \quad 0 < x < p^{\ell-1}, \quad 0 < y < p^{\ell-1}, \quad p \nmid xy, \quad x^4 \equiv y^4 [p^{\ell-1}].$$

Since p does not divide $4xy$, this solution is non-singular, and above it, there are exactly p non-singular solutions of $x_1^4 \equiv y_1^4 [p^\ell]$, whence 3.1.12.

From 3.1.12, we get

$$\begin{aligned} \sum_{\ell \geq 2} \frac{N_1(p^\ell)}{p^{2\ell}} &= \frac{N_1(p)}{p^3} + \sum_{\ell \geq 3} \frac{pN_1(p^{\ell-1})}{p^{2\ell}} \\ &= \frac{N_1(p)}{p^3} + \frac{1}{p} \sum_{m \geq 2} \frac{N_1(p^m)}{p^{2m}}, \end{aligned}$$

which implies

$$\left(1 - \frac{1}{p}\right) \sum_{\ell \geq 2} \frac{N_1(p^\ell)}{p^{2\ell}} = \frac{N_1(p)}{p^3},$$

whence 3.1.13. □

LEMMA 3.1.3. *We have*

$$(3.1.17) \quad N_1(2) = 1, \quad N_1(4) = 4, \quad N_1(8) = 16$$

$$(3.1.18) \quad N_1(2^k) = 2^{k+2} \text{ for } k \geq 4$$

$$(3.1.19) \quad \sum_{\ell \geq 1} \frac{N_1(2^\ell)}{2^{2\ell}} = \frac{5}{4}.$$

PROOF OF LEMMA 3.1.3: We prove 3.1.17 by a direct computation. By noticing that $(x + 2^{k-3})^4$ is congruent to x^4 modulo 2^{k-1} , but not modulo 2^k when $k \geq 5$ and x odd, we show by induction on k that there are 2^{k-4} odd biquadratic residues modulo 2^k and that each of them is obtained 8 times, for $k \geq 4$. This leads to 3.1.18. The relation 3.1.19 is then straightforward. \square

LEMMA 3.1.4. *We have*

$$(3.1.20) \quad \sum_{\ell \geq 1} \frac{N(2^\ell)}{2^{2\ell}} \leq \frac{37}{12}$$

$$(3.1.21) \quad \sum_{\ell \geq 1} \frac{N(3^\ell)}{3^{2\ell}} \leq \frac{23}{18}$$

$$(3.1.22) \quad \sum_{p \geq 5} \sum_{\ell \geq 2} \frac{N(p^\ell)}{p^{2\ell}} \leq 1$$

PROOF OF LEMMA 3.1.4. It is a straightforward application of the relation 3.1.5 and the three previous lemmas. For the prime 2, we get from 3.1.6, 3.1.8 and 3.1.19 the relation

$$\sum_{\ell \geq 1} \frac{N(2^\ell)}{2^{2\ell}} \leq \frac{1}{2} + \frac{4}{3} + \frac{5}{4} = \frac{37}{12}$$

which is 3.1.20.

When p is odd, we get from 3.1.8 and 3.1.13 the relation

$$(3.1.23) \quad \sum_{\ell \geq 2} \frac{N(p^\ell)}{p^{2\ell}} \leq \frac{4}{p^2 - 1} + \frac{N_1(p)}{p^2(p - 1)}.$$

When $p = 3$, relations 3.1.23 and 3.1.11 lead to

$$\sum_{\ell \geq 1} \frac{N(3^\ell)}{3^{2\ell}} \leq \frac{4}{8} + \frac{4}{18} + \frac{1}{9} + \frac{4}{9} = \frac{23}{18},$$

which is 3.1.21.

For a prime $p \geq 5$, relations 3.1.23 and 3.1.11 lead to

$$\sum_{\ell \geq 2} \frac{N(p^\ell)}{p^{2\ell}} \leq \frac{4}{p^2 - 1} + \frac{4}{p^2} \leq \frac{8.2}{p^2}$$

so that we have

$$\begin{aligned} \sum_{p \geq 5} \sum_{\ell \geq 2} \frac{N(p^\ell)}{p^{2\ell}} &\leq 8.2 \sum_{p \geq 5} \frac{1}{p^2} \\ &\leq 8.2 \left(\frac{1}{25} + \frac{1}{49} + \frac{1}{121} + \sum_{n \geq 6} \frac{1}{(2n)^2} \right) \leq 1, \end{aligned}$$

whence 3.1.22 is proven. □

PROOF OF PROPOSITION 3.1.1. By inequality 3.1.2, we have

$$\begin{aligned} A(X) &\leq 4X^2 \sum_{0 < k < X} \frac{N(k)}{k^2} \\ &\leq 4X^2 \prod_{p \leq X} \left(1 + \sum_{\ell \geq 1} \frac{N(p^\ell)}{p^{2\ell}} \right) \\ &\leq 4X^2 \exp \left(\sum_{p \leq X} \text{Log} \left(1 + \sum_{\ell \geq 1} \frac{N(p^\ell)}{p^{2\ell}} \right) \right) \end{aligned}$$

By Lemma 3.1.4, this leads to

$$A(X) \leq 40X^2 \exp \left(\sum_{5 \leq p \leq X} \frac{N(p)}{p^2} + 1 \right).$$

From 3.1.6 and 3.1.11, we have $N(p) \leq 4p$, so that we get

$$(3.1.24) \quad A(X) \leq 40X^2 \exp \left(4 \sum_{5 \leq p \leq X} \frac{1}{p} + 1 \right).$$

Combined with the relation (3.18) from Rosser and Schoenfeld [22], our last relation 3.1.24 leads to

$$A(X) \leq 40X^2 \exp(4 \text{Log Log } X - 1.28) \leq 12X^2 (\text{Log } X)^4,$$

from where Proposition 3.1 easily follows. □

3.2 Divisor sums

In this section, we give some upper bounds for mean-values of divisor functions.

