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Homogenization and Corrector
for the Wave Equation
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1. - Introduction

In this paper we study the homogenization of the wave equation with Dirichlet boundary conditions in perforated domains with small holes. Let Ω be a fixed bounded domain of \( \mathbb{R}^n \) \( (n \geq 2) \). Denote by \( \Omega_\varepsilon \) the domain obtained by removing from \( \Omega \) a set \( S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S^i_\varepsilon \) of \( N(\varepsilon) \) closed subsets of \( \Omega \) (here, \( \varepsilon > 0 \) denotes a parameter which takes its values in a sequence which tends to zero while \( N(\varepsilon) \) tends to infinity). Finally let \( T > 0 \) be fixed. We consider the wave equation

\[
\begin{ alignedat}{2}
&u''_\varepsilon - \Delta u_\varepsilon = f_\varepsilon & \quad & \text{in } \Omega_\varepsilon \times (0, T) \\
&u_\varepsilon = 0 & \quad & \text{on } \partial\Omega_\varepsilon \times (0, T) \\
&u_\varepsilon(0) = u^0_\varepsilon, \quad u'_\varepsilon(0) = u^1_\varepsilon & \quad & \text{in } \Omega_\varepsilon.
\end{ alignedat}
\]  

(1.1)

Our aim is to describe the convergence of the solutions \( u_\varepsilon \), to identify the equation satisfied by the limit \( u \) and to give corrector results.

In the whole of the present paper the sets \( \Omega_\varepsilon \) will be assumed to satisfy the requirements of the abstract framework introduced by D. Cioranescu and F. Murat [6] (see assumption (2.1) below) for the study of the homogenization of elliptic problems in perforated domains with Dirichlet boundary data. The model case (see Figure 1 on the next page) is provided by a domain periodically perforated (with a period \( 2\varepsilon \) in the direction of each coordinate axis) by holes of size \( r_\varepsilon \), where \( r_\varepsilon \) is asymptotically equal to or smaller than a “critical size”.
This critical size $a_\varepsilon$ is given by

$$\begin{cases}
    a_\varepsilon = C_0 \varepsilon^{n/(n-2)} & \text{for } n \geq 3 \\
    a_\varepsilon = \delta_\varepsilon \exp(-C_0/\varepsilon^2) & \text{for } n = 2
\end{cases}$$

where $C_0 > 0$ is fixed and $\varepsilon^2 \log \delta_\varepsilon \to 0$ as $\varepsilon \to 0$ (see Section 2 below).

In this abstract framework, let $v_\varepsilon$ be the solution of the problem

$$\begin{cases}
    -\Delta v_\varepsilon = g & \text{in } \Omega_\varepsilon \\
    v_\varepsilon = 0 & \text{on } \partial \Omega_\varepsilon
\end{cases}$$

where $g$ is given in $H^{-1}(\Omega)$. Denote by $\tilde{v}_\varepsilon$ the extension of $v_\varepsilon$ to the whole of $\Omega$ defined by

$$\tilde{v}_\varepsilon = \begin{cases}
    v_\varepsilon & \text{in } \Omega_\varepsilon \\
    0 & \text{in the holes } S_\varepsilon
\end{cases}$$

It has been shown in [6] that $\tilde{v}_\varepsilon$ weakly converges in $H^1_0(\Omega)$ to the solution $v$ of the problem

$$\begin{cases}
    -\Delta v + \mu v = g & \text{in } \Omega \\
    v = 0 & \text{on } \partial \Omega
\end{cases}$$

where $\mu$ is a nonnegative Radon measure belonging to $H^{-1}(\Omega)$. This measure appears in the abstract framework and relies on the behaviour of the capacity of the set $S_\varepsilon$ as $\varepsilon \to 0$. In the model case described above, $\mu$ is a constant which is strictly positive when the size of the holes is the critical one. In such case the additional zero order term $\mu v$ appears in the limit equation.

As a first hypothesis on the data in (1.1) we assume that

$$u^0_\varepsilon \in H^1_0(\Omega_\varepsilon), \quad u^1_\varepsilon \in L^2(\Omega_\varepsilon), \quad f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon))$$
As it is naturally expected, the extension \( \tilde{u}_e \) of the solution \( u_e \) of (1.1) weakly * converges in \( L^\infty(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \) to the solution \( u \) of the problem

\[
\begin{aligned}
&u'' - \Delta u + \mu u = f \quad \text{in } \Omega \times (0, T) \\
u = 0 \quad &\text{on } \partial \Omega \times (0, T) \\
u(0) = u^0, &\quad u'(0) = u^1 \quad \text{in } \Omega.
\end{aligned}
\]

This result is proved in Theorem 3.1 of Section 3.

In Section 4 we prove corrector results for the problem (1.1) by following ideas similar to those used by S. Brahim-Ostmane, G.A. Francfort and F. Murat [1], who adapted to the wave equation the ideas introduced by L. Tartar [18] in the elliptic case. Under a special assumption on the data (see (1.9) below) we prove in Theorem 4.1 that \( \tilde{u}_e \) can be decomposed in

\[
\tilde{u}_e = uw_e + R_e.
\]

In this decomposition \( u \) is the solution of (1.6), \( w_e \) are functions which appear in the abstract framework (they are related to the capacitary potential of the holes) and the remainder \( R_e \) satisfies the strong convergence property

\[
R_e \to 0 \quad \text{strongly in } C^0([0, T]; W^{1,1}(\Omega));
\]

we also prove that

\[
\tilde{u}_e' \to u' \quad \text{strongly in } C^0([0, T]; L^2(\Omega)).
\]

The term \( uw_e \) is then a good approximation ("the corrector") of the solution \( u_e \).

In order to obtain (1.7)-(1.8) we have to make special assumptions on the data; to be precise, we will assume that there exists \( g_e \in H^{-1}(\Omega) \) such that

\[
\begin{aligned}
&-\Delta u^0_e = g_e \quad \text{in } \mathcal{D}'(\Omega_e) \text{ with } g_e \to g \text{ strongly in } H^{-1}(\Omega) \\
u^1_e \to u^1 \quad &\text{strongly in } L^2(\Omega) \\
_e \to f \quad &\text{strongly in } L^1(0, T; L^2(\Omega)).
\end{aligned}
\]

Note that the initial condition \( u^0_e \) has to satisfy the very special hypothesis (1.9a). The meaning of this hypothesis is that \( u^0_e \) has to be well adapted to the
asymptotic behaviour of the holes. In general (1.9a) implies only the weak (and not strong) convergence in $H_0^1(\Omega)$ of $\tilde{u}_\varepsilon^0$ (see Remark 4.1 below).

Assumption (1.9) on the data turns to imply the convergence of the energies

$$\frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u_\varepsilon(x,t)|^2 + |u_\varepsilon'(x,t)|^2) \, dx$$

to the energy of the limit problem

$$\frac{1}{2} \int_{\Omega} (|\nabla u(x,t)|^2 + |u'(x,t)|^2) \, dx + \frac{1}{2} \int_{\Omega} |u(x,t)|^2 \, d\mu(x).$$

This convergence of energies is at the root of the proof of the corrector result (1.7)-(1.8).

We also consider in Section 6 the case of initial data which have a regularity weaker than (1.4)-(1.5), i.e., the case where

$$u_\varepsilon^0 \in L^2(\Omega_\varepsilon), \quad u_\varepsilon^1 \in H^{-1}(\Omega_\varepsilon), \quad f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)).$$

We then prove that under suitable convergence assumptions on $u_\varepsilon^0$, $u_\varepsilon^1$, and $f_\varepsilon$, the extension $\tilde{u}_\varepsilon$ of the solution of (1.1) weakly * converges in $L^{\infty}(0,T;L^2(\Omega))$ to the solution $u$ of the problem (1.6) (see Theorem 6.2). In this setting we also obtain, under special assumptions on the data, the following corrector result (see Theorem 6.3):

$$\tilde{u}_\varepsilon \rightarrow u \quad \text{strongly in } C^0([0,T];L^2(\Omega)).$$

Besides their own interest, the previous results have interesting applications to the exact boundary controllability problem for the wave equation in domains with small holes, see D. Cioranescu, P. Donato and E. Zuazua [4], [5], where Theorems 3.1, 4.1 and 5.1 are used as a tool combined to the "Hilbert Uniqueness Method" introduced by J.-L. Lions [14].

The present paper is only concerned with homogeneous Dirichlet boundary data. Let us mention that the case of homogeneous Neumann data leads to completely different results, the critical size being in this case $a_\varepsilon = \varepsilon$ (see D. Cioranescu and P. Donato [3] for the homogenization of this problem).

The present paper is organized as follows:

Section 2 is divided into three parts. We first recall (Subsection 2.1) the abstract framework of [6] on the geometry of the holes, as well as the results dealing with the homogenization of the elliptic problem in this setting. The counterparts of some of these results for the time dependent case are given in
Subsection 2.3, where a quasi-extension operator is also introduced. Subsection 2.2 presents some compactness results in the spaces \(L^p(0, T; X)\).

In Section 3 we prove the weak * convergence in \(L^\infty(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))\) of the extension \(\tilde{u}_\varepsilon\) of the solution \(u_\varepsilon\) of (1.1) to the solution \(u\) of (1.6). Lower semicontinuity of the corresponding energy is also proved.

In Section 4 the corrector result (1.7)-(1.8) is proved when the assumption (1.9) on the data is made.

In Section 5 we consider the case where the size of the holes is smaller than the critical one. In this situation \(\mu = 0\) and the convergence of \(\tilde{u}_\varepsilon\) to \(u\) is proved to take place in the strong topology of \(L^\infty(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))\).

In Section 6 we prove the convergence of the solutions and the corrector result (1.11) in the case where the data only meet a weaker regularity assumption (see (1.10)).

Finally the Appendix is devoted to the proof of the density of \(D(\Omega)\) in \(H^1_0(\Omega) \cap L^2(\Omega; d\mu)\) when \(\mu\) is a nonnegative and finite Radon measure which belongs to \(H^{-1}(\Omega)\).

2. - Geometric setting. A review of the elliptic case and preliminary results

This section is divided into three parts. In the first one (Subsection 2.1) we describe the geometry of the problem and the abstract framework introduced by D. Cioranescu and F. Murat [6] in which the present work will be carried out; we also recall the homogenization and corrector results obtained in this framework when dealing with elliptic problems. Subsection 2.2 deals with compactness results in the spaces \(L^p(0, T; X)\). In Subsection 2.3 we introduce a "quasi-extension" operator and prove some pointwise (with respect to the time variable) lower semicontinuity results of the energy for the time dependent case. These latest results are in some sense the time dependent counterparts of the results presented in Subsection 2.1 for the elliptic case.

2.1 The geometry of the problem. A review of homogenization and corrector results in the elliptic case.

Let \(\Omega\) be a bounded domain of \(\mathbb{R}^n\) \((n \geq 2)\) (no regularity is assumed on the boundary \(\partial \Omega\)), and let \(\Omega_\varepsilon\) be the domain obtained by removing from \(\Omega\) a set \(S_\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} S^i_\varepsilon\) of \(N(\varepsilon)\) closed subsets of \(\mathbb{R}^n\) (the holes). Here \(\varepsilon\) is a parameter which takes its values in a sequence which tends to zero while \(N(\varepsilon)\) is an integer which tends to infinity.

Instead of making direct geometric assumptions on the holes \(S^i_\varepsilon\), we adopt here the abstract framework introduced by D. Cioranescu and F. Murat in [6] where the assumption on the geometry of the holes is made by assuming the
existence of a suitable family of test functions. Precisely we will assume that

\[
\begin{aligned}
\text{(i) } w_e &\in H^1(\Omega) \cap L^\infty(\Omega), \quad \|w_e\|_{L^\infty(\Omega)} \leq M_0 \\
\text{(ii) } w_e &= 0 \text{ on } S_e \\
\text{(iii) } w_e &\rightharpoonup 1 \text{ weakly in } H^1(\Omega) \text{ and a.e. in } \Omega \\
\text{(iv) } -\Delta w_e &= \mu_e - \gamma_e \quad \text{where } \mu_e, \gamma_e \in H^{-1}(\Omega), \\
\mu_e &\rightharpoonup \mu \text{ strongly in } H^{-1}(\Omega) \\
\langle \gamma_e, v_e \rangle_\Omega &= 0 \text{ for any } v_e \in H_0^1(\Omega) \text{ such that } v_e = 0 \text{ on } S_e.
\end{aligned}
\]

In (2.1) and henceforth \( \langle \cdot, \cdot \rangle_\Omega \) denotes the duality pairing between \( H^{-1}(\Omega) \) and \( H_0^1(\Omega) \), while \( \langle \cdot, \cdot \rangle_{\Omega_e} \) will denote the duality pairing between \( H^{-1}(\Omega_e) \) and \( H_0^1(\Omega_e) \).

**Remark 2.1.** Hypothesis (2.1) differs from the original framework proposed in [6] by the fact that in (2.1) \( \mu \) is only assumed to belong to \( H^{-1}(\Omega) \) while in [6] \( \mu \) was assumed to belong to \( W^{-1,\infty}(\Omega) \). Nevertheless the present variation allows one to prove the same type of results in the elliptic case (see H. Kacimi and F. Murat [12, Paragraphe 2]). Note that the framework adopted here is of the type \( (H5)' \) in the notation of [6] (see [6, Remarque 1.6]) which for the present case is more convenient than the hypothesis \( (H5) \) of [6].

