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Fractional Powers of Non-densely Defined Operators

CELSO MARTINEZ - MIGUEL SANZ*

1. - Introduction and notation

In [1] Balakrishnan extended the concept of fractional powers to closed linear operators A, defined on a Banach space X, such that $]-\infty, 0[$ is included in the resolvent set $\rho(A)$ and the resolvent operator satisfies

$$\sup\{\|\lambda(\lambda + A)^{-1}\|; \ \lambda > 0\} < \infty.$$

(Following Komatsu's terminology in [10], we shall call these operators non-negative).

Balakrishnan defined the power with base A and exponent a complex number α ($\operatorname{Re}\alpha > 0$) as the closure of a closable operator, J_A^α, whose expression is:

For $0 < \operatorname{Re}\alpha < 1$, $D(J_A^\alpha) = D(A)$ and

$$J_A^\alpha \phi = \frac{\sin \alpha\pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + A)^{-1} A\phi \mathrm{d}\lambda.$$

For $0 < \operatorname{Re}\alpha < 2$, $D(J_A^\alpha) = D(A^2)$ and

$$J_A^\alpha \phi = \frac{\sin \alpha\pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} \left((\lambda + A)^{-1} - \frac{\lambda}{1+\lambda^2} \right) A\phi \mathrm{d}\lambda + \sin\left(\frac{\alpha\pi}{2}\right) A\phi.$$

For $n < \operatorname{Re}\alpha < n+1$, $D(J_A^\alpha) = D(A^{n+1})$ and $J_A^\alpha \phi = J_A^{\alpha-n} A^n \phi$.

For $n < \operatorname{Re}\alpha \leq n+1$, $D(J_A^\alpha) = D(A^{n+2})$ and $J_A^\alpha \phi = J_A^{\alpha-n} A^n \phi$.

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The theory of fractional powers was studied and completed by other authors (Hövel-Westphal [5], Kato [6]-[7], Komatsu [8]-[11], Krasnosel'skii-Sobolevskii [12], Nollau [16], Watanabe [17], Westphal [18], Yosida [19],...). An extensive bibliography can be found in Fattorini's book [4].

This theory worked satisfactorily when the operator A is densely defined since in this case the following properties hold:

(P1) If $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, then $A^{\alpha+\beta} = A^\alpha A^\beta$ (Additivity);

(P2) $A^n = AA\dots A$ (n times);

(P3) $\sigma(A^\alpha) = \{z^\alpha; z \in \sigma(A)\}$ if $\operatorname{Re}\alpha > 0$ (Spectral Mapping Theorem);

(P4) If $0 < \alpha < 1$, then A^α is non-negative and for $\operatorname{Re}\beta > 0$

$$(A^\alpha)^\beta = A^{\alpha\beta} \qquad \text{(Multiplicativity).}$$

In fact, a coherent theory of powers should satisfy (P1) to (P4).

However, if A is not densely defined and $0 \notin \rho(A)$, there was no definition of fractional powers satisfying properties (P2), (P3) and (P4), until the publication of [13].

There are simple examples of non-negative operators of this kind; for instance: the indefinite integral operator on $L^1(]0,\infty[)$ defined in its natural domain, and the derivative operator on $L^\infty(]0,\infty[)$ whose domain is the Sobolev space $W_0^{1,\infty}(]0,\infty[)$. Moreover, we can obtain a lot of non-negative non-densely defined operators, by considering the inverse of a one-to-one non-negative operator and the adjoint of a densely defined non-negative operator, which are not, in general, densely defined.

Many non-negative non-densely defined differential operators have been studied by Da Prato-Sinestrari in [3].

In [13], Martinez-Sanz-Marco developed a new theory of powers for non-negative operators which, apart from providing a considerable simplification of Balakrishnan's theory, is also valid (in the sense that fractional powers satisfy (P1)-(P4)) for non-densely defined operators. This theory is an extension of all other theories developed up to the moment, although it coincides with the one introduced in [1] in the case of dense domains.