PROPOSITION 3.2. *Let $P \geq 10^{74}$; for $0 \leq \lambda \leq 0.5$, we have*

$$(3.2.1) \quad \sum_{\ell \geq 1} \left(\sum_{0 < h < P} \sum_{2P < y \leq 4P} \sum_{\substack{0 < k \leq P^2/4 \\ hyk \leq P^4}} d(k)^\lambda \right)^2 \\ \leq \frac{28.3}{1 + \lambda} 3^{4\lambda} P^4 (\text{Log } P)^{5+6\lambda};$$

and, for $1 \leq \kappa \leq 2$, we have

$$(3.2.2) \quad \sum_{k \geq 1} \left(\sum_{h_1 \leq P} \sum_{\substack{h_2 \leq P \\ h_1 h_2 = k}} 1 \right)^\kappa \leq 1.01 \cdot 2^{k-1} P^2 (\text{Log } P)^{\kappa-1}.$$

These results are by no means best possible, but their derivation is completely elementary and easy, and furthermore, they are not far from being best possible in the range where we shall use them. The reader should not have any difficulty in providing a proof for the following lemma; if needed, a reference can be the work of Chen [7].

LEMMA 3.2.1. *Let $X \geq 1$ and $j = 1$ or 2 . We have*

$$(3.2.3) \quad \sum_{n \leq X} d^j(n) \leq A_j X (\text{Log } X + j)^{2j-1}$$

and

$$(3.2.4) \quad \sum_{n \leq X} \frac{d^j(n)}{n} \leq 2^{-j} A_j (\text{Log } X + j + 1)^{2j}$$

where

$$(3.2.5) \quad A_1 = 1 \text{ and } A_2 = 1/3,$$

PROOF OF 3.2.2. It is enough to prove 3.2.2 for $\kappa = 1$ and $\kappa = 2$, and then apply Hölder's inequality. The case $\kappa = 1$ is fairly straightforward (inversion of

summations); for $\kappa = 2$, we have

$$\begin{aligned} \sum_{k \leq 1} \left(\sum_{h_1 \leq P} \sum_{\substack{h_2 \leq P \\ h_1 h_2 = k}} 1 \right)^2 &= \sum_{k \leq P^2} \left(\sum_{\substack{d|k \\ d \leq P, k/d \leq P}} 1 \right)^2 \\ &= \sum_{k \leq P^2} \sum_{\substack{d_1|k \\ d_1 \leq P, k/d_1 \leq P}} \sum_{\substack{d_2|k \\ d_2 \leq P, k/d_2 \leq P}} 1 \\ &\leq 2 \sum_{d_1 \leq P} \sum_{d_2 \leq d_1} \sum_{\substack{n \leq P d_2 \\ d_1|n, d_2|n}} 1 \\ &\leq 2P \sum_{d_1 \leq P} \sum_{d_2 \leq d_1} \frac{d_2}{\text{lcm}(d_1, d_2)} \\ &\leq 2P \sum_{d_1 \leq P} \sum_{\ell|d_1} \sum_{m \leq d_1/\ell} \frac{m\ell}{d_1 m} \leq 2P \sum_{d_1 \leq P} d(d_1). \end{aligned}$$

By the case $j = 1$ of 3.2.3, the sum under consideration is at most

$$2P^2(\text{Log } P + 1),$$

which is less than

$$1.01 \times 2P^2 \text{Log } P,$$

since $\text{Log } P > 100$. □

LEMMA 3.2.2. *For $X \geq 1$ and $0 \leq \lambda \leq 0.5$, we have*

$$(3.2.6) \quad \sum_{n \leq X} \frac{1}{n} \left(\sum_{\delta|n} d(\delta)^\lambda \right)^2 \leq \frac{4^{-\lambda}}{12} (\text{Log } X + 3)^{4+4\lambda}$$

PROOF OF LEMMA 3.2.2. By Hölder's inequality, it is enough to verify the cases $\lambda = 0$ and $\lambda = 1/2$.

The case $\lambda = 0$ of 3.2.6 reduces to the case $j = 2$ of 3.2.4, since $\sum_{\delta|n} d(\delta)^0 = d(n)$.

For $\lambda = 1/2$, we notice that Cauchy's inequality implies

$$(3.2.7) \quad \left(\sum_{\delta|n} d(\delta)^{1/2} \right)^2 \leq d(n) \left(\sum_{\delta|n} d(\delta) \right);$$

we thus have

$$\begin{aligned}
 (3.2.8) \quad & \sum_{n \leq X} \frac{1}{n} \left(\sum_{\delta|n} d(\delta)^{1/2} \right)^2 \leq \sum_{n \leq X} \frac{d(n)}{n} \sum_{\delta|n} d(\delta) \\
 & \leq \sum_{\delta \leq X} d(\delta) \sum_{\substack{n \leq X \\ n=0[\delta]}} \frac{d(n)}{n} \leq \sum_{\delta \leq X} \frac{d^2(\delta)}{\delta} \sum_{m \leq X} \frac{d(m)}{m},
 \end{aligned}$$

where the last inequality comes from the submultiplicativity of the divisor function. The case $\lambda = 1/2$ of Lemma 3.2.2 now follows directly from the cases $j = 1$ and $j = 2$ of 3.2.4. □

PROOF OF 3.2.1. Let us denote by S_λ the LHS of 3.2.1.