**Example 2.1.** Let us present the typical example where assumption (2.1) is satisfied. Consider the case where \( \Omega \) is periodically perforated, with a period \( 2\varepsilon \) in the direction of each coordinate axis, by small holes \( S^i_\varepsilon \) of form \( S \) and size \( a_\varepsilon \) obtained from a model hole \( S \) by a translation and an \( a_\varepsilon \)-homothety. To be precise \( S^i_\varepsilon \) is given by

\[
S^i_\varepsilon = a_\varepsilon S + 2\varepsilon \sum_{k=1}^n i_k e_k
\]

where \( (i_1, i_2, \ldots, i_n) \) is a multi-index of \( \mathbb{Z}^n \), \( \{e_1, \ldots, e_n\} \) is the canonical basis of \( \mathbb{R}^n \), \( S \) is a closed set contained in the ball \( B_1 \) of radius 1 centered at the origin (in the case \( n = 2 \), \( S \) is assumed to contain a ball centered at the origin, see [12, Remarque 2.2]), and \( a_\varepsilon < \varepsilon \) satisfies

\[
\begin{aligned}
e^2 \log a_\varepsilon &\to -C_0 & \text{if } n = 2 \\
a_\varepsilon e^{-n/(n-2)} &\to C_0 & \text{if } n \geq 3
\end{aligned}
\]

for a given \( C_0 > 0 \).
The model case is then
\begin{align}
    \begin{cases}
        a_\varepsilon = \delta_\varepsilon \exp \left( -\frac{C_0}{\varepsilon^2} \right) & \text{if } n = 2, \text{ with } \varepsilon^2 \log \delta_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \\
        a_\varepsilon = C_0 \varepsilon^{n/(n-2)} & \text{if } n \geq 3,
    \end{cases}
\end{align}

the model hole \( S \) being chosen as the unit ball of \( \mathbb{R}^n \).

When \( a_\varepsilon \) is given by (2.3) or (2.4), it is possible to construct "explicitly" \( w_\varepsilon \) (see [6, Exemple modè le 2.1]) on the cube \( P_\varepsilon^n \) of size \( 2\varepsilon \) of center \( x_\varepsilon = 2\varepsilon \sum_{k=1}^{n} i_k e_k \); consider the function \( w_\varepsilon \in H^1(P_\varepsilon^n) \) defined by
\begin{align}
    \begin{cases}
        \Delta w_\varepsilon = 0 & \text{in } B_\varepsilon^i \setminus S_\varepsilon^i \\
        w_\varepsilon = 0 & \text{in } S_\varepsilon^i \\
        w_\varepsilon = 1 & \text{in } P_\varepsilon \setminus B_\varepsilon^i
    \end{cases}
\end{align}

where \( B_\varepsilon^i \) is the ball of center \( x_\varepsilon^i \) and radius \( \varepsilon \). (When \( S \) is a ball the function \( w_\varepsilon \) can be easily computed in radial coordinates). The function \( w_\varepsilon \) defined by (2.5) in each \( P_\varepsilon^n \) satisfies (2.1) with
\begin{align}
    \begin{cases}
        \mu = \frac{1}{C_0} \frac{\pi}{2} & \text{if } n = 2 \\
        \mu = C_0 \frac{\varepsilon^{n-2}}{2^n} \text{Cap}(S, \mathbb{R}^n) & \text{if } n \geq 3.
    \end{cases}
\end{align}

For the proof see [6, Exemple modè le 2.1] and [12, Théorème 2.1]; in (2.6)
\begin{align*}
\text{Cap}(S, \mathbb{R}^n) = \inf_{\varphi \in \mathcal{D}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dx
\end{align*}
is the capacity in \( \mathbb{R}^n \) of the closed set \( S \).

We refer the reader to [6] and to H. Kacimi [11, Chapitre 1] for other examples of holes where assumption (2.1) is satisfied.

\begin{remark}
In the above Example 2.1, the size \( a_\varepsilon \) defined in (2.3) is critical in the following sense: when the size of the holes is \( r_\varepsilon \) with \( r_\varepsilon << a_\varepsilon \), i.e., when
\begin{align}
    \begin{cases}
        \varepsilon^2 \log r_\varepsilon \to -\infty & \text{if } n = 2 \\
        \varepsilon^{n/(n-2)}/r_\varepsilon \to +\infty & \text{if } n \geq 3,
    \end{cases}
\end{align}
hypothesis (2.1) is satisfied, but in (iii) \( w_\varepsilon \) now converges strongly to 1 in \( H^1(\Omega) \) and in (iv), \( \mu_\varepsilon \) and \( \gamma_\varepsilon \) strongly converge to 0; thus \( \mu = 0 \) in this case. On the other hand, if \( a_\varepsilon << r_\varepsilon \) (which corresponds to replace \( \infty \) by 0 in (2.7)), it can be proved that there is no sequence satisfying assertions (i), (ii) and (iii).
of (2.1). The size \( a_e \) given by (2.3) is therefore the only one for which (2.1) holds with weak (and not strong) convergence of \( w^\varepsilon \) to 1 in (iii).

In the abstract framework of hypothesis (2.1), the following results can be proved (see [6, Chapitre 1] and [12, Paragraphe 2]).

**Lemma 2.1.** If (2.1) holds true, the distribution \( \mu \) which appears in (iv) is given by

\[
(2.8) \quad \langle \mu, \varphi \rangle_\Omega = \lim_{\varepsilon \to 0} \int_{\Omega} \varphi |\nabla w^\varepsilon|^2 dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).
\]

Thus \( \mu \) is a positive Radon measure as well as an element of \( H^{-1}(\Omega) \); moreover \( \mu(\Omega) \) is finite.

A result of J. Deny [7] (see also H. Brézis and F.E. Browder [2]) then asserts that any function \( v \in H^1_0(\Omega) \) is measurable with respect to the measure \( d\mu \) and belongs to \( L^1(\Omega; d\mu) \), namely

\[
(2.9) \quad \begin{cases} 
\text{if } \mu \in H^{-1}(\Omega), \mu \geq 0 \text{ and } v \in H^1_0(\Omega), \\
\text{then } v \in L^1(\Omega; d\mu) \text{ and } \langle \mu, v \rangle_\Omega = \int_{\Omega} v d\mu.
\end{cases}
\]

This allows one to define without ambiguity the space

\[
(2.10) \quad V = H^1_0(\Omega) \cap L^2(\Omega; d\mu)
\]

which is a Hilbert space for the scalar product

\[
(2.11) \quad a(u, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} uv d\mu.
\]

Finally, for any \( v \in L^2(\Omega) \) define \( \tilde{v} \) as the extension of \( v \) by zero outside \( \Omega_e \), i.e.

\[
(2.12) \quad \tilde{v}(x) = \begin{cases} 
v(x) & \text{if } x \in \Omega_e \\
0 & \text{if } x \in \partial \Omega_e.
\end{cases}
\]

Of course one has

\[
(2.13) \quad \|\tilde{v}\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)} \quad \text{if } v \in L^2(\Omega_e).
\]

Note that \( \tilde{v} \) belongs to \( H^1_0(\Omega) \) if \( v \) belongs to \( H^1_0(\Omega_e) \) and that

\[
(2.14) \quad \|\tilde{v}\|_{H^1_0(\Omega)} = \|v\|_{H^1_0(\Omega_e)} \quad \text{if } v \in H^1_0(\Omega_e).
\]

We recall the following result on the homogenization of elliptic problems.
THEOREM 2.2. Assume that (2.1) holds true and consider the solutions \(v_e\) of the Dirichlet problems

\[
\begin{align*}
\begin{cases}
-\Delta v_e = g_e & \text{in } \mathcal{D}'(\Omega_e) \\
v_e \in H^1_0(\Omega_e)
\end{cases}
\end{align*}
\]

where \(g_e \in H^{-1}(\Omega)\) is such that

\[
g_e \rightharpoonup g \quad \text{strongly in } H^{-1}(\Omega).
\]

The sequence \(\tilde{v}_e\) (obtained from the solution \(v_e\) of (2.15) by the extension (2.12)) satisfies

\[
\begin{align*}
\begin{cases}
\tilde{v}_e \rightharpoonup v & \text{weakly in } H^1_0(\Omega) \\
\int_{\Omega_e} |\nabla v_e|^2 dx \to \int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} |v|^2 d\mu
\end{cases}
\end{align*}
\]

where \(v = v(x)\) is the unique solution of

\[
\begin{align*}
\begin{cases}
-\Delta v + \mu v = g & \text{in } \mathcal{D}'(\Omega) \\
v \in V
\end{cases}
\end{align*}
\]

(see Remark 2.3 below). Moreover

\[
\begin{align*}
\begin{cases}
\tilde{v}_e = w_e v + r_e \\
r_e \to 0 & \text{strongly in } W^{1,1}_0(\Omega).
\end{cases}
\end{align*}
\]

Finally if \(v\) belongs to \(H^1_0(\Omega) \cap C^0(\overline{\Omega})\), the convergence of \(r_e\) in (2.19) takes place in the strong topology of \(H^1_0(\Omega)\).

REMARK 2.3. Note that the variational formulation associated to (2.18) is (see (2.10), (2.11) for the definitions of \(V\) and \(a\))

\[
\begin{align*}
\begin{cases}
a(v, y) = (g, y)_\Omega, & \text{for all } y \in V \\
v \in V.
\end{cases}
\end{align*}
\]

REMARK 2.4. Assertion (2.19) is a corrector result for the solution \(v_e\) of the Dirichlet problem (2.15), since it allows one to replace \(\tilde{v}_e\) by the "explicit" expression \(w_e v\), up to the remainder \(r_e\) which strongly converges to zero.

It is finally worth mentioning the following lower semicontinuity of the energy.
THEOREM 2.3. Assume that (2.1) holds true and consider a sequence \( z_\varepsilon \) such that
\[
\begin{cases}
  z_\varepsilon \in H_0^1(\Omega) \\
  z_\varepsilon = 0 & \text{on } S_e \\
  z_\varepsilon \rightharpoonup z & \text{weakly in } H_0^1(\Omega).
\end{cases}
\]
(2.21)

Then
\[
\begin{cases}
  z \in V \\
  \liminf_{\varepsilon \to 0} \int_\Omega |\nabla z_\varepsilon|^2 \, dz \geq \int_\Omega |\nabla z|^2 \, dz + \int_\Omega |z|^2 \, d\mu.
\end{cases}
\]
(2.22)

Moreover when \( z_\varepsilon \) also satisfies
\[
\int_\Omega |\nabla z_\varepsilon|^2 \, dz \to \int_\Omega |\nabla z|^2 \, dz + \int_\Omega |z|^2 \, d\mu
\]
(2.23)

one has
\[
\begin{cases}
  z_\varepsilon = v_\varepsilon z + r_\varepsilon \\
  r_\varepsilon \to 0 & \text{strongly in } W_{0,1}^1(\Omega).
\end{cases}
\]
(2.24)

Finally if \( z \) belongs to \( H_0^1(\Omega) \cap C^0(\overline{\Omega}) \), the convergence of \( r_\varepsilon \) in (2.24) takes place in the strong topology of \( H_0^1(\Omega) \).

REMARK 2.5. Note that, in view of (2.9), any element of \( H_0^1(\Omega) \) belongs to \( L^1(\Omega; d\mu) \); the first assertion of (2.22) thus claims that \( z \) actually belongs to \( L^2(\Omega; d\mu) \).

2.2 Some compactness lemmas.

Let \( X \) and \( Y \) be two reflexive Banach spaces such that \( X \subset Y \) with continuous and dense embedding. Denote by \( X' \) (resp. \( Y' \)) the dual space of \( X \) (resp. \( Y \)) and by \( (\cdot, \cdot)_{X,X'} \) (resp. \( (\cdot, \cdot)_{Y,Y'} \)) the duality pairing between \( X \) and \( X' \) (resp. \( Y \) and \( Y' \)). We will use the following space introduced in [16, Chapitre 3, Paragraphe 8.4]:
\[
C^0_\infty(0,T;Y) = \{ f \in L^\infty(0,T;Y) : t \mapsto (f(t),v)_{Y,Y'} \text{ is continuous} \}
\]
from \([0,T]\) into \( \mathbb{R} \) for any fixed \( v \in Y' \).