In the present paper we have chosen a different point of view: taking as a starting-point Balakrishnan's theory about densely defined operators, we shall give a definition of power for non-densely defined operators, and thus we shall, in a straightforward manner, obtain a theory satisfying properties (P1) to (P4). In Section 3 we prove that there is a unique family of closed linear operators, $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$, satisfying properties (P1)-(P2) and the auxiliary condition that the operator $P(\alpha)$ is an extension of the operator J_A^α. Consequently we obtain that the definition presented here is equivalent to the one given in [13]. In this section we study the equivalence between the uniqueness of a family of operators satisfying properties (P1) to (P4) and the uniqueness (studied in [15] and [16]) of non-negative n-th roots of a non-negative operator. We prove that there exist non-negative operators A for which there are families of operators, different from the family of fractional powers of A, satisfying (P1) to (P4).

Finally, we prove that, if A is a one-to-one non-negative operator, its inverse A^{-1} is non-negative and $(A^{-1})^\alpha = (A^\alpha)^{-1}$ and if A is densely defined, its adjoint A^* is non-negative and $(A^*)^\alpha = (A^\alpha)^*$ for $\alpha \in \mathbb{C} : \operatorname{Re}\alpha > 0$. The previous theories to the one developed in [13] did not satisfy these properties. The first result can be applied to prove that the fractional derivative of Riemann-Liouville is the inverse of the fractional integral of Riemann-Liouville, both considered as operators defined on suitable functional spaces. The second property allows us to prove that the fractional integral of Weyl is the adjoint of the fractional integral of Riemann-Liouville on $L^p(]0,\infty[)$, and so we obtain in this way a formula for fractional integration by parts.

2. - Construction of fractional powers. Additivity

Throughout this paper, A shall be a non-negative operator defined on a Banach space X such that $\overline{D(A)} \neq X$ and α shall be a complex number with $\operatorname{Re}\alpha > 0$.

Let us consider the Banach space $X_1 = \overline{D(A)}$ and the operator A_1 whose domain is $D(A_1) = \{\phi \in D(A) : A\phi \in X_1\}$ defined as $A_1\phi = A\phi$ for $\phi \in D(A_1)$.

The operator A_1 is obviously a non-negative densely defined operator (note that $D(A^2) \subset D(A_1)$, and the property $\lim_{\lambda\to\infty} \lambda(\lambda + A)^{-1}\phi = \phi$ for $\phi \in \overline{D(A)}$, valid for any non-negative operator, implies that $\overline{D(A^2)} = \overline{D(A)}$) on the Banach space X_1.

DEFINITION 2.1. We define the operator A^α as $A^\alpha\phi = (1+A)A_1^\alpha(1+A)^{-1}\phi$ on

$$D(A^\alpha) = \{\phi \in X : (1+A)^{-1}\phi \in D(A_1^\alpha) \text{ and } A_1^\alpha(1+A)^{-1}\phi \in D(A)\},$$

where A_1^α is the Balakrishnan power $(\overline{J_{A_1}^\alpha})$ of the (densely defined) operator A_1.

REMARK 2.2. The first resolvent formula easily shows that, replacing the resolvent operator $(1+A)^{-1}$ by $(\lambda + A)^{-1}$ $(\lambda > 0)$ in definition 2.1, would yield the same definition. Moreover, it is clear that A^α is an extension of A_1^α.

THEOREM 2.3. *A^α is a closed operator.*

PROOF. This is a straightforward consequence of the definition of A^α and the fact that A_1^α is a closed operator. □

LEMMA 2.4. *Both of the following assertions hold:*

(i) $D(A) \subset D(A_1^\alpha)$ *if* $0 < \operatorname{Re}\alpha < 1$.

(ii) *If* $\phi \in D(A_1^\alpha)$ *and* $A_1^\alpha \phi \in D(A)$, *then* $\phi \in D(A_1)$.