We have

$$S_\lambda \leq \sum_{1 \leq \ell \leq P^4} \left(\sum_y \sum_k d(k)^\lambda \right)^2,$$

where the variables y and k satisfy

$$(3.2.9) \quad 2P < y \leq 4P, \quad 0 < k \leq P^2/4, \quad yk|\ell, \quad \ell \leq Pyk;$$

expanding the square, we find

$$(3.2.10) \quad S_\lambda \leq 2 \sum_{y_1} \sum_{k_1} d(k_1)^\lambda \sum_{\substack{y_2 \\ y_2 k_2 \leq y_1 k_1}} \sum_{k_2} d(k_2)^\lambda \sum_{\substack{\ell \leq P y_2 k_2 \\ y_1 k_1 | \ell}} 1.$$

We let $m = (y_1 k_1, y_2 k_2)$, $y_i = m_i r_i$, $k_i = n_i s_i$ with $m = m_i n_i$; we have

$$(3.2.11) \quad S_\lambda \leq 2 \sum_{m \leq P^3} \sum_{m_i, n_i} \sum_{r_i, s_i} d(n_1 s_1)^\lambda d(n_2 s_2)^\lambda \sum_\ell 1,$$

where the summation conditions are

$$(3.2.12) \quad \begin{cases} m_i \leq 4P, \quad n_i \leq P^2/4, \quad m_i n_i = m, \quad \frac{2P}{m_i} < r_i \leq \frac{4P}{m_i}, \quad s_i \leq \frac{P^2}{4n_i}, \\ \ell \leq P m_2 r_2 n_2 s_2 \text{ and } \ell \equiv 0 \pmod{r_1 s_1 m_2 r_2 n_2 s_2}. \end{cases}$$

We thus have

$$\begin{aligned}
 (3.2.13) \quad & S_\lambda \leq 2P \sum_{m \leq P^3} \sum_{m_i, n_i} d(n_1)^\lambda d(n_2)^\lambda \\
 & \sum_{r_1, r_2} \frac{1}{r_1} \sum_{s_1 \leq P^2/4} \frac{d(s_1)^\lambda}{s_1} \sum_{s_2 \leq P^2/4n_2} d(s_2)^\lambda;
 \end{aligned}$$

for the sum over s_2 we use Hölder's inequality and 3.2.3:

$$(3.2.14) \quad \sum_{s \leq X} d(s)^\lambda \leq X(\text{Log } X + 1)^\lambda;$$

we now sum this relation by parts, which leads to

$$(3.2.15) \quad \sum_{s \leq X} \frac{d(s)^\lambda}{s} \leq \frac{1}{1 + \lambda} (\text{Log } X + 2)^{1+\lambda};$$

in 3.2.13 the sum over r_2 is trivial, and that over r_1 is dealt with, thanks to the following easy inequality

$$(3.2.16) \quad \sup_{z > 0} \sum_{z < r \leq 2z} 1/r \leq 1.$$

Indeed, if one denotes by $g(z)$ the *LHS* of 3.2.16, it is easy to check that g is constant on any interval $\left[\frac{n}{2}, \frac{n+1}{2}\right]$, that one has

$$g(0) \leq g\left(\frac{1}{2}\right) = 1, \quad g(1) \leq g\left(\frac{3}{2}\right) = \frac{5}{6}, \quad g(2) \leq g\left(\frac{5}{2}\right) = \frac{47}{60},$$

$$g(3) \leq g\left(\frac{7}{2}\right) = g\left(\frac{e}{e-2}\right), \quad \text{and for } z \geq \frac{e}{e-2} \text{ one has}$$

$$g(z) \leq \int_{z-1}^{2z} \frac{dt}{t} = \text{Log}\left(\frac{2z}{z-1}\right) \leq 1.$$

By 3.2.14, 3.2.15, 3.2.16, the inequality 3.2.13 becomes

$$(3.2.17) \quad S_\lambda \leq \frac{2}{1+\lambda} P \cdot 4P \cdot \frac{1}{4} P^2 \sum_{m \leq P^3} \sum_{m_1, n_1} \frac{d(n_1)^\lambda d(n_2)^\lambda}{m_2 n_2} \left(\text{Log } \frac{P^2}{4} + 2\right)^{1+2\lambda}$$

Since $P \geq 10^{74}$, we deduce from 3.2.17 that

$$(3.2.18) \quad S_\lambda \leq 1.01 \frac{2^{2+2\lambda}}{1+\lambda} P^4 (\text{Log } P)^{1+2\lambda} \sum_{m \leq P^3} \frac{1}{m} \left(\sum_{n|m} d(n)^\lambda\right)^2.$$

We now use Lemma 3.2.2 and get

$$S_\lambda \leq \frac{28.3}{1+\lambda} 3^{4\lambda} P^4 (\text{Log } P)^{5+6\lambda},$$

which is 3.2.1. □

3.3 Hua's inequality

In this section, we prove Theorem 3.

Let us denote by I_0 the interval $]P - \varepsilon/2, 2P - \varepsilon/2[$.

1st step. By Parseval's identity, we have

$$(3.3.1) \quad \int_0^1 |S_\varepsilon(\alpha)|^2 d\alpha = \sum_{x \in I_0} 1 = P$$

2nd step. We have

$$(3.3.2) \quad \begin{aligned} |S_\varepsilon(\alpha)|^2 &= \sum_{x,y} e(\alpha((2x + \varepsilon)^4 - (2y + \varepsilon)^4)) \\ &= \sum_{|h_1| < P} \sum_{y \in I_1} e(\alpha Q_1(h_1, y)), \end{aligned}$$

where

$$(3.3.3) \quad I_1 = I_1(h_1) = I_0 \cap (I_0 - h_1),$$

$$(3.3.4) \quad \begin{aligned} Q_1(h_1, y) &= (2(y + h_1) + \varepsilon)^4 - (2y + \varepsilon)^4 \\ &= 8h_1((2y + \varepsilon)^3 + 3h_1(2y + \varepsilon)^2 + 4h_1^2(2y + \varepsilon) + 2h_1^3); \end{aligned}$$

we may thus write

$$(3.3.5) \quad |S_\varepsilon(\alpha)|^2 = \sum_{h \in \mathbb{Z}} c_{h,1} e(\alpha h),$$

where

$$(3.3.6) \quad c_{h,1} = \#\{(h_1, y) / |h_1| < P, y \in I_1, Q_1(h_1, y) = h\}.$$