LEMMA 2.4. Consider a sequence \( g_\varepsilon \) such that
\[
\begin{cases}
  g_\varepsilon \rightharpoonup g & \text{weakly } * \text{ in } L^\infty(0,T;X) \\
  g_\varepsilon \to g & \text{strongly in } C^0([0,T];Y).
\end{cases}
\]
(2.25)
Then \( g_e \) strongly converges to \( g \) in \( C^0_\nu([0,T]; X) \), i.e., for every \( v \in X' \) the function

\[
(2.26) \quad h_e : t \mapsto \langle g_e(t), v \rangle_{X',X}
\]

belongs to \( C^0([0,T]) \) and satisfies

\[
(2.27) \quad h_e \to h \quad \text{strongly in } C^0([0,T])
\]

where \( h \) is defined by

\[
(2.28) \quad h : t \mapsto \langle g(t), v \rangle_{X,X'}.
\]

**PROOF.** According to a result of J.-L. Lions and E. Magenes \[16, Lemme 8.1, p. 297\] we have

\[
L^\infty(0,T; X) \cap C^0_\nu([0,T]; Y) = C^0_\nu([0,T]; X).
\]

Hence \( g_e \) belongs to \( C^0_\nu([0,T]; X) \) by (2.25) and therefore \( h_e \) defined by (2.26) lies in \( C^0([0,T]) \).

To prove (2.27), it is sufficient to show that \( h_e \) is a Cauchy sequence in \( C^0([0,T]) \). For a given \( \tilde{v} \in Y' \) introduce the function

\[
\tilde{h}_e : t \mapsto \langle g_e(t), \tilde{v} \rangle_{Y',Y'}.
\]

We have

\[
|h_e(t) - h_{e'}(t)| \leq |h_e(t) - \tilde{h}_e(t)| + |\tilde{h}_e(t) - \tilde{h}_{e'}(t)|
+ |\tilde{h}_{e'}(t) - h_{e'}(t)|
= |\langle g_e(t), v - \tilde{v} \rangle_{X,X'}| + |\langle g_e(t) - g_{e'}(t), \tilde{v} \rangle_{Y,Y'}|
+ |\langle g_{e'}(t), v - \tilde{v} \rangle_{X,X'}|
\leq (\|g_e\|_{L^\infty(0,T;X)} + \|g_{e'}\|_{L^\infty(0,T;X)}) \|v - \tilde{v}\|_{X'}
+ \|g_e - g_{e'}\|_{C^0([0,T];Y)} \|	ilde{v}\|_{Y'}.
\]

Combining (2.25), (2.29) and the density of \( Y' \) in \( X' \), we easily deduce that \( h_e \) is a Cauchy sequence in \( C^0([0,T]) \).

**PROPOSITION 2.5.** Assume further that the embedding \( X \subset Y \) is compact. Let \( g_e \) be a sequence such that

\[
(2.30) \quad g_e \rightharpoonup g \quad \text{weakly in } L^1(0,T; X)
\]

\[
(2.31) \quad g'_e \rightharpoonup g' \quad \text{weakly in } L^1(0,T; Y).
\]

Then

\[
(2.32) \quad g_e \to g \quad \text{strongly in } C^0([0,T]; Y).
\]
PROOF. In order to obtain the result it is sufficient to prove that (see, e.g. J. Simon [17, Theorem 3])

\begin{equation}
\|g_e(\cdot + h) - g_e(\cdot)\|_{L^\infty([0,T-h];Y)} \to 0 \quad \text{as } h \to 0 \quad \text{uniformly in } \varepsilon.
\end{equation}

On the first hand

\begin{equation}
\|g_e(\cdot + h) - g_e(\cdot)\|_{L^\infty([0,T-h];Y)} \leq \sup_{t \in [0,T-h]} \int_t^{t+h} \|g_e'(s)\|_Y \, ds.
\end{equation}

On the other hand Theorem 4 of J. Diestel and J.J. Uhl, [8, p. 104] states that the norm (in Y) of a sequence which converges weakly in \(L^1(0,T;Y)\) is uniformly integrable on \([0,T]\). This implies that the right hand side of (2.34) converges to zero as \(h\) converges to 0, uniformly in \(\varepsilon\), which yields (2.33) and completes the proof. \(\Box\)

REMARK 2.6. The convergence (2.32) is proved in [17, Corollary 4] under the further assumption (compare with (2.31)) that for some \(p > 1\)

\begin{equation}
g'_e \rightharpoonup g' \quad \text{weakly in } L^p(0,T;Y).
\end{equation}

Observe also that compactness (2.32) is false in general when (2.31) is replaced by the boundedness of \(g'_e\) in \(L^1(0,T;Y)\): consider, e.g. the case where \(X = Y = \mathbb{R}, \ g_e(t) = t/\varepsilon\) if \(0 \leq t \leq \varepsilon\), \(g_e(t) = 1\) if \(\varepsilon \leq t \leq 1\) and \(g(t) = 1\) if \(0 \leq t \leq 1\). \(\Box\)

As a consequence of Lemma 2.4 and Proposition 2.5 we have the following result.

COROLLARY 2.6. Assume that the embedding \(X \subset Y\) is compact. Let \(g_e\) be a sequence satisfying

\begin{equation}
\begin{cases}
g_e \rightharpoonup g & \text{weakly } \ast \text{ in } L^\infty(0,T;X) \\
g'_e \rightharpoonup g' & \text{weakly in } L^1(0,T;Y).
\end{cases}
\end{equation}

Then \(g_e\) strongly converges to \(g\) in \(C^0([0,T];X)\), i.e.

\begin{equation}
\langle g_e(\cdot), v \rangle_{X,X'} \to \langle g(\cdot), v \rangle_{X,X'} \quad \text{strongly in } C^0([0,T])
\end{equation}

for all \(v \in X'\).

REMARK 2.7. Note that (2.37) implies in particular that

\(g_e(t) \rightharpoonup g(t) \quad \text{weakly in } X\) for any fixed \(t \in [0,T]\)

and not only almost everywhere in \([0,T]\). \(\Box\)
2.3 Counterparts of some elliptic results. A “quasi-extension” operator.

**Proposition 2.7.** Assume that (2.1) holds true and consider a sequence of functions \( v_\varepsilon \) in \( L^\infty(0, T; H^1_0(\Omega_\varepsilon)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \) satisfying

\[
\begin{align*}
\bar{v}_\varepsilon &\rightharpoonup v \quad \text{weakly * in } L^\infty(0, T; H^1_0(\Omega)) \\
\bar{v}'_\varepsilon &\rightharpoonup v' \quad \text{weakly * in } L^\infty(0, T; L^2(\Omega)).
\end{align*}
\]

Then

\[
(\theta, \bar{v}_\varepsilon(\cdot))_\Omega \to (\theta, v(\cdot))_\Omega \quad \text{strongly in } C^0([0, T]) \quad \text{for any } \theta \in H^{-1}(\Omega)
\]

and on the other hand

\[
v \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(\Omega)).
\]

**Remark 2.8.** From (2.38) we know that

\[
v \in L^\infty(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega));
\]

property (2.40) then asserts that the limit \( v \) further satisfies

\[
v \in L^\infty(0, T; L^2(\Omega; d\mu)). \quad \square
\]

**Proof of Proposition 2.7.** Convergence (2.39) is a direct consequence of Corollary 2.6 applied to \( X = H^1_0(\Omega) \) and \( Y = L^2(\Omega) \).

In view of Remark 2.7, we have in particular

\[
\bar{v}_\varepsilon(t) \rightharpoonup v(t) \quad \text{weakly in } H^1_0(\Omega) \quad \text{for any fixed } t \in [0, T].
\]

Applying Theorem 2.3 to \( \bar{v}_\varepsilon(t) \) and \( v(t) \) we obtain that for any fixed \( t \in [0, T] \)

\[
\begin{align*}
\|v(t)\|_V^2 &= \int_\Omega |\nabla v(x, t)|^2 \, dx + \int_\Omega |v(x, t)|^2 \, d\mu(x) \\
&\leq \liminf_{\varepsilon \to 0} \int_\Omega |\nabla \bar{v}_\varepsilon(x, t)|^2 \, dx.
\end{align*}
\]

Since for some \( C_0 < +\infty \) we have

\[
\sup_{t \in [0, T]} \text{ess sup} \int_\Omega |\nabla \bar{v}_\varepsilon(x, t)|^2 \, dx = \|\bar{v}_\varepsilon\|_{L^\infty(0, T; H^1_0(\Omega))} \leq C_0,
\]

we have proved that \( v(t) \) belongs to \( V \) for any \( t \) and that

\[
\sup_{t \in [0, T]} \|v(t)\|_V^2 \leq C_0.
\]
This implies (2.40) once the measurability of the function \( v : [0, T] \rightarrow L^2(\Omega; d\mu) \) is proved, since we know from (2.38) that \( v \) belongs to \( L^\infty(0, T; H^1_0(\Omega)) \) \( \cap W^{1,\infty}(0, T; L^2(\Omega)) \).

Since \( L^2(\Omega; d\mu) \) is separable, it is sufficient to prove, using Pettis’ measurability Theorem (see [8, Theorem 2, p. 42]), that \( v \) is weakly measurable, i.e. that for any \( \varphi \in L^2(\Omega; d\mu) \), the function \( t \mapsto \int_\Omega v(x, t)\varphi(x)d\mu(x) \) is measurable. We already know that \( v \) belongs to \( L^\infty(0, T; H^1_0(\Omega)) \) and therefore, by (2.9), to \( L^\infty(0, T; L^1(\Omega; d\mu)) \). Thus the function \( t \mapsto \int_\Omega v(x, t)\psi(x)d\mu(x) \) is measurable for any \( \psi \in C^0(\overline{\Omega}) \). Approximate now \( \varphi \in L^2(\Omega; d\mu) \) by a sequence \( \psi_n \in C^0(\Omega) \). Since \( v(t) \) belongs to \( L^2(\Omega; d\mu) \) for any \( t \in [0, T] \), we have

\[
\int_\Omega v(x, t)\psi_n(x)d\mu(x) \rightarrow \int_\Omega v(x, t)\varphi(x)d\mu(x) \quad \text{for any } t \in [0, T].
\]

This proves the measurability of the function \( t \mapsto \int_\Omega v(x, t)\varphi(x)d\mu(x) \) and completes the proof of Proposition 2.7. \( \square \)

In the following Proposition we prove the existence of some quasi-extension operators that will be useful in the sequel.

**PROPOSITION 2.8.** Assume that (2.1) holds true and define the operator \( P^e \) by

\[
P^e\psi = w^e \tilde{\psi} \quad \text{in } \Omega \quad \text{for all } \psi \in L^2(\Omega^e)
\]

where \( \tilde{\psi} \) is the extension of \( \psi \) by zero in the holes \( S^e \) defined by (2.12). Then

\[
P^e \in \mathcal{L}(L^2(\Omega^e); L^2(\Omega)) \quad \text{and} \quad \|P^e\|_{\mathcal{L}(L^2(\Omega^e); L^2(\Omega))} \leq M_0.
\]

Moreover the operator \( P^e \) extends to an operator defined on \( H^{-1}(\Omega^e), \) and for any \( \varepsilon > 0 \) and any \( q \in (1, n/(n - 1)) \)

\[
P^e \in \mathcal{L}(H^{-1}(\Omega^e); W^{-1,q}(\Omega)) \quad \text{and} \quad \|P^e\|_{\mathcal{L}(H^{-1}(\Omega^e); W^{-1,q}(\Omega))} \leq C_q.
\]

**REMARK 2.9.** The operator \( P^e \) is not an extension operator since \( P^e\psi \) does not coincide with \( \psi \) in \( \Omega^e \): indeed \( w^e \) does not coincide with 1 on this set. However for any \( \varphi \) in \( L^2(\Omega) \) we have

\[
P^e\varphi \rightarrow \varphi \quad \text{strongly in } L^2(\Omega)
\]

which shows that \( P^e \) acts as a “quasi-extension” operator. \( \square \)
PROOF OF PROPOSITION 2.8. In view of (2.1), the operator \( P_\varepsilon \) defined by (2.41) clearly enjoys properties (2.42) and (2.44) (use Lebesgue’s dominated convergence theorem to prove (2.44)).