PROOF. (i) If $\phi \in D(A)$, then $\lim\limits_{\mu \to +\infty} \mu(\mu + A)^{-1}\phi = \phi$. Moreover, $(\mu + A)^{-1}\phi \in D(A^2)$ and so $(\mu + A)^{-1}\phi \in D(A_1)$ and

$$A_1^\alpha(\mu + A)^{-1}\phi = \frac{\sin \alpha\pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + A_1)^{-1} A_1 (\mu + A)^{-1}\phi d\lambda$$

$$= (\mu + A)^{-1} \left[\frac{\sin \alpha\pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + A)^{-1} A\phi d\lambda \right] = (\mu + A)^{-1} J_A^\alpha \phi.$$

The definition of the operator J_A^α implies that $\operatorname{Ran} J_A^\alpha \subset \overline{D(A)}$. Then, by using the former equality and the fact that A is a non-negative operator we obtain that

$$\lim_{\mu \to +\infty} A_1^\alpha[\mu(\mu + A)^{-1}\phi] = \lim_{\mu \to +\infty} \mu(\mu + A)^{-1} J_A^\alpha \phi = J_A^\alpha \phi.$$

Since A_1^α is a closed operator on X_1, then $\phi \in D(A_1^\alpha)$ and $A_1^\alpha \phi = J_A^\alpha \phi$.

(ii) Let β be a complex number such that $0 < \operatorname{Re}\beta < 1$ and $\operatorname{Re}(\alpha+\beta) > 1$. If $\phi \in D(A_1^\alpha)$ and $A_1^\alpha \phi \in D(A)$ we conclude, owing to (i), that $A_1^\alpha \phi \in D(A_1^\beta)$. The additivity of the fractional powers of A_1 implies that

$$\phi \in D(A_1^{\alpha+\beta}) = D(A_1^{\alpha+\beta-1} A_1)$$

and consequently $\phi \in D(A_1)$. □

THEOREM 2.5. (*Additivity*). *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, *then* $A^{\alpha+\beta} = A^\alpha A^\beta$.

PROOF. It is clear that $A^{\alpha+\beta}$ is an extension of the operator $A^\alpha A^\beta$. Now, given $\phi \in D(A^{\alpha+\beta})$ we know that

$$(1 + A)^{-1}\phi \in D(A_1^{\alpha+\beta}) = D(A_1^\alpha A_1^\beta)$$

and $A_1^\alpha A_1^\beta (1 + A)^{-1}\phi \in D(A)$. Applying part (ii) of Lemma 2.4 we obtain

$$A_1^\beta (1 + A)^{-1}\phi \in D(A).$$

Thus $\phi \in D(A^\beta)$ and applying the definition of A^α we conclude that $A^\beta \phi \in D(A^\alpha)$ and $A^\alpha A^\beta \phi = A^{\alpha+\beta}\phi$. □

COROLLARY 2.6. *Given* $n \in \mathbb{N}$, *then* $A^n = AA \ldots A$ *(n times).*

COROLLARY 2.7. *If* $\operatorname{Re}\alpha > 0$, *then* $\overline{D(A^\alpha)} = \overline{D(A)}$.

PROOF. Let us suppose first that $0 < \operatorname{Re}\alpha < 1$. If $\phi \in D(A^\alpha)$, then $A_1^\alpha(1+A)^{-1}\phi \in D(A)$ and according to (i) of Lemma 2.4,

$$A_1^\alpha(1+A)^{-1}\phi \in D(A_1^{1-\alpha})$$

and consequently $\phi \in \overline{D(A)}$. The other inclusion is clear.

When $\operatorname{Re}\alpha \geq 1$, we have $\overline{D(A)} = \overline{D(A^{1/2})} \supset \overline{D(A^\alpha)}$. □

COROLLARY 2.8. *The following statements are equivalent:*

(i) $\phi \in D(\overline{J_A^\alpha})$;

(ii) $\phi \in D(A_1^\alpha)$;

(iii) $\phi \in D(A^\alpha)$ *and* $A^\alpha\phi \in \overline{D(A)}$.

Then $\overline{J_A^\alpha}\phi = A_1^\alpha\phi = A^\alpha\phi$. *Moreover, the operators* $\overline{J_A^\alpha}$ *satisfy the additivity property:* $\overline{J_A^{\alpha+\beta}} = \overline{J_A^\alpha}\,\overline{J_A^\beta}$ *if* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$.