If $h = 0$, we have $c_{0,1} = \int_0^1 |S_\varepsilon(\alpha)|^2 d\alpha$, so that 3.3.1 leads to

$$(3.3.7) \quad c_{0,1} = P.$$

If $h \neq 0$, relation 3.3.4 implies that h_1 has the same sign as h , so that

$$(3.3.8) \quad c_{h,1} \leq \#\{(h_1, y) / hh_1 > 0, |h_1| < P, y \in I_1, Q_1(h_1, y) = h\}.$$

For a given positive h_1 , the function $y \mapsto (2y + \varepsilon)^3 + 3h_1(2y + \varepsilon)^2 + 4h_1^2(2y + \varepsilon) + 2h_1^3$ is strictly increasing on I_0 ; we have

$$(3.3.9) \quad \sum_{h>0} c_{h,1}^2 \leq \#\{(k_1, h_1, y), 0 < k_1 < P, 0 < h_1 < P, y \in I_1, k_1 | Q_1(h_1, y)\}.$$

From 3.3.5 we get

$$(3.3.10) \quad c_{-h,1} = c_{h,1}$$

which leads to

$$(3.3.11) \quad \sum_{h \neq 0} c_{h,1}^2 \leq \#\{(k, x, y)/0 < k < P, x \in I_0, y \in I_0, (2x + \varepsilon)^4 \equiv (2y + \varepsilon)^4 \pmod{k}\}$$

and so we have

$$(3.3.12) \quad \sum_{h \neq 0} c_{h,1}^2 \leq \#\{(k, \ell, m)/0 < k < P, \ell \in]2P, 4P], m \in]2P, 4P], \ell^4 \equiv m^4 \pmod{k}\}.$$

From 3.3.5 and Parseval's equality, we get

$$(3.3.13) \quad \int_0^1 |S_\varepsilon(\alpha)|^4 d\alpha = \sum_h c_{h,1}^2,$$

and finally, Proposition 3.1 and relations 3.3.13, 3.3.7, and 3.3.12 lead to

$$(3.3.14) \quad \int_0^1 |S_\varepsilon(\alpha)|^4 d\alpha \leq 60 P^2 (\text{Log } P)^4.$$

We may notice here that Greaves has proven that indeed $\int_0^1 |S_\varepsilon(\alpha)|^4 d\alpha = O(P^2)$; however his result, though effective, would imply a tremendous constant; on the other hand, 3.3.14 will enter our final result only as a negligible error-term.

3rd step. On one hand, we write

$$(3.3.15) \quad |S_\varepsilon(\alpha)|^4 = \sum_k b_{k,2} e(\alpha),$$

so that we have

$$(3.3.16) \quad b_{0,2} = \int_0^1 |S_\varepsilon(\alpha)|^4 d\alpha,$$

and by Parseval's relation

$$(3.3.17) \quad \sum_k b_{0,k}^2 = \int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha.$$

On the other hand, by applying Cauchy's inequality to 3.3.2, we get

$$(3.3.18) \quad \begin{aligned} |S_\varepsilon(\alpha)|^4 &\leq 2P \sum_{|h_1| < P} \left| \sum_{y \in I_1} e(\alpha Q_1(h_1, y)) \right|^2 \\ &\leq 2P \sum_{|h_1| < P} \sum_{|h_2| < P - |h_1|} \sum_{y \in I_2} e(\alpha Q_2(h_1, h_2, y)), \end{aligned}$$

where

$$(3.3.19) \quad I_2 = I_2(h_1, h_2) = I_1 \cap (I_1 - h_2),$$

$$(3.3.20) \quad \begin{aligned} Q_2(h_1, h_2, y) &= Q_1(h_1, y + h_2) - Q_1(h_1, y) \\ &= 16(3(2y + \varepsilon + h_1 + h_2)^2 + h_1^2 + h_2^2)h_1 h_2. \end{aligned}$$

We may thus write

$$(3.3.21) \quad |S_\varepsilon(\alpha)|^4 \leq 2P \sum_{h \in \mathbb{Z}} c_{h,2} e(\alpha h),$$

where

$$(3.3.22) \quad c_{h,2} := \#\{(h_1, h_2, y) / |h_1| + |h_2| < P, y \in I_2, Q_2(h_1, h_2, y) = h\}$$

From 3.3.20, we see that $Q_2(h_1, h_2, y)$ vanishes only when h_1 or h_2 is zero, so that

$$(3.3.23) \quad c_{0,2} \leq 4P^2$$

If h is not divisible by 16, then $c_{h,2} = 0$. If h is divisible by 16, and different from 0, we write $h = 16\ell$; the coefficient $c_{16\ell,2}$ is at most twice the number of ways to write $|\ell|$ as a product of three positive integers: indeed, if $h_1 = \pm u$, $h_2 = \pm \frac{h}{|h|}v$, we have to solve $3(2y + \varepsilon + h_1 + h_2)^2 + h_1^2 + h_2^2 = w$

with $2y + \varepsilon + h_1 + h_2 > 0$, which admits at most one solution. We thus have

$$(3.3.24) \quad \sum_{h \neq 0} c_{h,2}^2 \leq 2 \sum_{\substack{0 < |h_1| + |h_2| < P \\ h_1 h_2 \neq 0}} \sum_{y \in I_2} d_3 \left(\frac{1}{16} |Q_2(h_1, h_2, y)| \right) \leq B_\varepsilon(P),$$

where $B_\varepsilon(P)$ satisfies relations 3.0.1 and 3.0.2. By 3.3.15 and 3.3.21, we have

$$(3.3.25) \quad \begin{aligned} \int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha &\leq 2P \sum_{h \in \mathbb{Z}} c_{h,2} b_{h,2} \\ &\leq 2P \left(c_{0,2} b_{0,2} + \sum_{h \neq 0} c_{h,2} b_{h,2} \right) \\ &\leq 2P \left(c_{0,2} b_{0,2} + \left(\sum_{h \neq 0} c_{h,2}^2 \right)^{1/2} \left(\sum_{h \neq 0} b_{h,2}^2 \right)^{1/2} \right). \end{aligned}$$