Let now \( p > n \) be fixed; define on \( W^{1,p}_0(\Omega) \) the operator

\[
R_\varepsilon \varphi = \varphi w_\varepsilon |_{\Omega_\varepsilon} \quad \text{for all } \varphi \in W^{1,p}_0(\Omega).
\]

Since \( \nabla (R_\varepsilon \varphi) = w_\varepsilon \nabla \varphi + \varphi \nabla w_\varepsilon \) in \( \Omega_\varepsilon \), we have

\[
\begin{align*}
\int_{\Omega_\varepsilon} |\nabla (R_\varepsilon \varphi)|^2 \, dx & \leq 2 \int_{\Omega_\varepsilon} |w_\varepsilon|^2 |\nabla \varphi|^2 \, dx + 2 \int_{\Omega_\varepsilon} |\varphi|^2 |\nabla w_\varepsilon|^2 \, dx \\
& \leq 2M_0^2 \int_{\Omega_\varepsilon} |\nabla \varphi|^2 \, dx + 2 \|\varphi\|_{L^p(\Omega)}^2 \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 \, dx \\
& \leq C_\varepsilon \|\varphi\|_{W^{1,p}_0(\Omega)}^2,
\end{align*}
\]

where the constant \( C_\varepsilon \) does not depend on \( \varepsilon \). We thus have

\[
\begin{align*}
R_\varepsilon & \in \mathcal{L}(W^{1,p}_0(\Omega); H^1_0(\Omega_\varepsilon)) \\
\|R_\varepsilon\|_{\mathcal{L}(W^{1,p}_0(\Omega); H^1_0(\Omega_\varepsilon))} & \leq C_\varepsilon.
\end{align*}
\]

Consider the operator \( R_\varepsilon^* \) defined on \( H^{-1}(\Omega_\varepsilon) \) by

\[
(R_\varepsilon^* \psi, \varphi)_{W^{-1}(\Omega_\varepsilon), W^{1,p}_0(\Omega)} = (\psi, R_\varepsilon \varphi)_{H^{-1}(\Omega_\varepsilon), H^1_0(\Omega_\varepsilon)}
\]

for all \( \psi \in H^{-1}(\Omega_\varepsilon) \) and \( \varphi \in W^{1,p}_0(\Omega) \),

where \( q \) is given by \( \frac{1}{p} + \frac{1}{q} = 1 \). The identity

\[
(\psi, R_\varepsilon \varphi)_{H^{-1}(\Omega_\varepsilon), H^1_0(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \psi \varphi w_\varepsilon \, dx = \int_{\Omega_\varepsilon} P_\varepsilon \psi \varphi \, dx
\]

for all \( \psi \in L^2(\Omega_\varepsilon) \), and \( \varphi \in W^{1,p}_0(\Omega) \),

proves that \( R_\varepsilon^* = P_\varepsilon \) is an extension of \( P_\varepsilon \) defined by (2.41) and (2.47) immediately implies (2.43). \( \square \)

REMARK 2.10. An interesting consequence of the construction of the operators \( P_\varepsilon \) is that it makes possible to perform the homogenization of the elliptic problem (2.15) when the sequence \( g_\varepsilon \) only satisfies

\[
g_\varepsilon \in H^{-1}(\Omega_\varepsilon), \quad \|g_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)} \leq C,
\]

where \( C > 0 \) is a constant independent of \( \varepsilon \). Under this hypothesis the extension \( \tilde{v}_\varepsilon \) of the solution \( v_\varepsilon \) of (2.15) is still bounded in \( H^1_0(\Omega) \).
On the other hand, from Proposition 2.8 we deduce that \( P_g g_\varepsilon \) is uniformly bounded in \( W^{-1,q}(\Omega) \). Thus, by passing to a subsequence (denoted by \( \varepsilon' \)), we have

\[
(2.50) \quad P_\varepsilon g_\varepsilon \rightharpoonup g^* \quad \text{weakly in } W^{-1,q}(\Omega).
\]

Using \( w_\varepsilon \varphi \) (with \( \varphi \in \mathcal{D}(\Omega) \) and \( w_\varepsilon \) defined by (2.1)) as test function in the variational formulation of (2.15) and following exactly the proof of [6, Théorème 1], one easily proves that

\[
(2.51) \quad \tilde{v}_\varepsilon \rightharpoonup v \quad \text{weakly in } H^1_0(\Omega),
\]

where \( v \) solves

\[
(2.52) \quad \begin{cases}
\int_\Omega \nabla v \nabla \varphi \, dx + \int_\Omega v \varphi \, d\mu = \langle g^*, \varphi \rangle, \\
v \in V.
\end{cases}
\]

Since \( \mathcal{D}(\Omega) \) is dense in \( V \) (see Appendix below), (2.52) actually holds for any \( \varphi \) in \( V \). Since the left hand side of (2.52) is a continuous linear form on \( \varphi \in V \), we actually have

\[
(2.53) \quad g^* \in V'
\]

and not only in \( W^{1,q}(\Omega) \) for \( q \in (1, n/(n-1)) \) as obtained in (2.50).

Finally note that in order to pass to the limit in (2.15), for some right hand side \( g_\varepsilon \) defined on \( \Omega \) and bounded in \( H^{-1}(\Omega) \) (this assumption is slightly stronger than (2.49) where \( g_\varepsilon \) is only defined on \( \Omega_\varepsilon \), one has to consider a subsequence \( \varepsilon' \) such that \( P_\varepsilon g_\varepsilon \) weakly converges to some \( g^* \) in \( H^{-1}(\Omega) \) (see (2.50)) and not a subsequence such that \( g_\varepsilon \) weakly converges to some \( g^{**} \) in \( H^{-1}(\Omega) \).

We conclude this Section with the following result, which proves that averaging in space provides some compactness in time.

PROPOSITION 2.9. Assume that (2.1) holds true and let \( P_\varepsilon \) be the quasi-extension operator defined in Proposition 2.8. Consider a sequence \( v_\varepsilon \) in \( L^\infty(0,T;L^2(\Omega_\varepsilon)) \cap W^{1,1}(0,T;H^{-1}(\Omega_\varepsilon)) \) satisfying

\[
(2.54) \quad \begin{cases}
\tilde{v}_\varepsilon \rightharpoonup v \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)) \\
P_\varepsilon v_\varepsilon \rightharpoonup v' \quad \text{weakly in } L^1(0,T;W^{-1,q}(\Omega))
\end{cases}
\]

for some \( q \in (1, n/(n-1)) \). Then for all \( \varphi \in L^2(\Omega) \)

\[
(2.55) \quad \int_\Omega \tilde{v}_\varepsilon(x, \cdot) \varphi(x) \, dx \to \int_\Omega v(x, \cdot) \varphi(x) \, dx \quad \text{strongly in } C^0([0,T]).
\]
PROOF. Combining (2.1), (2.42) and (2.54a) one can easily prove that

\[ P_\varepsilon \varphi = w_\varepsilon \varphi \rightharpoonup \varphi \quad \text{weakly} \quad * \quad \text{in} \quad L^\infty(0,T,L^2(\Omega)). \]

Corollary 2.6 applied to the sequence \( g_\varepsilon = P_\varepsilon \varphi \) with \( X = L^2(\Omega), \ Y = W^{-1,q}(\Omega) \), therefore implies that

\[
\int_\Omega w_\varepsilon(x)\tilde{v}_\varepsilon(x, \cdot)\varphi(x)dx \to \int_\Omega v(x, \cdot)\varphi(x)dx \quad \text{strongly in} \quad C^0([0,T])
\]

for all \( \varphi \in L^2(\Omega) \).

Convergence (2.55) is then deduced from (2.56) and from

\[
\int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi dx = \int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi w_\varepsilon dx + \int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi(1 - w_\varepsilon)dx
\]

since

\[
\int_\Omega \tilde{v}_\varepsilon(\cdot)\varphi(1 - w_\varepsilon)dx \leq \| \tilde{v}_\varepsilon \|_{L^\infty(0,T,L^2(\Omega))} \| \varphi(1 - w_\varepsilon) \|_{L^1(\Omega)} \to 0
\]

in view of (2.54), (2.1) and Lebesgue’s dominated convergence theorem. \( \square \)

3. - The homogenization result for the wave equation

The goal of this Section is to prove the homogenization result for the wave equation (Theorem 3.1). Lower semicontinuity of the energy is also proved (Theorem 3.2).

Consider a bounded domain \( \Omega \) of \( \mathbb{R}^n \) \( (n \geq 2) \) and the domain \( \Omega_\varepsilon \) obtained by removing from \( \Omega \) a set \( S_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} S_i^\varepsilon \) of “small” holes for which hypothesis (2.1) holds true. Let \( T > 0 \) be a given real number. Consider the wave equation

\[
\begin{cases}
\varepsilon u_{\varepsilon}'' - \Delta u_\varepsilon = f_\varepsilon & \text{in} \ Q_\varepsilon = \Omega_\varepsilon \times (0,T) \\
 u_\varepsilon = 0 & \text{on} \ \Sigma_\varepsilon = \partial\Omega_\varepsilon \times (0,T)
\end{cases}
\]

(3.1a)

\[
\begin{cases}
u_\varepsilon(0) = u_\varepsilon^0 & \text{in} \ \Omega_\varepsilon \\
v_\varepsilon'(0) = u_\varepsilon^1 & \text{in} \ \Omega_\varepsilon
\end{cases}
\]

(3.1b)

where the data \( u_\varepsilon^0, u_\varepsilon^1, f_\varepsilon \) are assumed to satisfy

\[
u_\varepsilon^0 \in H^1_0(\Omega_\varepsilon), \ u_\varepsilon^1 \in L^2(\Omega_\varepsilon), \ f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)).
\]

(3.2)
Classical results (see, e.g. [16] or [14]) provide the existence and uniqueness of a solution $u_e = u_e(x, t)$ of (3.1) which satisfies

\begin{equation}
    u_e \in C^0([0, T]; H^1_0(\Omega_e)) \cap C^1([0, T]; L^2(\Omega_e)).
\end{equation}

Moreover defining, for any $t \in [0, T]$, the energy $E_e(\cdot)$ by

\begin{equation}
    E_e(t) = \frac{1}{2} \| u'_e(t) \|^2_{L^2(\Omega_e)} + \frac{1}{2} \| \nabla u_e(t) \|^2_{L^2(\Omega_e)}
\end{equation}

one has the following energy identity

\begin{equation}
    E_e(t) = E_e(0) + \int_0^t \int_{\bar{\Omega}_e} f_e(x, s) u'_e(x, s) \, dx \, ds.
\end{equation}

Recall that $\bar{\cdot}$ denotes the extension by zero outside of $\Omega_e$ and that $V$ is the space $H^1_0(\Omega) \cap L^2(\Omega; d\mu)$ (see (2.12) and (2.10)). We have the following homogenization result for the wave equation (3.1).

**Theorem 3.1.** Assume that (2.1) holds true and consider a sequence of data which satisfy

\begin{equation}
    \begin{cases}
        \tilde{f}_e \rightharpoonup f & \text{weakly in } L^1(0, T; L^2(\Omega)) \\
        \tilde{u}^0_e \rightharpoonup u^0 & \text{weakly in } H^1_0(\Omega) \\
        \tilde{u}^1_e \rightharpoonup u^1 & \text{weakly in } L^2(\Omega).
    \end{cases}
\end{equation}

The sequence $u_e$ of solutions of (3.1) then satisfies

\begin{equation}
    \begin{cases}
        \bar{u}_e \rightharpoonup u & \text{weakly * in } L^\infty(0, T; H^1_0(\Omega)) \\
        \bar{u}'_e \rightharpoonup u' & \text{weakly * in } L^\infty(0, T; L^2(\Omega))
    \end{cases}
\end{equation}

where $u = u(x, t)$ is the unique solution of the homogenized wave equation

\begin{align}
    &\begin{cases}
        u'' - \Delta u + \mu u = f & \text{in } Q = \Omega \times (0, T) \\
        u = 0 & \text{on } \Sigma = \partial \Omega \times (0, T)
    \end{cases} \\
    &\begin{cases}
        u(0) = u^0 & \text{in } \Omega \\
        u'(0) = u^1 & \text{in } \Omega
    \end{cases}
\end{align}

\begin{equation}
    u \in C^0([0, T]; V) \cap C^1([0, T]; L^2(\Omega)).
\end{equation}
REMARK 3.1. In view of definition (2.11) of the scalar product $a(\cdot, \cdot)$ of $V$, the variational formulation of the wave equation (3.8a) is

\[
\begin{align*}
\frac{d^2}{dt^2} \int_{\Omega} u(x,t)v(x)dx + a(u(t), v) &= \int_{\Omega} f(x,t)v(x)dx \quad \forall v \in V \\
u &\in L^\infty(0,T; V) \cap W^{1,\infty}(0,T; L^2(\Omega)).
\end{align*}
\]

Note that according to Theorem 2.3, the function $u^0$ (which is the weak limit in $H^1(\Omega)$ of functions $\tilde{u}_e^0$ vanishing on the holes $S_\varepsilon$) belongs to $V$, so there is no contradiction between the two assertions $u(0) = u^0$ and $u \in C^0([0,T]; V)$.

On the other hand, observe that classical results (see e.g. [16]) provide existence and uniqueness of a solution of (3.8). In fact, uniqueness holds in the larger class $L^\infty(0,T; V) \cap W^{1,\infty}(0,T; L^2(\Omega))$.

Finally note that $f_\varepsilon$ is assumed to converge weakly in $L^1(0,T; L^2(\Omega))$, which is a stronger assumption than to be bounded in this space. \qed

PROOF OF THEOREM 3.1. We proceed in four steps.

\textit{First step: a priori estimates.}

From (3.5) we have

\[
E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon(x,s)u_\varepsilon'(x,s)dx \, ds \leq E_\varepsilon(0) + \sqrt{2} \int_0^t \|f_\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}(E_\varepsilon(s))^{1/2}ds,
\]

which by Gronwall's inequality implies

\[
(E_\varepsilon(t))^{1/2} \leq (E_\varepsilon(0))^{1/2} + \frac{1}{\sqrt{2}} \int_0^t \|f_\varepsilon(s)\|_{L^\infty(\Omega_\varepsilon)}ds.
\]

In view of (3.6), the right hand side of (3.10) is bounded independently of $t \in [0,T]$ and $\varepsilon$. Using the properties of the extension by zero outside $\Omega_\varepsilon$ (see (2.12)) this implies that $\tilde{u}_\varepsilon$ is bounded in $L^\infty(0,T; H^1_0(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$. Extracting a subsequence (still denoted by $\varepsilon$, since in the fourth step the whole sequence will be proved to converge) one has

\[
\begin{align*}
\tilde{u}_\varepsilon &\rightharpoonup u \quad \text{weakly * in } L^\infty(0,T; H^1_0(\Omega)) \\
\tilde{u}_\varepsilon' &\rightharpoonup u' \quad \text{weakly * in } L^\infty(0,T; L^2(\Omega)).
\end{align*}
\]

On the other hand, in view of Proposition 2.7 we have

\[
\begin{align*} u \in L^\infty(0,T; V) \cap W^{1,\infty}(0,T; L^2(\Omega)).
\end{align*}
\]
Second step: passing to the limit in the wave equation (3.1a).