PROOF. (i) $\Longrightarrow$ (ii) follows directly from the definitions of the operators J_A^α and A_1^α, and the fact that A_1^α is a closed operator. (ii) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (i). Let n be an integer number such that $n > \operatorname{Re}\alpha$. Since A is a non-negative operator, we get $\lim_{\mu\to\infty}[\mu(\mu+A)^{-1}]^n\psi = \psi$ for $\psi \in \overline{D(A)}$, and by Corollary 2.7 we obtain that

$$\lim_{\mu\to\infty}[\mu(\mu+A)^{-1}]^n\phi = \phi$$

and

$$\lim_{\mu\to\infty} J_A^\alpha[\mu(\mu+A)^{-1}]^n\phi = [\mu(\mu+A)^{-1}]^n A^\alpha\phi = A^\alpha\phi.$$

Therefore, we conclude that $\phi \in D(\overline{J_A^\alpha})$ and $\overline{J_A^\alpha}\phi = A^\alpha\phi$.

As the operators $\overline{J_A^\alpha}$ coincide with the operators A_1^α, the additivity for the operators $\overline{J_A^\alpha}$ follows from the corresponding property for the fractional powers of the (densely defined) operator A_1. □

REMARK 2.9. The additivity for the operators $\overline{J_A^\alpha}$, although A was not densely defined, was already known (see [8], [13] and [16]).

3. - Uniqueness

In this section we study the relationship between the theory of fractional powers given in this paper and the one constructed in [13]. We shall see that they are both equivalent; our proof will consist in showing that this equivalence is true for any two definitions of fractional powers such that additivity holds, the power with exponent equal to one is the base operator and both definitions are extensions of the operator J_A^α defined by Balakrishnan.

DEFINITION 3.1. A family of closed linear operators $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$ is an additive class of operators associated to A if

(i) $P(\alpha+\beta) = P(\alpha)P(\beta)$ for $\alpha, \beta : \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0$.

(ii) $P(1) = A$.

PROPOSITION 3.2. *If $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$ is an additive class of operators associated to A, then*

(i) *$P(\alpha)$ commutes with $(1+A)^{-1}$.*

(ii) *If $[A(1+A)^{-1}]^n\phi \in D[P(\alpha)]$ for some integer numbers n, then*

$$\phi \in D[P(\alpha)].$$

PROOF. (i) First, let us suppose that $0 < \operatorname{Re}\alpha < 1$. If $\phi \in D[P(\alpha)]$, then the element $A(1+A)^{-1}\phi = \phi - (1+A)^{-1}\phi$ belongs to $D[P(\alpha)]$, since

$$D(A) = D[P(1)] = D[P(1-\alpha)P(\alpha)] \subset D[P(\alpha)].$$

Thus $(1+A)^{-1}\phi \in D[P(\alpha)A] = D[AP(\alpha)]$ and

$$P(\alpha)A(1+A)^{-1}\phi = AP(\alpha)(1+A)^{-1}\phi,$$

from which we conclude that $P(\alpha)(1+A)^{-1}\phi = (1+A)^{-1}P(\alpha)\phi$.

Given any complex number α with $\operatorname{Re}\alpha > 0$, let n be an integer such that $n > \operatorname{Re}\alpha$. Applying additivity and commutativity of $P(\alpha/n)$ with $(1+A)^{-1}$, we obtain that $P(\alpha)$ commutes with $(1+A)^{-1}$.