We now use 3.3.16 and 3.3.17 for estimating the $b_{h,2}$'s and 3.3.23 and 3.3.24, for the $c_{h,2}$'s. We get

$$(3.3.26) \quad \int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha - 2PB_\varepsilon^{\frac{1}{2}} \left(\int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha \right)^{1/2} - 8P^3 \int_0^1 |S_\varepsilon(\alpha)|^4 d\alpha \leq 0,$$

where we let

$$(3.3.27) \quad B_\varepsilon = B_\varepsilon(P).$$

By 3.3.14, we get

$$(3.3.28) \quad \begin{aligned} \left(\int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha \right)^{1/2} &\leq PB_\varepsilon^{1/2} + \{P^2 B_\varepsilon + 8P^3 60P^2 (\text{Log } P)^4\}^{1/2} \\ &\leq PB_\varepsilon^{1/2} \{1 + (1 + 480P^3 (\text{Log } P)^4 B_\varepsilon^{-1})^{1/2}\}. \end{aligned}$$

We now use 3.0.1; for $P \geq 10^{74}$, we get

$$(3.3.29) \quad \int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha \leq (4 + 10^{-6}) B_\varepsilon(P) \cdot P^2.$$

4th step. As we start to be used to, we write

$$(3.3.30) \quad |S_\varepsilon(\alpha)|^8 = \sum_k b_{k,3} e(\alpha k),$$

so that we have

$$(3.3.31) \quad b_{0,3} = \int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha,$$

and

$$(3.3.32) \quad \sum_k b_{k,3}^2 = \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha.$$

In connection with 3.3.18, we introduce the following notation. For an integer k , we let

$$(3.3.33) \quad \mathcal{E}(k) = \mathcal{E}_P(k) = \{(h_1, h_2) | h_1 h_2 = k, 0 < |h_1| + |h_2| < P\}.$$

If $\mathcal{E}(k)$ is non empty, we choose a triple (h_1, h_2, I_2) that we denote $(h_1(k), h_2(k), I(k))$, such that

$$(3.3.34) \quad (h_1, h_2) \in \mathcal{E}(k), I_2 = I_2(h_1, h_2), \left| \sum_{y \in I_2} e(\alpha Q_2(h_1, h_2, y)) \right| \text{ maximal.}$$

If $\mathcal{E}(k)$ is empty, which is surely the case when $4k > P^2$, we define

$$(3.3.35) \quad (h_1(k), h_2(k), I_2(k)) = (1, 1, \emptyset).$$

We further write

$$(3.3.36) \quad E(k) = E_P(k) := \text{Card } \mathcal{E}_P(k).$$

and we discard the value $k = 0$ in S_ε by defining

$$(3.3.37) \quad T_\varepsilon(\alpha) = |S_\varepsilon(\alpha)|^4 - 8P^3.$$

From 3.3.18 we deduce

$$(3.3.38) \quad T_\varepsilon(\alpha) \leq 2P \sum_{k \neq 0} E(k) \left| \sum_{y \in I_2(k)} e(\alpha Q_2(h_1(k), h_2(k), y)) \right|$$

and by Cauchy's inequality, we get for $0 \leq \lambda \leq 2$

$$(3.3.39) \quad T_\varepsilon^2(\alpha) \leq 4P^2 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \sum_{k \neq 0} E^\lambda(k) \left| \sum_{y \in I_2(k)} e(\alpha Q_2(h_1(k), h_2(k), y)) \right|^2.$$

The expression $\left| \sum_y \dots \right|^2$ may be rewritten as

$$(3.3.40) \quad \sum_{h_3} \sum_{y \in I_3} e(\alpha Q_3(k, h_3, y)),$$

where

$$(3.3.41) \quad 0 < |h_1(k)| + |h_2(k)| + |h_3| < P,$$

$$(3.3.42) \quad I_3 = I_3(k, h_3) = I_2(k) \cap (I_2(k) - h_3),$$

and

$$(3.3.43) \quad Q_3(k, h_3, y) = 192 kh_3(2y + \varepsilon + h_1(k) + h_2(k) + h_3);$$

with this notation, we have

$$(3.3.44) \quad T_\varepsilon^2(\alpha) \leq 4P^2 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \left(\sum_{h \in \mathbb{Z}} c_{h,3} e(\alpha h) \right),$$

where

$$(3.3.45) \quad c_{h,3} = \sum_{k \neq 0} E^\lambda(k) \text{ Card } \{(h_3, y) \mid |h_1| + |h_2| + |h_3| < P, y \in I_3, Q_3(k, h_3, y) = h\}.$$

We shall retain from 3.3.45 only the following relations

$$(3.3.46) \quad 0 \leq c_{0,3} \leq P \sum_{k \neq 0} E^\lambda(k)$$

$$(3.3.47) \quad c_{h,3} = 0 \text{ if } 192 \nmid h$$

and, for $\ell \neq 0$:

$$(3.3.48) \quad 0 \leq c_{192\ell,3} \leq 2 \sum_{0 < k \leq P^2/4} d^\lambda(k) \sum_{0 < h \leq P} \sum_{\substack{2P < z \leq 4P \\ khz = |\ell|}} 1.$$