Using in (3.1a) the test function \( \psi(t) \varphi(x) w_\varepsilon(x) \) where \( \psi \in \mathcal{D}((0,T)) \), \( \varphi \in \mathcal{D}(\Omega) \) and \( w_\varepsilon \) is defined in hypothesis (2.1), we obtain after integration by parts

\[
\int_{Q_\varepsilon} u_\varepsilon \psi' \varphi w_\varepsilon \, dx \, dt + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla \varphi w_\varepsilon \, dx \, dt + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla \varphi w_\varepsilon \, dx \, dt \\
= \int_{Q_\varepsilon} f_\varepsilon \varphi w_\varepsilon \, dx \, dt.
\]

Extension by zero outside \( \Omega_\varepsilon \), Fubini’s Theorem and an integration by parts in the second term give, in view of the identity \( -\Delta u_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon \) (see (2.1)(iv))

\[
\begin{align*}
\int_{\Omega} \varphi w_\varepsilon & \left( \int_0^T \psi'' \bar{u}_\varepsilon \, dt \right) \, dx + \left( \mu_\varepsilon - \gamma_\varepsilon, \varphi \left( \int_0^T \psi \bar{u}_\varepsilon \, dt \right) \right)_{\Omega} \\
- \int_{\Omega} \nabla w_\varepsilon \nabla \varphi & \left( \int_0^T \psi \bar{u}_\varepsilon \, dt \right) \, dx + \int_{\Omega} w_\varepsilon \nabla \varphi \left( \int_0^T \psi \bar{u}_\varepsilon \, dt \right) \, dx \\
= & \int_{\tilde{Q}} f_\varepsilon \varphi w_\varepsilon \, dx \, dt.
\end{align*}
\]

(3.13)

Consider the function \( U_\varepsilon \in H^1_0(\Omega) \) defined by

\[
U_\varepsilon(x) = \int_0^T \psi(t) \bar{u}_\varepsilon(x,t) \, dt.
\]

Convergences (3.11) imply that the sequence \( U_\varepsilon \) satisfies

\[
(3.14) \quad \begin{cases} 
U_\varepsilon \rightharpoonup \int_0^T \psi(t) u(x,t) \, dt & \text{ weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega) \\
U_\varepsilon = 0 & \text{ on } S_\varepsilon 
\end{cases}
\]

and that

\[
(3.15) \quad \begin{cases} 
\int_0^T \psi''(t) \bar{u}_\varepsilon(x,t) \, dt \to \int_0^T \psi''(t) u(x,t) \, dt & \text{ strongly in } L^2(\Omega) \\
\int_0^T \psi''(t) \bar{u}_\varepsilon(x,t) \, dt = 0 & \text{ on } S_\varepsilon.
\end{cases}
\]

It is now easy to pass to the limit in each term of (3.13), using hypothesis (2.1)(iii) and (iv) (note that \( \langle \gamma_\varepsilon, \varphi U_\varepsilon \rangle_{\Omega} = 0 \)). Using Fubini's and Deny's Theorems
(see (2.9)), we obtain
\begin{equation}
(3.16) \quad \left\{ \begin{array}{l}
\int_0^T \frac{\psi''}{\Omega} \left( \int_0^T u(\varphi) \, dx \right) \, dt + \int_0^T \psi \left( \int_0^\infty \varphi_u \, d\mu \right) \, dt + \int_0^T \psi \left( \int \nabla u \nabla \varphi \, dx \right) \, dt \\
\quad = \int_0^T \psi \left( \int f \varphi \, dx \right) \, dt.
\end{array} \right.
\end{equation}

Since \( \psi \in \mathcal{D}((0,T)) \) is arbitrary, we have proved that
\begin{equation}
(3.17) \quad \frac{d^2}{dt^2} \int_\Omega u(x,t) \varphi(x) \, dx + a(u, \varphi) = \int_\Omega f \varphi \, dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall \varphi \in \mathcal{D}(\Omega).
\end{equation}

In view of (3.12), the density of \( \mathcal{D}(\Omega) \) in \( V \) (see the Appendix below) allows one to extend (3.17) to every test function \( \varphi \in V \).

We thus have proved (3.9) which (see Remark 3.1) is equivalent to (3.8a).

\textit{Third step: passing to the limit in the initial data.}

From (3.11) and Proposition 2.7 we deduce that
\begin{equation}
(3.18) \quad (\theta, \bar{u}_e(\cdot))_\Omega \rightarrow (\theta, u(\cdot))_\Omega \quad \text{strongly in } C^0([0,T])
\end{equation}
for any \( \theta \in H^{-1}(\Omega) \). Since \( u_e(0) = \bar{u}_e \) tends to \( u^0 \) weakly in \( H^1_0(\Omega) \) (see (3.6)), we obtain
\begin{equation}
(3.19) \quad u(0) = u^0.
\end{equation}

In order to prove that \( u'(0) = u^1 \) we will apply Proposition 2.9 to \( v = u_e \).

Let us first check that
\begin{equation}
(3.20) \quad P_e u_e'' \rightharpoonup u'' \quad \text{weakly in } L^1(0,T;W^{-1,q}(\Omega)).
\end{equation}

Indeed observe that
\begin{equation*}
u'_e = \Delta u_e + f_e \quad \text{in } Q_e = \bar{\Omega} \times (0,T)
\end{equation*}
and thus
\begin{equation*}
P_e u''_e = P_e \Delta u_e + P_e f_e.
\end{equation*}

In view of (3.11a), we have \( \|\Delta u_e\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C \) and thus by Proposition 2.8,
\begin{equation*}
\|P_e \Delta u_e\|_{L^\infty(0,T;W^{-1,q}(\Omega))} \leq C_q \quad \text{for any } q \in (1, n/(n-1)).
\end{equation*}

On the other hand, \( P_e f_e = w_e \bar{f}_e \) is relatively compact in the weak topology of \( L^1(0,T;L^2(\Omega)) \) in view of (3.6) and (2.1(i)). This follows from Dunford's
Theorem (see [8, Theorem 1, p. 101]) since $L^2(\Omega)$ enjoys the Radon-Nikodym property (see [8, Corollary 13, p. 76]), since $w_\varepsilon f_\varepsilon$ is bounded in $L^1(0,T;L^2(\Omega))$, since $\int E w_\varepsilon f_\varepsilon \, dx$ is relatively compact in the weak topology of $L^2(\Omega)$ for any measurable subset $E$ of $[0,T]$, and finally since the function $t \mapsto \|w_\varepsilon f_\varepsilon(t)\|_{L^2(\Omega)}$ is uniformly integrable on $[0,T]$ (indeed, the functions $t \mapsto \|f_\varepsilon(t)\|_{L^2(\Omega)}$ are uniformly integrable on $[0,T]$ since the sequence $f_\varepsilon$ is relatively compact in $L^1(0,T;L^2(\Omega))$, see [8, Theorem 4, p. 104]).

Consequently $P_\varepsilon u_\varepsilon^\prime = P_\varepsilon \Delta u_\varepsilon + P_\varepsilon f_\varepsilon$ is relatively compact in the weak topology of $L^1(0,T;W^{-1,q}(\Omega))$.

Finally, combining (2.1) and (3.11b), it is easy to prove that

$$P_\varepsilon u_\varepsilon' = w_\varepsilon u_\varepsilon' \rightharpoonup u' \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)).$$

Since $(P_\varepsilon u_\varepsilon')' = P_\varepsilon u_\varepsilon''$ because of the definition of $P_\varepsilon$, this implies that (3.20) is satisfied.

Combining (3.11b) and (3.20), Proposition 2.9 ensures that

$$\int \bar{u}_\varepsilon'(x,\cdot)\varphi(x)\,dx \to \int \bar{u}'(x,\cdot)\varphi(x)\,dx \quad \text{strongly in } C^0([0,T])$$

for any $\varphi \in L^2(\Omega)$. Since $\bar{u}_\varepsilon'(0) = \bar{u}_\varepsilon'$ tends to $u'$ weakly in $L^2(\Omega)$ (see (3.6)), we deduce

$$u'(0) = u^1.$$

The limit $u = u(x,t)$ therefore satisfies the initial conditions (3.8b).

Fourth step: end of the proof.

In the second and third step we have proved that, up to the extraction of a subsequence (still denoted by $\varepsilon$), the sequence $u_\varepsilon$ satisfies (3.11) where the limit $u$ belongs to $L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and satisfies (3.8a)-(3.8b).

The uniqueness of the solution of (3.8a)-(3.8b) in $L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega))$ (see Remark 3.1) allows us to deduce that the whole sequence $u_\varepsilon$ satisfies (3.7) and that the limit $u$ satisfies (3.8c).

This completes the proof of Theorem 3.1. □

We have also the following pointwise (in time) convergence result and lower semicontinuity property of the energy.

**THEOREM 3.2.** Assume that the hypotheses of Theorem 3.1 are fulfilled. Then for any fixed $t \in [0,T]$

$$\bar{u}_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } H_0^1(\Omega)$$

$$\bar{u}_\varepsilon'(t) \rightharpoonup u'(t) \quad \text{weakly in } L^2(\Omega)$$
and where $E_\varepsilon$ is defined by (3.4) while

$$E(t) = \frac{1}{2} \| u'(t) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\Omega)} + \frac{1}{2} \| u(t) \|^2_{L^2(\Omega, \mu)}.$$  

PROOF. The convergences (3.23) and (3.24) have been proved in the third step of the proof of Theorem 3.1 (see (3.18) and (3.22)). From Theorem 2.3 we have (3.25) while (3.26) is straightforward. This immediately implies (3.27).

In the next Section we shall show that, under special convergence assumptions on the data (which are quite stronger than (3.6)), we have

$$E_\varepsilon(t) \to E(t) \quad \text{strongly in } C^0([0, T]).$$

This convergence property of the energies will play a crucial role when proving the corrector result.

4. - Corrector for the wave equation in a domain with small holes

This Section is devoted to state and prove the corrector result when special assumptions are satisfied by the data. The proof follows along the lines of S. Brahim-Otsmane, G.A. Francfort and F. Murat [1], who adapted to the wave equation the ideas introduced by L. Tartar [18] in the elliptic case. One of the main steps of the proof is the strong convergence of the energy in $C^0([0, T])$.

Concerning the initial condition $u^0_\varepsilon$ we shall assume that

$$u^0_\varepsilon \in H^1_0(\Omega_\varepsilon)$$

there exists $g_\varepsilon \in H^{-1}(\Omega)$ such that

$$-\Delta u^0_\varepsilon = g_\varepsilon \quad \text{in } D'(\Omega_\varepsilon)$$

$$g_\varepsilon \to g \quad \text{strongly in } H^{-1}(\Omega).$$
As a consequence of Theorem 2.2 we deduce that

\[(4.2) \quad \tilde{u}_e^0 \rightharpoonup u^0 \quad \text{weakly in } H^1_0(\Omega)\]

where \(u^0 = u^0(x)\) is the solution of

\[
\begin{cases}
-\Delta u^0 + \mu u^0 = g & \text{in } \mathcal{D}'(\Omega) \\
u^0 \in V.
\end{cases}
\]

**THEOREM 4.1.** Assume that (2.1) holds true and consider a sequence of data \(u^0_e, u^1_e, f_e\) that satisfy (4.1) and

\[(4.4) \quad \hat{f}_e \to f \quad \text{strongly in } L^1(0,T;L^2(\Omega))\]

\[(4.5) \quad \tilde{u}_e^1 \to u^1 \quad \text{strongly in } L^2(\Omega)).\]

If \(u\) denotes the unique solution of the homogenized equation (3.8), the sequence \(u_e\) of solutions of (3.1) satisfies

\[(4.6) \quad \tilde{u}_e^l \to u' \quad \text{strongly in } C^0([0,T];L^2(\Omega))\]

\[(4.7) \quad \tilde{u}_e = u w_e + R_e\]

with

\[(4.8) \quad R_e \to 0 \quad \text{strongly in } C^0([0,T];W^{1,1}_0(\Omega)).\]

Moreover, if \(u \in C^0(\overline{\Omega} \times [0,T])\), then

\[(4.9) \quad R_e \to 0 \quad \text{strongly in } C^0([0,T];H^1_0(\Omega)).\]

**REMARK 4.1.** In (4.4) and (4.5) we have assumed the strong convergence of the data and not only the weak convergence as in (3.6).