(ii) The proof is similar to that of [13, Lemma 3.1]. We shall give a brief outline. By induction on n it is sufficient to prove it for $n = 1$. Let $A(1+A)^{-1}\phi \in D[P(\alpha)]$ and m be an integer number such that $2^m > \operatorname{Re}\alpha$. By using (i) we obtain

$$\phi - [(1+A)^{-1}]^{2^m}\phi = \prod_{p=0}^{m-1}(1 + [(1+A)^{-1}]^{2^p})(\phi - (1+A)^{-1}\phi).$$

The right-hand side belongs to $D[P(\alpha)]$ and since

$$[(1+A)^{-1}]^{2^m}\phi \in D(A^{2^m}) \subset D[P(\alpha)],$$

we conclude that $\phi \in D[P(\alpha)]$. □

THEOREM 3.3. *Let $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$ and $\{Q(\alpha),\ \operatorname{Re}\alpha > 0\}$ be two additive classes of operators associated to A. If $P(\alpha_0)$ and $Q(\alpha_0)$ are extensions of the operator $J_A^{\alpha_0}$, then $P(\alpha_0) = Q(\alpha_0)$.*

PROOF. Let n be an integer number such that $n > \operatorname{Re}\alpha_0$. Given $\phi \in D[P(\alpha_0)]$, Proposition 3.2 (i) gives

$$(3.1)\qquad \begin{aligned}(1+A)^{-n}P(\alpha_0)\phi &= P(\alpha_0)(1+A)^{-n}\phi = J_A^{\alpha_0}(1+A)^{-n}\phi \\ &= Q(\alpha_0)(1+A)^{-n}\phi\end{aligned}$$

and so $Q(\alpha_0)(1+A)^{-n}\phi \in D(A^n) = D[Q(\alpha_0)Q(n-\alpha_0)]$, from which

$$A^n(1+A)^{-n}\phi = Q(n-\alpha_0)Q(\alpha_0)(1+A)^{-n}\phi \in D[Q(\alpha_0)]$$

and by property (ii) of Proposition 3.2 we get that $\phi \in D[Q(\alpha_0)]$, and commuting with $(1+A)^{-n}$ in (3.1), we conclude that $P(\alpha_0)\phi = Q(\alpha_0)\phi$.

Taking as a starting point $\phi \in D[Q(\alpha_0)]$ we would argue in a similar way. □

REMARK 3.4. If $\{Q(\alpha),\ \operatorname{Re}\alpha > 0\}$ satisfies all the hypothesis of Theorem 3.3, to prove that $Q(\alpha_0)$ is an extension of $P(\alpha_0)$ we only need suppose that the operator $P(\alpha_0)$ commutes with the resolvent operator $(1+A)^{-1}$ and that $P(\alpha_0)$ is an extension of $J_A^{\alpha_0}$.

COROLLARY 3.5. *If A is not densely defined, the fractional powers as defined in this paper coincide with the ones given in* [13].

REMARK 3.6. In [13, Theorem 4.1, Corollary 4.1 and Theorem 4.2] it has been proved that the definition of fractional powers presented there, satisfied the spectral mapping theorem (property (P3)) and the multiplicativity (property (P4)). Both properties were established for any non-negative operator and their proofs were independent of its domain being dense or not.

When the operator is densely defined, both properties had been proved before the publication of [13]. Balakrishnan [1, Theorem 3.1] gave a proof for densely defined non-negative operators and his proof was based on giving an integral representation of the resolvent of the fractional power. On the other hand, Watanabe [17, Theorem 1] proved the multiplicativity property for the case that the base operator is densely defined.

We can prove directly both properties for non-densely defined operators, starting from the definition presented here. The proofs are very simple and probably easier than those of [13], but depend on the validity of the corresponding properties for densely defined operators.

REMARK 3.7. In Theorem 3.3 we have proved that an additive class of operators $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$ associated to a non-negative operator A and satisfying that the operators $P(\alpha)$ extended to the operators J_A^α, coincides with the class $\{A^\alpha,\ \operatorname{Re}\alpha > 0\}$ of fractional powers of A. The question that naturally arises is whether the last property can be substituted by the following two properties:

(3.2) $\sigma[P(\alpha)] = \{z^\alpha;\ z \in \sigma(A)\}$ (Spectral Property);

(3.3) If $0 < \alpha < 1$, then $P(\alpha)$ is non-negative and $[P(\alpha)]^\beta = [P(\alpha\beta)]$ for

$\beta \in \mathbb{C} : \operatorname{Re}\beta > 0$ (Multiplicativity Property);

to claim the uniqueness of a such additive class.