From 3.3.37 and 3.3.44, we get

$$(3.3.49) \quad |S_\varepsilon(\alpha)|^8 \leq 4P^2 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \left(\sum_{h \in \mathbb{Z}} c_{h,3} e(\alpha h) \right) + 16P^3 |S_\varepsilon(\alpha)|^4.$$

Combined with 3.3.30, this leads to

$$(3.3.50) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 4P^2 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \left(\sum_h c_{h,3} b_{h,3} \right) + 16 \int_0^1 |S_\varepsilon(\alpha)|^{12} d\alpha,$$

and Cauchy and Schwarz inequalities allow to write

$$(3.3.51) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 4P^2 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \left\{ c_{0,3} b_{0,3} + \left(\sum_{h \neq 0} c_{h,3}^2 \right)^{1/2} \left(\sum_{h \neq 0} b_{h,3}^2 \right)^{1/2} \right\} + 16P^3 \left(\int_0^1 |S_\varepsilon(\alpha)|^8 d\alpha \right)^{1/2} \left(\int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \right)^{1/2}.$$

Together with 3.3.46, 3.3.32, 3.3.31 and 3.3.29, this leads to

$$(3.3.52) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 16.1 B_\varepsilon P^5 \left(\sum_{k \neq 0} E^{2-\lambda}(k) \right) \left(\sum_{k \neq 0} E^\lambda(k) \right) + \left(\int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \right)^{1/2} \left(4P^2 \left(\sum_{h \neq 0} E^{2-\lambda}(k) \right) \left(\sum_{h \neq 0} c_{h,3}^2 \right)^{1/2} + 32.1 P^4 B_\varepsilon^{1/2} \right).$$

In order to solve the quadratic inequality 3.3.52, we notice that for $x \geq 0$ we have

$$(3.3.53) \quad 2 + x + 2\sqrt{1+x} \leq 2 + x + 2\left(1 + \frac{x}{2}\right) = 4 + 2x,$$

so that, for $a \geq 0$ and $b \geq 0$ we have

$$(3.3.54) \quad (b + \sqrt{b^2 + a})^2 \leq 4b^2 + 2a.$$

From 3.3.52 and 3.3.54, we get

$$(3.3.55) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 8.05 B_\varepsilon P^5 \left(\sum_{k \neq 0} E_P^{2-\lambda}(k) \right) \left(\sum_{k \neq 0} E_P^\lambda(k) \right) \\ + \left(4P^2 \left(\sum_{k \neq 0} E_P^{2-\lambda}(k) \right) \left(\sum_{h \neq 0} c_{h,3}^2 \right)^{1/2} + 32.1 P^4 B_\varepsilon^{1/2} \right)^2$$

We use 3.3.47 and 3.3.48 to majorize the $c_{h,3}$'s:

$$(3.3.56) \quad \sum_{h \neq 0} c_{h,3}^2 \leq 8 \sum_{1 \leq \ell \leq P^4} \left(\sum_{0 < h \leq P} \sum_{2P < y \leq 4P} \sum_{\substack{0 < k \leq P^2/4 \\ hyk = \ell}} d^\lambda(k) \right)^2$$

We have now reduced the majorization of $\int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha$ to that of divisor sums, for which we apply Proposition 3.2. We get, for $0 \leq \lambda \leq 0.5$

$$(3.3.57) \quad \int_0^1 |S_\varepsilon(\alpha)|^{16} d\alpha \leq 66 \cdot 2^{-\lambda} B_\varepsilon(P) P^9 (\text{Log } P)^{1-\lambda} \\ + \left(\frac{244 \cdot 3^{2\lambda}}{2^\lambda \sqrt{1+\lambda}} P^6 (\text{Log } P)^{3.5+2\lambda} + 32.1 P^4 B_\varepsilon^{1/2} \right)^2,$$

which leads to 3.0.4, by using the upper bound 3.0.1 for B_ε in the second term of the *RHS*. □

§4. - Upper bound for $\int_0^1 |S_0(\alpha)|^u |S_1(\alpha)|^v d\alpha$

We combine Theorem 3 with the main result of [13], in order to derive the following

THEOREM 4. *Let P be larger than 10^{87} , and u and v be two integers such that $4 \leq v \leq 16$ and $u + v = 16$. We have*

$$(4.0.1) \quad \int_0^1 \left| \sum_{\substack{2P < x \leq P \\ x \text{ even}}} e(\alpha x^4) \right|^u \left| \sum_{\substack{2P < x \leq 4P \\ x \text{ odd}}} e(\alpha x^4) \right|^v d\alpha \\ \leq 0.36 P^{12} (\text{Log } P)^{9.88} (\text{Log Log } P)^2$$

We quote the main result of [13].

PROPOSITION 4.0. *With the notation of Theorem 3, we may take, for $P \geq 10^{80}$*

$$(4.0.2) \quad B_\varepsilon(P) = C_\varepsilon P^3 (\text{Log } P)^9 (\text{Log Log } P)^2,$$

with $C_0 = 1.734 \cdot 10^{-2}$ and $C_1 = 8.18 \cdot 10^{-4}$.

PROOF OF THEOREM 4. Thanks to Hölder's inequality we have

$$(4.0.3) \quad \int_0^1 |S_0(\alpha)|^u |S_1(\alpha)|^v d\alpha \leq \left(\int_0^1 |S_0(\alpha)|^{16} d\alpha \right)^{u/16} \left(\int_0^1 |S_1(\alpha)|^{16} d\alpha \right)^{v/16},$$

and we can use Hua's inequality to get a bound for each integral in the *RHS* of 4.0.3 (cf. 4.0.7 and 4.0.10).