For what concerns (4.1), note that this assumption is \(-\Delta u_0^e = g_e \in \mathcal{D}'(\Omega_e)\) and not \(\tilde{\Delta u}_e = g_e \in \mathcal{D}'(\Omega)\). Note also that (4.1) is quite different of assuming that

\(\tilde{u}_e^0 \to u^0 \quad \text{strongly in } H^1_0(\Omega).\)

Indeed in view of Theorem 2.2, (4.1) implies that

\[
\int_{\Omega} |\nabla \tilde{u}_e^0|^2 \, dx \to \int_{\Omega} |\nabla u^0|^2 \, dx + \int_{\Omega} |u^0|^2 \, d\mu
\]

which prevents in general the strong convergence of \(\tilde{u}_e^0\) to \(u^0\). Nevertheless, this convergence of energies to the homogenized energy is exactly what is necessary
in order to prove the corrector result of Theorem 4.1. This is a natural substitute to the strong convergence of \( \tilde{u}_e^0 \), which is not the convenient hypothesis here.

Before proving Theorem 4.1 we prove the convergence of the energy. Let us recall definitions (3.4) and (3.28)

\[
E_e(t) = \frac{1}{2} \| u_e'(t) \|_{L^2(\Omega_e)}^2 + \frac{1}{2} \| \nabla u_e(t) \|_{L^2(\Omega_e)}^2,
\]

\[
E(t) = \frac{1}{2} \| u'(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla u(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u(t) \|_{L^2(\Omega_{d\mu})}^2.
\]

**Proposition 4.2.** Assume that the hypotheses of Theorem 4.1 hold true. Then

\[
E_e(\cdot) \rightarrow E(\cdot) \quad \text{strongly in } C^0([0, T]).
\]

**Remark 4.2.** Let us observe that combining (3.25) and (3.26) with (4.10) we have, for any fixed \( t \in [0, T] \),

\[
\begin{align*}
\| u_e'(t) \|_{L^2(\Omega_e)}^2 & \rightarrow \| u'(t) \|_{L^2(\Omega)}^2, \\
\| \nabla u_e(t) \|_{L^2(\Omega_e)}^2 & \rightarrow \| \nabla u(t) \|_{L^2(\Omega)}^2 + \| u(t) \|_{L^2(\Omega_{d\mu})}^2.
\end{align*}
\]

On the other hand, from (3.24) and (4.11a), we obtain that

\[
\tilde{u}_e'(t) \rightarrow u'(t) \quad \text{strongly in } L^2(\Omega)
\]

for any fixed \( t \in [0, T] \); this statement is not as strong as (4.6), but is a first attempt in this direction.

**Remark 4.3.** Further to (4.10) and (4.11) one actually has

\[
\begin{align*}
\| \tilde{u}_e'(\cdot) \|_{L^2(\Omega_e)}^2 & \rightarrow \| u'(\cdot) \|_{L^2(\Omega)}^2, \\
\| \nabla \tilde{u}_e(\cdot) \|_{L^2(\Omega_e)}^2 & \rightarrow \| \nabla u(\cdot) \|_{L^2(\Omega)}^2 + \| u(\cdot) \|_{L^2(\Omega_{d\mu})}^2.
\end{align*}
\]

indeed (4.13) follows from (4.10), (4.6) and from the definitions of \( E_e \) and \( E \).

**Proof of Proposition 4.2.** We have the identities

\[
E_e(t) = E_e(0) + \int_0^t \int_{\Omega_e} f_e(x, s) u_e'(x, s) dx ds
\]
In view of Theorem 3.1 and hypothesis (4.4) we have for any \( t \in [0,T] \)

\[
(4.15) \quad E(t) = E(0) + \int_{0}^{t} \int_{\Omega} f(x,s)u'(x,s)dx \, ds
\]

with

\[
E_{\varepsilon}(0) = \frac{1}{2} \int_{\Omega_{\varepsilon}} |u_{\varepsilon}^{1}|^2 dx + \frac{1}{2} \int_{\Omega_{\varepsilon}} |\nabla u_{\varepsilon}^{0}|^2 dx
\]

\[
E(0) = \frac{1}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u^{0}|^2 dx + \frac{1}{2} \int_{\Omega} |u^{0}|^2 d\mu.
\]

On the other hand, assumptions (4.1) and (4.5) imply that (see (2.17) in Theorem 2.2)

\[
(4.16) \quad \int_{0}^{t} \int_{\Omega_{\varepsilon}} f_{\varepsilon}(x,s)u_{\varepsilon}' dx \, ds \to \int_{0}^{t} \int_{\Omega} f(x,s)u'(x,s)dx \, ds.
\]

The statements (4.18), (4.19) and Ascoli-Arzela's Theorem imply (4.10).

Therefore

\[
(4.17) \quad E_{\varepsilon}(0) \to E(0).
\]

Moreover, given any \( t \in [0,T] \) and \( h > 0 \) small enough, we have

\[
|E_{\varepsilon}(t + h) - E_{\varepsilon}(t)| \leq \int_{t}^{t+h} \int_{\Omega} |\tilde{f}_{\varepsilon}(x,s)| |\tilde{u}_{\varepsilon}'(x,s)| dx \, ds
\]

\[
\leq ||\tilde{u}_{\varepsilon}'||_{L^{\infty}(0,T;L^{2}(\Omega))} \int_{t}^{t+h} ||\tilde{f}_{\varepsilon}(s)||_{L^{1}(\Omega)} ds.
\]

Since \( \tilde{u}_{\varepsilon}' \) is bounded in \( L^{\infty}(0,T;L^{2}(\Omega)) \) and since \( \tilde{f}_{\varepsilon} \) strongly converges in \( L^{1}(0,T;L^{2}(\Omega)) \), this inequality implies that

\[
(4.19) \quad |E_{\varepsilon}(t + h) - E_{\varepsilon}(t)| \to 0 \quad \text{as } h \to 0 \quad \text{uniformly in } \varepsilon.
\]

The statements (4.18), (4.19) and Ascoli-Arzela's Theorem imply (4.10).

Defining

\[
(4.20) \quad e_{\varepsilon}(v)(t) = \frac{1}{2} ||v'(t)||_{L^2(\Omega_{\varepsilon})}^2 + \frac{1}{2} ||\nabla v(t)||_{L^2(\Omega_{\varepsilon})}^2
\]
for \( v \in C^0([0, T]; H^1_0(\Omega_e)) \cap C^1([0, T]; L^2(\Omega_e)) \) and

\[
(4.21) \quad e(v)(t) = \frac{1}{2} \| v'(t) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla v(t) \|^2_{L^2(\Omega)} + \frac{1}{2} \| u(t) \|^2_{L^2(\Omega; dm)}
\]

for \( v \in C^0([0, T]; V) \cap C^1([0, T]; L^2(\Omega)) \), we have the following result:

**PROPOSITION 4.3.** Assume that the hypotheses of Theorem 4.1 hold true. Then

\[
(4.22) \quad e_e(u_e - w_e \varphi)(\cdot) \to e(u - \varphi)(\cdot) \quad \text{strongly in } C^0([0, T])
\]

for every \( \varphi \in \mathcal{D}(Q) \).

**REMARK 4.4.** If \( u \) belongs to \( \mathcal{D}(Q) \), Theorem 4.1 is a direct consequence of (4.22). Indeed, (4.7) and (4.9) immediately follow from (4.22); (4.6) is also a consequence of (4.22) by virtue of decomposition (4.42) below.

When \( u \) is not in \( \mathcal{D}(Q) \), Theorem 4.1 cannot be obtained so simply: in the proof of Theorem 4.1 below we will approximate \( u \) by a sequence of smooth functions \( \varphi \) and deduce (4.6) and (4.7)-(4.8) from (4.22). \( \square \)

**PROOF OF PROPOSITION 4.3.** We have

\[
(4.23) \quad \begin{cases}
    e_e(u_e - w_e \varphi)(t) = e_e(u_e)(t) + e_e(w_e \varphi)(t) - \int_\Omega \bar{u}_e(x, t) \varphi'(x, t) dx \\
    &- \int_\Omega \nabla \bar{u}_e(x, t) \nabla (w_e(x) \varphi(x, t)) dx.
\end{cases}
\]

We will pass successively to the limit in each term of the right hand side of (4.23).

**First term.** Since \( e_e(u_e)(t) = E_e(t) \), we have from Proposition 4.2

\[
(4.24) \quad e_e(u_e)(\cdot) \to e(u)(\cdot) \quad \text{strongly in } C^0([0, T]).
\]

**Second term.** Using (2.1)(iii) we obtain that (differentiating in time proves that the function is bounded in \( W^{1, \infty}(0, T) \)),

\[
(4.25) \quad \| w_e \varphi'(\cdot) \|^2_{L^2(\Omega_e)} = \| w_e \varphi'(\cdot) \|^2_{L^2(\Omega)} \to \| \varphi'(\cdot) \|^2_{L^2(\Omega)} \quad \text{in } C^0([0, T]).
\]

On the other hand,

\[
\| \nabla (w_e \varphi)(t) \|^2_{L^2(\Omega)} = \| \nabla (w_e \varphi(t)) \|^2_{L^2(\Omega)}
\]

\[
= - (\Delta w_e, w_e \varphi(t))_\Omega - 2 \int_\Omega \nabla w_e \nabla \varphi(t) w_e \varphi(t) dx - \int_\Omega |w_e|^2 \varphi(t) \Delta \varphi(t) dx.
\]
In view of (2.1)(iii) and (iv) we can pass to the limit in each term of the right hand side to get (note that each term is bounded in $W^{1,\infty}(0,T)$)

\begin{equation}
(4.26) \quad -\int_\Omega |w_e|^2 \varphi(t) \Delta \varphi(t) dx \to \int_\Omega \varphi(t) \Delta \varphi(t) dx \quad \text{strongly in } C^0([0,T])
\end{equation}

\begin{equation}
(4.27) \quad -2 \int_\Omega \nabla w_e \nabla \varphi(t) w_e \varphi(t) dx \to 0 \quad \text{strongly in } C^0([0,T])
\end{equation}

\begin{equation}
(4.28) \quad -(\Delta w_e, w_e \varphi^2(t))_\Omega = (\mu_e, w_e \varphi^2(t))_\Omega \to (\mu, \varphi^2(t))_\Omega \quad \text{strongly in } C^0([0,T]).
\end{equation}

Combining (4.25)-(4.28) we deduce that

\begin{equation}
(4.29) \quad e_e(w_e \varphi)(\cdot) \to e(\varphi)(\cdot) \quad \text{strongly in } C^0([0,T]).
\end{equation}

**Third term.** In view of (3.22) we have

\[
\int_\Omega \bar{u}_e'(x, \cdot) \psi(x) dx \to \int_\Omega u'(x, \cdot) \psi(x) dx \quad \text{strongly in } C^0([0,T]),
\]

for all $\psi \in L^\infty(\Omega)$, from which we deduce

\begin{equation}
(4.30) \quad \int_\Omega \bar{u}_e'(x, \cdot) w_e(x) \psi(x) dx \to \int_\Omega u'(x, \cdot) \psi(x) dx \quad \text{strongly in } C^0([0,T]),
\end{equation}

since

\[
\sup_{t \in [0,T]} \left| \int_\Omega \bar{u}_e'(x, t)(w_e(x) - 1) \psi(x) dx \right| \leq \left\| \bar{u}_e' \right\|_{L^\infty([0,T]; L^1(\Omega))} \left\| \psi \right\|_{L^\infty(\Omega)} \left\| w_e - 1 \right\|_{L^1(\Omega)} \to 0
\]

in view of (2.1)(iii).