In Theorem 3.9 we shall prove that, under certain analyticity hypothesis, this question is similar to the uniqueness, for each integer $n \geq 2$, of a non-negative n-th root of A with spectrum $\sigma(A)$ equal to $\{z^{1/n};\ z \in \sigma(A)\}$.

In [15] we have found infinitely many counterexamples that prove the non-uniqueness of non-negative n-th roots of a non-negative operator A. Consequently, there exist, in general, additive classes of operators associated to A, different from the class $\{A^\alpha,\ \operatorname{Re}\alpha > 0\}$ of fractional powers of A, satisfying the spectral and multiplicativity properties.

In our next proposition we study the analyticity conditions that fractional powers satisfy.

PROPOSITION 3.8. *If* $\phi \in D(A^{\alpha_0})$ *with* $\operatorname{Re}\alpha_0 > 0$, *the mapping* $\alpha \to A^\alpha\phi$ *is analytic for* $0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_0$.

PROOF. In [1, Lemma 2.2] it has been proved that if $\phi \in D(A^n)$, the mapping $\alpha \to J_A^\alpha\phi$ is analytic for $0 < \operatorname{Re}\alpha < n$. This fact is a direct consequence of the definition of the operator J_A^α.

Let $\alpha_1 \in \mathbb{C} : 0 < \operatorname{Re}\alpha_1 < \operatorname{Re}\alpha_0$ and $\phi \in D(A^{\alpha_0})$. Choosing an integer $n > \operatorname{Re}\alpha_0$ and a complex number α_2 satisfying that $0 < \operatorname{Re}\alpha_2 < \operatorname{Re}\alpha_1$ and $\operatorname{Im}\alpha_2 = \operatorname{Im}\alpha_0$, the operator $B = A^{(\alpha_0-\alpha_2)/n}$ is a non-negative operator, as a consequence of the multiplicativity of fractional powers, and $A^{\alpha_2}\phi \in D(B^n)$. Applying [1, Lemma 2.2] we obtain that the mapping

$$\alpha \to A^\alpha\phi = B^{n(\alpha-\alpha_2)/(\alpha_0-\alpha_2)}A^{\alpha_2}\phi = J_B^{n(\alpha-\alpha_2)/(\alpha_0-\alpha_2)}A^{\alpha_2}\phi$$

is analytic in the region $0 < \operatorname{Re} n(\alpha-\alpha_2)/(\alpha_0-\alpha_2) < n$, and so in α_1. □

THEOREM 3.9. *Let* $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$ *be an additive class associated to* A, *satisfying the spectral property* (3.2), *the multiplicativity property* (3.3) *and the property of analyticity:*

(3.4) *If* $\phi \in D[P(\alpha_0)]$, *the mapping* $\alpha \to P(\alpha)\phi$ *is analytic in*

$0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_0$.

Then $P(\alpha) = A^\alpha$ *for* $\operatorname{Re}\alpha > 0$ *if and only if there exists a unique non-negative operator* B *with spectrum* $\sigma(B) = \{z^{1/n};\ z \in \sigma(A)\}$ *satisfying that* $B^n = A$, *for any integer* n.

PROOF. Let B be a non-negative n-th root of A, different from $A^{1/n}$, satisfying that $\sigma(B) = \{z^{1/n};\ z \in \sigma(A)\}$. It can be easily seen that

$$\{P(\alpha) = B^{n\alpha},\ \operatorname{Re}\alpha > 0\}$$

is an additive class associated to A for which the conditions of analyticity (3.4) and the spectral property (3.2) hold. Moreover, this class satisfies the multiplicativity property (3.3) (see Watanabe [17, Theorem 1] and Komatsu [8, Theorem 10.6]) and obviously $P(\alpha)$ does not coincide with A^α for any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$.