In the case when $\varepsilon = 0$, we use Theorem 3 with $\lambda = 0.24$. By 3.0.4 and 4.0.2, we get

$$(4.0.4) \quad \int_0^1 |S_0(\alpha)|^{16} d\alpha \leq 99700 P^{12} (\text{Log } P)^{7.96} \\ + 0.970 P^{12} (\text{Log } P)^{9.76} (\text{Log Log } P)^2;$$

since $\text{Log } P \geq 200$, we have

$$(4.0.5) \quad 99700 (\text{Log } P)^{-1.8} (\text{Log Log } P)^{-2} \leq 0.260,$$

and so, from 4.0.4 and 4.0.5, we get

$$(4.0.6) \quad \int_0^1 |S_0(\alpha)|^{16} d\alpha \leq 1.23P^{12}(\text{Log } P)^{9.76}(\text{Log Log } P)^2.$$

In order to compare it with the integral of $S_1(\alpha)$, we shall use the following

$$(4.0.7) \quad \int_0^1 |S_0(\alpha)|^{16} d\alpha \leq 0.66P^{12}(\text{Log } P)^{9.88}(\text{Log Log } P)^2.$$

In the case when $\varepsilon = 1$, we use Theorem 3 with $\lambda = 0.12$. By 3.0.4 and 4.0.2, we get

$$(4.0.8) \quad \int_0^1 |S_1(\alpha)|^{16} d\alpha \leq 76900 P^{12}(\text{Log } P)^{7.48} \\ + 0.0497 P^{12}(\text{Log } P)^{9.88}(\text{Log Log } P)^2;$$

since $\text{Log } P \geq 200$, we have

$$(4.0.9) \quad 76900(\text{Log } P)^{-2.4}(\text{Log Log } P)^{-2} \leq 0.0083,$$

and, from 4.0.8 and 4.0.9, we get

$$(4.0.10) \quad \int_0^1 |S_1(\alpha)|^{16} d\alpha \leq 0.058 P^{12}(\text{Log } P)^{9.88}(\text{Log Log } P)^2$$

We come back to 4.0.3. Since $v \geq 4$, and the *RHS* of 4.0.10 is smaller than that of 4.0.7, we have

$$(4.0.11) \quad \int_0^1 |S_0(\alpha)|^u |S_1(\alpha)|^u d\alpha \\ \leq (0.66)^{3/4} (0.058)^{1/4} P^{12}(\text{Log } P)^{9.88}(\text{Log Log } P)^2,$$

which leads to 4.0.1. □

§5. - Upper bound for Weyl sums on the minor arcs

THEOREM 5. *With the notation of section 2.0, we have, for $\alpha \in m$ and $P \geq 10^{80}$:*

$$(5.0.1) \quad |S_\varepsilon(\alpha)| \leq 15.7 P^{0.884} (\text{Log } P)^{0.25},$$

where $\varepsilon = 0$ or 1 .

By Dirichlet's approximation theorem, for any $\alpha \in m$, we may find a pair (a, q) of coprime integers with

$$(5.0.2) \quad P^{1/2} < q \leq \frac{P^3}{974} \quad \text{and} \quad \left| a - \frac{\alpha}{q} \right| \leq \frac{975}{P^3}$$

When q is larger than $4 \cdot 10^6 P$, the main result of [14] tells us precisely that 5.0.1 holds. It may be interesting at that point to underline the quality of 5.0.1, which implies that for $P = 10^{91}$, we have $|S_\varepsilon(\alpha)| \leq 394 P^{7/8}$ on the minor arcs, it being possible to reduce this bound through the improvements explained, but not performed in [14].

Proposition 5.1 will close the proof of Theorem 5.

5.1 Contribution of the minor arcs associated to small q 's

PROPOSITION 5.1. *Let α be a real number for which one can find integers P, q, a , such that*

$$(5.1.1) \quad P \geq 10^{80}, \quad P^{1/2} \leq q \leq 4 \cdot 10^6 P, \quad (a, q) = 1, \quad \left| a - \frac{\alpha}{q} \right| \leq \frac{975}{qP^3};$$

for $\varepsilon = 0$ or 1 we have

$$(5.1.2) \quad |S_\varepsilon(\alpha)| \leq 20P^{7/8}.$$

PROOF OF PROPOSITION 5.1. From 2.4.2 in Proposition 2.4, we have

$$(5.1.3) \quad \left| S_\varepsilon(\alpha) - \frac{1}{2q} G_\varepsilon(a, q; 0) I(\beta, N) \right| \leq 2.7 \cdot 10^6 q^{1/4} P^{1/2} + 61 q^{3/4} (\text{Log } q + 1).$$

We then use Proposition 2.2, and rewrite 2.2.3 as

$$(5.1.4) \quad |G_\varepsilon(a, q; \nu)| \leq 18q^{3/4},$$

as well as the trivial upper bound

$$(5.1.5) \quad |I(\beta, N)| \leq 2P_0 \leq 2P + 2.$$

We get

$$\begin{aligned}
 |S_\varepsilon(\alpha)| &\leq 18.5 Pq^{-1/4} + 2.7 \cdot 10^6 q^{1/4} P^{1/2} + 61q^{3/4}(\text{Log } q + 1) \\
 (5.1.6) \quad &\leq 18.5 P^{7/8} + 1.3 \cdot 10^8 P^{3/4} + 7 \cdot 10^6 P^{3/4} \text{Log } P \\
 &\leq 20P^{7/8},
 \end{aligned}$$

which is 5.1.2. □

§6. - Proof of the main theorem

The general route has been explained in the first chapter. With the notation of 1.0.1 we want to prove that $r(N) > 0$ for $N \geq 10^{364}$. Because of 2.0.2 and the fact that $\nu_0 \leq 151$, we have $P \geq 1.4 \cdot 10^{90}$, and we can apply Theorems 2, 4 and 5.