Approximating $\varphi'(x, t)$ in $C^0([0,T]; L^2(\Omega))$ by functions of the form $\sum_{i=1}^k \eta_i(t) \psi_i(x)$, where the $\eta_i$ are continuous functions on $[0,T]$ and the $\psi_i$ belong to $L^\infty(\Omega)$ for all $i \in \{1, \ldots, k\}$, we deduce, from (4.30) and from the $L^\infty(\Omega)$ bound of $w_e$ (see (2.1)(i)), that

\begin{equation}
(4.31) \quad \int_\Omega \bar{u}_e'(x, \cdot) w_e(x) \varphi'(x, \cdot) dx \to \int_\Omega u'(x, \cdot) \varphi'(x, \cdot) dx \quad \text{strongly in } C^0([0,T]).
\end{equation}

**Fourth term.** Let us now consider the last term of (4.23). We have

\[
(4.32) \quad \left\{ \begin{array}{l}
\int_\Omega \nabla \bar{u}_e(x, t) \nabla (w_e(x) \varphi(x, t)) dx = (-\Delta w_e, \bar{u}_e(t) \varphi(t))_\Omega \\
-2 \int_\Omega \bar{u}_e(x, t) \nabla w_e(x) \nabla \varphi(x, t) dx - \int_\Omega \bar{u}_e(x, t) w_e(x) \Delta \varphi(x, t) dx.
\end{array} \right.
\]
Consider the function
\[ t \rightarrow -2 \int_{\Omega} \bar{u}_e(x, t) \nabla w_e(x) \nabla \varphi(x, t) \, dx - \int_{\Omega} \bar{u}_e(x, t) w_e(x) \Delta \varphi(x, t) \, dx. \]

Since \( \bar{u}_e \) is bounded in \( W^{1,\infty}(0, T; L^2(\Omega)) \) (see Theorem 3.1), the above function is bounded in \( W^{1,\infty}(0, T) \), thus relatively compact in \( C^0([0, T]) \). This implies that
\[
\begin{aligned}
-2 \int_{\Omega} \bar{u}_e(x, \cdot) \nabla w_e(x) \nabla \varphi(x, \cdot) \, dx &= - \int_{\Omega} \bar{u}_e(x, \cdot) w_e(x) \Delta \varphi(x, \cdot) \, dx \\
&\rightarrow - \int_{\Omega} u(x, \cdot) \Delta \varphi(x, \cdot) \, dx \\
&= \int_{\Omega} \nabla u(x, \cdot) \nabla \varphi(x, \cdot) \, dx \quad \text{strongly in } C^0([0, T]).
\end{aligned}
\]

Consider now the remaining term \( (\Delta w_e, \bar{u}_e(t) \varphi(t))_\Omega \). Since \( \bar{u}_e \) vanishes on the holes, we have
\[ (\Delta w_e, \bar{u}_e(t) \varphi(t))_\Omega = (\mu_e, \bar{u}_e(t) \varphi(t))_\Omega. \]

On the other hand, since \( \mu \in H^{-1}(\Omega) \), there exists a sequence \( \nu_k \) in \( L^2(\Omega) \) such that
\[ \nu_k \rightarrow \mu \quad \text{strongly in } H^{-1}(\Omega). \]

We have
\[
\begin{aligned}
\left\| (\Delta w_e, \bar{u}_e \varphi)_\Omega - (\mu, u \varphi)_\Omega \right\|_{L^\infty(0, T)} &\leq \left\| (\mu_e - \mu, \bar{u}_e \varphi)_\Omega \right\|_{L^\infty(0, T)} \\
&\quad + \left\| (\mu - \nu_k, \varphi(\bar{u}_e - u))_\Omega \right\|_{L^\infty(0, T)} + \left\| (\nu_k, \varphi(\bar{u}_e - u))_\Omega \right\|_{L^\infty(0, T)}.
\end{aligned}
\]

By (2.1)(iv) and (3.7) one has
\[
\begin{aligned}
\lim_{\varepsilon \to 0} \left\| (\mu_e - \mu, \bar{u}_e \varphi)_\Omega \right\|_{L^\infty(0, T)} \\
&\leq \lim_{\varepsilon \to 0} \left( \left\| \bar{u}_e \varphi \right\|_{L^\infty(0, T; H^1_0(\Omega))} \right) \left\| \mu_e - \mu \right\|_{H^{-1}(\Omega)} = 0.
\end{aligned}
\]

On the other hand, (4.34) yields
\[
\begin{aligned}
\lim_{k \to \infty} \sup_{t \to 0} \left\| (\mu - \nu_k, \varphi(\bar{u}_e - u))_\Omega \right\|_{L^\infty(0, T)} \\
&\leq \lim_{k \to \infty} \sup_{t \to 0} \left\| \varphi(\bar{u}_e - u) \right\|_{L^\infty(0, T; H^1_0(\Omega))} \left\| \mu - \nu_k \right\|_{H^{-1}(\Omega)} = 0.
\end{aligned}
\]

Finally, from (3.7) and Proposition 2.5, we have
\[ \bar{u}_e \rightarrow u \quad \text{strongly in } C^0([0, T]; L^2(\Omega)). \]

Therefore, for \( k \) fixed,
\[
\begin{aligned}
\lim_{\varepsilon \to 0} \left\| (\nu_k, \varphi(\bar{u}_e - u))_\Omega \right\|_{L^\infty(0, T)} \\
&\leq \lim_{\varepsilon \to 0} \left( \left\| \nu_k \right\|_{L^2(\Omega)} \left\| \varphi \right\|_{L^\infty(\Omega)} \right) \left\| \bar{u}_e - u \right\|_{C^0([0, T]; L^2(\Omega))} = 0.
\end{aligned}
\]
Combining (4.35)-(4.38) we deduce that

\[ (\Delta w_e, \tilde{u}_e \varphi)_{\Omega} \to (\mu, u \varphi)_{\Omega} \quad \text{strongly in } C^0([0, T]). \]

From (4.23), (4.24), (4.29), (4.31), (4.32), (4.33) and (4.39) we get (4.22). The proof of Proposition 4.3 is complete. \[ \square \]

**PROOF OF THEOREM 4.1.** From Theorem 3.1 we know that

\[ u \in C^0([0, T]; V) \cap C^1([0, T]; L^2(\Omega)). \]

Let us consider a sequence \( \varphi_k \) in \( D(Q) \) such that

\[ \varphi_k \to u \text{ strongly in } C^0([0, T]; V) \cap C^1([0, T]; L^2(\Omega)) \text{ as } k \to 0. \]

From Proposition 4.3 we have

\[ \lim_{\varepsilon \to 0} \sup_k \left\{ \| (\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))}^2 + \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))}^2 \right\} \leq 2\| e(u - \varphi_k) \|_{L^2(0, T)} \]

and thus

\[ \lim_{\varepsilon \to 0} \sup_k \| (\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))}\]

\[ = \lim_{k \to \infty} \sup_{\varepsilon \to 0} \| (\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))} = 0. \]

We now observe that

\[ \| \tilde{u}_e - u' \|_{L^2(0, T; L^2(\Omega))} \leq \| \tilde{u}_e - w_e \varphi_k \|_{L^2(0, T; L^2(\Omega))} + \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))}. \]

Combining (4.40), (4.41), (4.42) and hypothesis (2.1), we easily deduce that

\[ \tilde{u}_e \to u' \quad \text{strongly in } C^0([0, T]; L^2(\Omega)). \]

Therefore (4.6) is proved.

On the other hand, we have

\[ \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))} \leq \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))} + \| \nabla(\tilde{u}_e \varphi_k - u) \|_{L^2(0, T; L^2(\Omega))} \]

\[ \leq C \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))} + \| \nabla(\tilde{u}_e \varphi_k - u) \|_{L^2(0, T; L^2(\Omega))} \]

\[ + \| \nabla w_e \|_{L^2(\Omega)} \| \varphi_k - u \|_{C^0([0, T]; L^2(\Omega))} \]

\[ \leq C \| \nabla(\tilde{u}_e - w_e \varphi_k) \|_{L^2(0, T; L^2(\Omega))} + \| \nabla w_e \|_{L^2(\Omega)} \| \varphi_k - u \|_{C^0([0, T]; L^2(\Omega))} \]

\[ + \| w_e \|_{L^2(\Omega)} \| \varphi_k - u \|_{C^0([0, T]; H^1(\Omega))}. \]
By (2.1), (4.22), (4.40) and (4.43), we conclude that
\[ \nabla R_e = \nabla (\bar{u}_e - w_e \varphi) \to 0 \quad \text{strongly in } C^0([0, T]; L^1(\Omega)).\]

Thus (4.8) is proved.

Let us finally consider the case where \( u \in C^0(\overline{\Omega} \times [0, T]). \) In such case, the approximating sequence \( \varphi_k \) may be chosen to satisfy, further to (4.40), the hypothesis
\[ (4.44) \quad \varphi_k \to u \quad \text{strongly in } C^0(\overline{\Omega} \times [0, T]). \]

In this case we can estimate \( \nabla (\bar{u}_e - w_e u) \) in \( L^\infty(0, T; L^2(\Omega)) \) and not only in \( L^\infty(0, T; L^1(\Omega)) \) as in (4.43): indeed, we have
\[
\| \nabla (w_e (\varphi_k - u)) \|_{L^\infty(0, T; L^2(\Omega))} \\
\leq \| (\varphi_k - u) \nabla w_e \|_{L^\infty(0, T; L^2(\Omega))} + \| w_e \nabla (\varphi_k - u) \|_{L^\infty(0, T; L^2(\Omega))} \\
\leq \| \varphi_k - u \|_{C^0(\overline{\Omega} \times [0, T])} \| \nabla w_e \|_{L^2(\Omega)} \\
+ \| w_e \|_{L^\infty(\Omega)} \| \varphi_k - u \|_{L^\infty(0, T; H^1_0(\Omega))}.
\]

Similarly to (4.43), this implies
\[ \nabla R_e = \nabla (\bar{u}_e - w_e u) \to 0 \quad \text{strongly in } C^0([0, T]; L^2(\Omega)) \]

which gives the desired result (4.9).

The proof of Theorem 4.1 is now complete. \( \square \)

5. - The case of holes smaller than the critical size

In this Section we consider the particular case where the holes are smaller than the critical size. This corresponds to the assumption that the functions \( w_e \) of hypothesis (2.1) strongly converge in \( H^1(\Omega) \), which implies \( \mu = 0 \). In this case all the results of Sections 3 and 4 hold true, but the corrector result of Theorem 4.1 can be improved by replacing \( w_e \) by 1 in the statement.

Let us assume that the holes \( S_e \) are such that

\[
\begin{cases}
(i) \quad w_e \in H^1(\Omega), \| w_e \|_{L^\infty(\Omega)} \leq M_0 \\
(ii) \quad w_e = 0 \quad \text{on } S_e \\
(iii) \quad w_e \to 1 \quad \text{strongly in } H^1(\Omega).
\end{cases}
\]

**Remark 5.1.** Once again the main assumption is not made directly on the form and size of the holes but in terms of the family of test functions \( w_e \).
The main difference between assumptions (2.1) and (5.1) is that in (5.1)(iii) we assume the strong convergence of \( w_\varepsilon \). In this case (2.1)(iv) is obviously satisfied with \( \gamma_\varepsilon = 0 \), \( \mu_\varepsilon = -\Delta w_\varepsilon \) and \( \mu = 0 \). As pointed out in Remark 2.2, in the model case (Example 2.1), assumption (5.1) signifies that the size of the holes is smaller than the critical one given by (2.4), i.e. that (2.7) holds.

Hypothesis (5.1) may also be understood in terms of the capacity of \( S_\varepsilon \) with respect to \( \Omega \). More precisely, let us denote by \( \text{Cap}(A, B) \) the capacity of the closed set \( A \subset B \) with respect to the open set \( B \), i.e.

\[
\text{Cap}(A, B) = \inf_{v \in \mathcal{D}(B)} \int_{B} |\nabla v(x)|^2 \, dx.
\]

It is easy to see that hypothesis (5.1) follows from the hypothesis

\[
\text{Cap}(S_\varepsilon, \Omega) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Indeed in this case the function \( w_\varepsilon \) can be constructed from the capacitary potential of \( S_\varepsilon \) by setting \( w_\varepsilon = 1 - p_\varepsilon \), where \( p_\varepsilon \in H_0^1(\Omega) \), \( p_\varepsilon = 1 \) on \( S_\varepsilon \), is the unique function which achieves the minimum in \( \text{Cap}(S_\varepsilon, \Omega) \).

Note finally that if we assume that (2.1) holds true with \( \mu = 0 \), the use of \( v_\varepsilon = \varphi w_\varepsilon \) with \( \varphi \in \mathcal{D}(\Omega) \) in (2.1)(iv) implies that

\[
w_\varepsilon \to 1 \quad \text{strongly in} \quad H^1_{\text{loc}}(\Omega).
\]

The assumption \( \mu = 0 \) in (2.1) is thus equivalent to a “local version” of (5.1), where \( H^1(\Omega) \) is replaced by \( H^1_{\text{loc}}(\Omega) \).

Let us mention two simple examples where (5.1) is satisfied.

**EXAMPLE 1.** As already pointed out in Example 2.1 and Remark 2.2, (5.1) is satisfied when \( \Omega \) is periodically perforated by holes \( S_\varepsilon \) of form \( S \), the size of which satisfies (2.7).

**EXAMPLE 2.** Another situation where (5.1) is satisfied is the case where \( S_\varepsilon \) is the union of a finite (and fixed) number \( N \) of vanishing holes, i.e. \( S_\varepsilon = \bigcup_{i=1}^{N} S_\varepsilon^i \) with \( S_\varepsilon^i \) closed sets such that \( S_\varepsilon^i \subset K \) for some \( K \) such that \( \overline{K} \subset \Omega \) and \( \text{diam}(S_\varepsilon^i) \to 0 \) as \( \varepsilon \to 0 \).