Conversely, let us now suppose the uniqueness, for any integer number n, of a non-negative n-th root of A with spectrum equal to $\{z^{1/n};\ z \in \sigma(A)\}$. Given an additive class associated to A, $\{P(\alpha),\ \operatorname{Re}\alpha > 0\}$, satisfying the hypothesis of the theorem, then $P(1/n) = A^{1/n}$. The additivity implies that $P(q) = A^q$ for any rational number q. The analiticity of the mappings $\alpha \to P(\alpha)\phi$ and $\alpha \to J_A^\alpha \phi$ in the region $0 < \operatorname{Re}\alpha < n$ for $\phi \in D(A^n)$, and the equality between the operators $P(q)$ and A^q for any rational q imply by analytic continuation that $P(\alpha)$ is an extension of the operator J^α, and applying Theorem 3.3 we obtain that $P(\alpha) = A^\alpha$ for any complex number α with $\operatorname{Re}\alpha > 0$. □

4. - Fractional powers of the inverse and the adjoint

The aim of this section is the study of the fractional powers of the inverse of a one-to-one non-negative operator and those of the adjoint of a densely defined non-negative operator. Note that, in general, these operators are non-densely defined and non-negative and consequently this is another reason to construct a satisfactory theory of fractional powers for non-negative operators, valid too for the case where the base operator is non-densely defined. We shall prove that fractional powers preserve the inverse and the adjoint, when these operators exist.

THEOREM 4.1. *If A is a one-to-one, non-negative operator, then A^{-1} is non-negative and $(A^{-1})^\alpha = (A^\alpha)^{-1}$ for $\alpha \in \mathbb{C} : \operatorname{Re}\alpha > 0$.*

PROOF. The identity $\lambda(\lambda + A^{-1})^{-1} = A(\lambda^{-1} + A)^{-1} = 1 - \lambda^{-1}(\lambda^{-1} + A)^{-1}$ proves that the operator A^{-1} is non-negative. Moreover, the operators A^α are one-to-one operators. In fact, if $\phi \in D(A^\alpha)$ and $A^\alpha\phi = 0$, then

$$A^n(1 + A)^{-n}\phi = A^{n-\alpha}A^\alpha(1 + A)^{-n}\phi = A^{n-\alpha}(1 + A)^{-n}A^\alpha\phi = 0$$

for any integer number n ($n > \operatorname{Re}\alpha$). Using that A and $(1+A)^{-1}$ are one-to-one operators we conclude that $\phi = 0$.

As a consequence of the additivity of the fractional powers it is sufficient to prove this theorem for $0 < \operatorname{Re}\alpha < 1$.

The additivity of the fractional powers of the operator A easily implies that the family of closed operators $\{(A^\alpha)^{-1} : \operatorname{Re}\alpha > 0\}$ is an additive class associated to A^{-1}.

Let $0 < \operatorname{Re}\alpha < 1$. If we show that the operator $(A^\alpha)^{-1}$ is an extension of the operator $J_{A^{-1}}^\alpha$ (associated to A^{-1}), then Theorem 3.3 will allow us to conclude that $(A^{-1})^\alpha = (A^\alpha)^{-1}$.

Let $\phi \in D(A^{-1})$. The equality $\phi = A^\alpha A^{1-\alpha}A^{-1}\phi = A^\alpha J_A^{1-\alpha}A^{-1}\phi$ implies that $(A^\alpha)^{-1}\phi = J_A^{1-\alpha}A^{-1}\phi$. Applying the definition of $J_A^{1-\alpha}$ we obtain

$$(A^\alpha)^{-1}\phi = J_A^{1-\alpha}A^{-1}\phi = \frac{\sin(1-\alpha)\pi}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}AA^{-1}\phi d\lambda$$

$$= \frac{\sin \alpha\pi}{\pi}\int_0^\infty \lambda^{-\alpha-1}(\lambda^{-1} + A^{-1})^{-1}A^{-1}\phi d\lambda$$

$$= \frac{\sin \alpha\pi}{\pi}\int_0^\infty \mu^{\alpha-1}(\mu + A^{-1})^{-1}A^{-1}\phi d\mu,$$

with $\mu = \lambda^{-1}$. This last integral is precisely $J_{A^{-1}}^\alpha\phi$. □

THEOREM 4.2. *If A is a densely defined non-negative operator, then A^* is a non-negative operator in the dual space X^* and $(A^*)^\alpha = (A^\alpha)^*$ for $\alpha \in \mathbb{C} : \operatorname{Re}\alpha > 0$.*

PROOF. The equality $[(\lambda+A)^{-1}]^* = (\lambda+A^*)^{-1}$ $(\lambda > 0)$ and the non-negativity of A^* follows easily from the definition of the adjoint operator.