By Theorem 4 and 5, we have

$$(6.0.1) \quad \int_m |S_0(\alpha)|^s |S_1(\alpha)|^t d\alpha \leq 1394 P^{14.652} (\text{Log } P)^{10.63} (\text{Log Log } P)^2,$$

and by Theorem 2, we have

$$(6.0.2) \quad \int_M S_0(\alpha)^s S_1(\alpha)^t e(-\alpha N) d\alpha \geq 0.0079 P^{15}$$

The ratio of the *RHS* of 6.0.2 by that of 6.0.1 is a function P which is increasing for $P \geq 1.4 \cdot 10^{90}$, and takes the value

$$1.0925 \dots > 1$$

for $P = 1.4 \cdot 10^{90}$. This implies that for $P \geq 1.4 \cdot 10^{90}$, we have

$$r(N) = \int_0^1 S_0^s(\alpha) S_1^t(\alpha) e(-\alpha N) d\alpha > 0,$$

and so every integer N larger than 10^{364} is the sum of 19 biquadrates. □

REFERENCES

- [1] F.C. AULUCK, *On Waring's problem for biquadrates*. Proc. Nat. Acad. Sci. India Sect. A **11** (1940), 437-450.
- [2] R. BALASUBRAMANIAN, *On Waring's problem: $g(4) \leq 21$* . Hardy-Ramanujan Journ. **2** (1979), 1-31.
- [3] R. BALASUBRAMANIAN, *On Waring's problem: $g(4) \leq 20$* . Hardy-Ramanujan Journ. **8** (1985), 1-40.
- [4] R. BALASUBRAMANIAN - J-M. DESHOUILERS - F. DRESS, *Problème de Waring pour les bicarrés, 1: Schéma de la solution*. C.R. Acad. Sci. Paris **303** (1986), 85-88.
- [5] R. BALASUBRAMANIAN - J-M. DESHOUILERS - F. DRESS, *Problème de Waring pour les bicarrés, 2: résultats auxiliaires pour le théorème asymptotique*. C.R. Acad. Sci. Paris **303** (1986), 161-163.
- [6] Z.I. BOREVICH - I.R. SHAFAREVICH, *Number Theory*. Acad. Press, New York, 1966.
- [7] CHEN JING-RUN, *Waring's problem for $g(5) = 37$* . Sci. Sinica **13** (1964), 1547-1568.
- [8] H. DAVENPORT, *Sur les sommes de puissances entières*. C.R. Acad. Sci. Paris **207** (1938), 1366-1368.
- [9] H. DAVENPORT, *On Waring's problem for fourth powers*. Ann. of Math. **40** (1939), 731-747.
- [10] F. DELMER - J-M. DESHOUILERS, *On the computation of $g(k)$ in Waring's problem*. Math. Comp. **54** (1990), 815-820.
- [11] J-M. DESHOUILERS, *Problème de Waring pour les bicarrés*. Sémin. Th. Nb. Bordeaux, année 1984-1985, exp. 14, 47 pp.
- [12] J-M. DESHOUILERS, *Le problème de Waring pour les bicarrés: le point en 1984*. Sémin. Th. anal. Nb. Paris, année 1984-1985.
- [13] J-M. DESHOUILERS - F. DRESS, *Sommes de diviseurs et structures multiplicatives des entiers*. Acta Arith. **49** (1988), 341-375.
- [14] J-M. DESHOUILERS, *Sur la majoration de sommes de Weyl biquadratiques* (to appear Annali Scuola Normale Superiore Pisa).
- [15] J-M. DESHOUILERS - F. DRESS, *Numerical results for sums of 5 and 7 biquadrates, and consequences for sums of 19 biquadrates* (submitted to Math. Comp.).
- [16] L.E. DICKSON, *Proof of the ideal Waring theorem for exponents 7-180, followed by Solution of Waring's problem*. Amer. Journ. Math. **58** (1936), 521-535.
- [17] G.H. HARDY - J.E. LITTLEWOOD, *Some problems in "Partitio Numerorum" (VI): Further researches on Waring's problem*. Math. Z. **23** (1925), 1-37.
- [18] HUA LOO-KENG, *On Waring's problem*. Quart. Journ. Math. **9** (1938), 199-202.
- [19] B. LANDREAU, *Moyennes de fonctions arithmétiques sur des suites de faible densité*. Thèse 3^{ème} cycle, Université Bordeaux I, 1987.
- [20] V.I. NEČAEV, V.L. TOPUNOV, *Estimation of the modulus of complete rational trigonometric sums of degree three and four (Russian)*. Trudy Mat. Inst. Steklov. **158** (1981), 125-129, 229.
- [21] S.S. PILLAI, *On Waring's problem*. Journ. Indian Math. Soc. **2** (1936), 16-44.
- [22] J.B. ROSSER - L. SCHOENFELD, *Approximate formulas for some functions of prime numbers*. Ill. Journ. Math. **6** (1962), 64-94.

- [23] H.E. THOMAS, *A numerical approach to Waring's problem for fourth powers*. Ph.D. thesis, University of Michigan, 1973.
- [24] H.E. THOMAS, *Waring's problem for 22 biquadrates*. Trans. Amer. Math. Soc. **193** (1974), 427-430.
- [25] E.C. TITCHMARSH, *The Theory of the Riemann Zeta Function*. Clarendon Press, Oxford, 1951.
- [26] I.M. VINOGRADOV, *A new solution of Waring's problem*. Dokl. Akad. Nauk SSSR (1934), vol. 2, 337-341.
- [27] I.M. VINOGRADOV, *On the upper bound of $G(n)$ in Waring's problem*. Izv. Akad. Nauk SSSR **10** (1934), 1455-1469.

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