According to Corollary 2.7, the operators A^α are densely defined. Using the additivity of fractional powers it is clear that the family

$$\{P(\alpha) = (A^\alpha)^* : \operatorname{Re}\alpha > 0\}$$

satisfies $P(1) = A^*$ and the property:

$$P(\alpha+\beta) \text{ is an extension of } P(\alpha)P(\beta). \tag{4.1}$$

To prove that $(A^*)^\alpha$ is an extension of $(A^\alpha)^*$ we will apply Remark 3.4. We have to check that $(A^\alpha)^*$ commutes with the resolvent operator $(1 + A^*)^{-1}$ and that $(A^\alpha)^*$ is an extension of the operator $J_{A^*}^\alpha$.

The first assertion is an immediate consequence of the commutativity of A^α with the resolvent operator $(1 + A)^{-1}$.

Let us now prove the second assertion. First, let us suppose $0 < \operatorname{Re}\alpha < 1$. As A is densely defined, we know that $(A^\alpha)^* = (\overline{J_A^\alpha})^* = (J_A^\alpha)^*$. Thus it is sufficient to check that $(J_A^\alpha)^*$ is an extension of $J_{A^*}^\alpha$. Let $\phi \in D(A^*)$ and $\psi \in D(A)$, then

$$(\phi, J_A^\alpha\psi) = \left(\phi, \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + A)^{-1}A\psi \mathrm{d}\lambda\right)$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\phi, (\lambda + A)^{-1}A\psi)\mathrm{d}\lambda$$

and as $\phi \in D(A^*)$ we obtain

$$(\phi, J_A^\alpha\psi) = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}((\lambda + A^*)^{-1}A^*\phi, \psi)\mathrm{d}\lambda$$

$$= \left(\frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + A^*)^{-1}A^*\phi \mathrm{d}\lambda, \psi\right) = (J_{A^*}^\alpha\phi, \psi),$$

from which $\phi \in D(J_A^\alpha)^*$ and $(J_A^\alpha)^*\phi = J_{A^*}^\alpha\phi$.

The general case for $\mathrm{Re}\,\alpha > 0$ is reduced to the former case by using the additivity of the operators $J_{A^*}^\alpha$ and property (4.1).

Let us now prove directly that $(A^\alpha)^*$ is an extension of $(A^*)^\alpha$. Let $\phi \in D[(A^*)^\alpha]$ and $\psi \in D(A^\alpha)$. Let n be an integer such that $n > \mathrm{Re}\,\alpha$. Then

$$\begin{aligned}(\phi, A^\alpha\psi) &= \lim_{\lambda\to\infty}(\phi, \lambda^n(\lambda + A)^{-n}A^\alpha\psi) = \lim_{\lambda\to\infty}(\lambda^n(\lambda + A^*)^{-n}\phi, A^\alpha\psi)\\ &= \lim_{\lambda\to\infty}(\lambda^n J_{A^*}^\alpha(\lambda + A^*)^{-n}\phi, \psi) = \lim_{\lambda\to\infty}(\lambda^n(A^*)^\alpha(\lambda + A^*)^{-n}\phi, \psi)\\ &= \lim_{\lambda\to\infty}(\lambda^n(\lambda + A^*)^{-n}(A^*)^\alpha\phi, \psi) = \lim_{\lambda\to\infty}((A^*)^\alpha\phi, \lambda^n(\lambda + A)^{-n}\psi)\\ &= ((A^*)^\alpha\phi, \psi),\end{aligned}$$

where the third equality is a consequence of the fact that $(A^\alpha)^*$ is an extension of $J_{A^*}^\alpha$. Hence $\phi \in D[(A^\alpha)^*]$ and $(A^\alpha)^*\phi = (A^*)^\alpha\phi$. □

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