Under assumption (5.1) all the results of Sections 3 and 4 obviously hold true, but strong convergence of the data now implies strong convergence of the solutions: indeed the corrector result of Theorem 4.1 holds true if \( w_\varepsilon \) is replaced by \( 1 \).
THEOREM 5.1. Assume that (5.1) holds true and consider a sequence of data that satisfy (3.6). The sequence \( u_\varepsilon \) of solutions of (3.1) then satisfies
\[
\begin{align*}
\tilde{u}_\varepsilon &\rightharpoonup u \quad \text{weakly * in } L^\infty(0,T;H^1_0(\Omega)) \\
\tilde{u}_\varepsilon' &\rightharpoonup u' \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega))
\end{align*}
\]
and
\[
\begin{align*}
\tilde{u}_\varepsilon(t) &\rightharpoonup u(t) \quad \text{weakly in } H^1_0(\Omega) \\
\tilde{u}_\varepsilon'(t) &\rightharpoonup u'(t) \quad \text{weakly in } L^2(\Omega)
\end{align*}
\]
for all \( t \in [0,T] \), where the limit \( u \) is the unique solution of
\[
\begin{align*}
\begin{cases}
u'' - \Delta u &= f &\text{in } Q = \Omega \times (0,T) \\
u &= 0 &\text{on } \Sigma = \partial \Omega \times (0,T) \\
 u(0) &= u^0, \ u'(0) = u^1 &\text{in } \Omega \\
 u &\in C^0([0,T];H^1_0(\Omega)) \cap C^1([0,T];L^2(\Omega)).
\end{cases}
\end{align*}
\]
Moreover, if the data satisfy the stronger assumption
\[
\begin{align*}
\tilde{f}_\varepsilon &\to f \quad \text{strongly in } L^1(0,T;L^2(\Omega)) \\
\tilde{u}_\varepsilon^0 &\to u^0 \quad \text{strongly in } H^1_0(\Omega) \\
\tilde{u}_\varepsilon^1 &\to u^1 \quad \text{strongly in } L^2(\Omega)
\end{align*}
\]
then
\[
\begin{align*}
\begin{cases}
\tilde{u}_\varepsilon &\rightharpoonup u \quad \text{strongly in } C^0([0,T];H^1_0(\Omega)) \\
\tilde{u}_\varepsilon' &\rightharpoonup u' \quad \text{strongly in } C^0([0,T];L^2(\Omega)).
\end{cases}
\end{align*}
\]
REMARK 5.2. When hypothesis (2.1) is replaced by hypothesis (5.1), assumption (5.3) is equivalent to the hypotheses of Theorem 4.1. Indeed, when (4.1) and (5.1) hold true, Theorem 2.2 implies the strong convergence of \( \tilde{u}_\varepsilon^0 \) to \( u^0 \) in \( H^1_0(\Omega) \) because \( \mu = 0 \) in (2.17). Conversely \( g_\varepsilon = -\Delta \tilde{u}_\varepsilon^0 \) and \( g = -\Delta u^0 \) clearly satisfy (4.1).

PROOF OF THEOREM 5.1. The first part of Theorem 5.1 is a mere rewriting of Theorems 3.1 and 3.2. On the other hand, hypothesis (5.3) implies that the hypotheses of Theorem 4.1 on the data \( u^0_\varepsilon, u^1_\varepsilon \) and \( f_\varepsilon \) are satisfied. Therefore (5.4b) is nothing but (4.6).

To prove (5.4a) we proceed as in the proof of Theorem 4.1. Let \( \varphi_k \in \mathcal{D}(Q) \) be a sequence verifying (4.40). We have
\[
\begin{align*}
\begin{cases}
\|\nabla(\tilde{u}_\varepsilon - u)\|_{L^\infty(0,T;L^2(\Omega))} &\leq \|\nabla(\tilde{u}_\varepsilon - w_\varepsilon \varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} \\
+\|\nabla((1 - w_\varepsilon)\varphi_k)\|_{L^\infty(0,T;L^2(\Omega))} &+ \|\nabla(\varphi_k - u)\|_{L^\infty(0,T;L^2(\Omega))}.
\end{cases}
\end{align*}
\]
Arguing as in the proof of Theorem 4.1 and using now the strong convergence of \( w_\varepsilon \) to 1 in the second term of (5.5) (this is the essential novelty here) we easily obtain (5.4a).
6. - Homogenization and corrector for non-smooth data

This Section is devoted to the study of the homogenization and corrector for the wave equation (3.1) with non-smooth data. We consider here the case where

\[(6.1) \quad u^0_\varepsilon \in L^2(\Omega_\varepsilon), \quad u^1_\varepsilon \in H^{-1}(\Omega_\varepsilon), \quad f_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)).\]

This functional framework naturally appears in the study of the exact controllability problems for the wave equation in domains with small holes (see D. Cioranescu and P. Donato [3], D. Cioranescu, P. Donato and E. Zuazua [4], [5]).

When the data only satisfy (6.1), the solution \( u_\varepsilon \) of (3.1) has to be defined by transposition: \( u_\varepsilon = u_\varepsilon(x,t) \) is said to be a solution of (3.1) in the transposition sense if

\[
\begin{cases}
    u_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon)) \\
    \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x,t)g_\varepsilon(x,t)dx \, dt = -\int_{\Omega_\varepsilon} u^0_\varepsilon(x)\theta_\varepsilon(x,0)dx + \langle u^1_\varepsilon, \theta_\varepsilon(0) \rangle_{\Omega_\varepsilon} \\
    + \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon(x,t)\theta_\varepsilon(x,t)dx \, dt
\end{cases}
\]

for all \( g_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)) \), where \( \theta_\varepsilon = \theta_\varepsilon(x,t) \) is the unique solution of

\[
\begin{cases}
    \theta''_\varepsilon - \Delta \theta_\varepsilon = g_\varepsilon & \text{in } Q_\varepsilon \\
    \theta_\varepsilon = 0 & \text{on } \Sigma_\varepsilon \\
    \theta_\varepsilon(T) = \theta'_\varepsilon(T) = 0 & \text{in } \Omega_\varepsilon.
\end{cases}
\]

Note that (6.3) admits a unique solution

\[
\theta_\varepsilon \in C^0([0,T];H_0^1(\Omega_\varepsilon)) \cap C^1([0,T];L^2(\Omega_\varepsilon)).
\]

Therefore each term in (6.2) has a meaning.

Following along the lines of J.-L. Lions [14, Chapitre I] it is easy to see that (6.2) admits a unique solution \( u_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon)) \); moreover this solution satisfies

\[
u_\varepsilon \in C^0([0,T];L^2(\Omega_\varepsilon)) \cap C^1([0,T];H^{-1}(\Omega_\varepsilon)).
\]

REMARK 6.1. Another (equivalent) way of defining the solution of the wave equation (3.1), when the data only satisfy (6.1), is to consider the equation satisfied by the primitive of \( u_\varepsilon \) with respect to time. This idea will be used below in the proof of Theorem 6.2. \( \square \)
LEMMA 6.1. Consider a sequence \( u_\varepsilon \) of solutions of (6.2) associated to data satisfying (6.1) and assume that

\[
\begin{align*}
\| f_\varepsilon \|_{L^1(0,T;L^2(\Omega_\varepsilon))} & \leq C \\
\| u_\varepsilon^0 \|_{L^2(\Omega_\varepsilon)} & \leq C \\
\| u_\varepsilon^1 \|_{H^{-1}(\Omega_\varepsilon)} & \leq C
\end{align*}
\]

Then

\[
\| u_\varepsilon \|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C.
\]

PROOF. In view of (6.2) we have

\[
\begin{align*}
\left| \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon g_\varepsilon \, dx \, dt \right| & \leq \| u_\varepsilon^0 \|_{L^2(\Omega_\varepsilon)} \| \theta_\varepsilon(0) \|_{L^2(\Omega_\varepsilon)} \\
& \quad + \| u_\varepsilon^1 \|_{H^{-1}(\Omega_\varepsilon)} \| \theta_\varepsilon(0) \|_{H^1(\Omega_\varepsilon)} \\
& \quad + \| f_\varepsilon \|_{L^1(0,T;L^2(\Omega_\varepsilon))} \| \theta_\varepsilon \|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \\
& \leq C \{ \| \theta_\varepsilon \|_{C^0([0,T];H^1(\Omega_\varepsilon))} + \| \theta_\varepsilon \|_{C^0([0,T];L^2(\Omega_\varepsilon))} \}.
\end{align*}
\]

By using for \( \theta_\varepsilon \) the a priori estimates obtained in the proof of Theorem 3.1 (see (3.10)), we deduce from (6.7) that for all \( g_\varepsilon \in L^1(0,T;L^2(\Omega_\varepsilon)) \) one has

\[
\left| \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon g_\varepsilon \, dx \, dt \right| \leq C \| g_\varepsilon \|_{L^1(0,T;L^2(\Omega_\varepsilon))}
\]

where the constant \( C \) does not depend on \( \varepsilon \). This proves (6.6). \( \square \)

We have the following homogenization result.

THEOREM 6.2. Assume that (2.1) holds true and consider a sequence of data which satisfy

\[
\begin{align*}
f_\varepsilon & \in L^1(0,T;L^2(\Omega_\varepsilon)) \quad \text{and} \quad \bar{f}_\varepsilon \to f \quad \text{weakly in } L^1(0,T;L^2(\Omega)) \\
u_\varepsilon^0 & \in L^2(\Omega_\varepsilon) \quad \text{and} \quad \bar{u}_\varepsilon^0 \to u^0 \quad \text{weakly in } L^2(\Omega) \\
u_\varepsilon^1 & \in H^{-1}(\Omega_\varepsilon), \quad \| u_\varepsilon^1 \|_{H^{-1}(\Omega_\varepsilon)} \leq C \\
P_\varepsilon u_\varepsilon^1 & \rightharpoonup u^1 \quad \text{weakly in } W^{-1,q}(\Omega) \text{ with } 1 < q < n/(n-1),
\end{align*}
\]

where \( P_\varepsilon \) is the quasi-extension operator defined in Proposition 2.8.
The solution $u_e$ of (6.2) then satisfies

\begin{equation}
\bar{u}_e \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T; L^2(\Omega))
\end{equation}

where $u = u(x,t)$ is the unique solution (in the transposition sense) of

\begin{equation}
\begin{cases}
u'' - \Delta u + \mu u = f \\ u = 0 \\ u(0) = u^0, \quad u'(0) = u^1 \\ u \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];V')
\end{cases}
\end{equation}

**Remark 6.2.** System (6.13) has to be understood in the transposition sense:

\begin{equation}
\begin{cases}
u \in L^\infty(0,T; L^2(\Omega)) \\ \frac{1}{0} T \int_0^T \int_\Omega u(x,t) \varphi(x,t) dx \, dt = - \int_\Omega u^0(x) \varphi(x,0) dx + (u^1, \varphi(0))_{V',V} \\
\quad + \int_0^T \int_\Omega f(x,t) \varphi(x,t) dx \, dt
\end{cases}
\end{equation}

for all $g \in L^1(0,T; L^2(\Omega))$, where $\vartheta = \vartheta(x,t)$ is the unique solution of

\begin{equation}
\begin{cases}
\vartheta'' - \Delta \vartheta + \mu \vartheta = g \\ \vartheta = 0 \\ \vartheta(T) = \vartheta'(T) = 0 \\ \vartheta \in C^0([0,T];V) \cap C^1([0,T]; L^2(\Omega)).
\end{cases}
\end{equation}

Another (equivalent) way of defining the solution $u$ of (6.13) is to consider the equation satisfied by the primitive of $u$ with respect to time (see below the proof of Theorem 6.2).

**Remark 6.3.** In view of (2.53) (Remark 2.10) we know that $u^1$ defined in (6.11) actually belongs to $V'$. Thus each term makes sense in (6.14).

**Proof of Theorem 6.2.** The main idea of the proof of Theorem 6.2 is to integrate the solution $u_e$ with respect to time, in order to work with a wave equation with usual (smooth) data where the results of the previous Sections apply. This argument was introduced by J.-L. Lions in [15] and was used in the context of exact controllability of the wave equation in perforated domains in D. Cioranescu and P. Donato [3].

Define

\begin{equation}
y_e(x,t) = \int_0^t u_e(x,s) ds + z_e(x)
\end{equation}
where \( z_e \) is the solution of the elliptic problem

\[
(6.17) \quad \begin{cases} -\Delta z_e = -u_1^1 & \text{in } \mathcal{D}'(\Omega_e) \\ z_e \in H^1_0(\Omega_e). \end{cases}
\]

Define also

\[
(6.18) \quad h_e(x, t) = \int_0^t f_e(x, s)ds.
\]

The function \( y_e \) is the unique solution of the following wave equation

\[
(6.19) \quad \begin{cases} y_e'' - \Delta y_e = h_e & \text{in } Q_e \\ y_e = 0 & \text{on } \Sigma_e \\ y_e(0) = z_e, \ y'_e(0) = u_0^1 & \text{in } \Omega_e 
\end{cases}
\]

and \( u_e \) is nothing but \( u_e = y'_e \).

From (6.11) we deduce (see Remark 2.10) that \( z_e \) is bounded in \( H^1_0(\Omega_e) \) and that

\[
\tilde{z}_e \rightharpoonup z \quad \text{weakly in } H^1_0(\Omega)
\]

where \( z \) is the solution of

\[
\begin{cases} -\Delta z + \mu z = -u_1^1 \\ z \in V
\end{cases}
\]

with \( u_1^1 \) defined by (6.11).

On the other hand, in view of (6.9), we have

\[
\tilde{h}_e \rightharpoonup h = \int_0^t f(x, s)ds \quad \text{weakly in } L^1(0, T; L^2(\Omega)).
\]

Applying Theorem 3.1, we deduce that

\[
\begin{cases} \tilde{y}_e \rightharpoonup y & \text{weakly * in } L^\infty(0, T; H^1_0(\Omega)) \\ \tilde{y}'_e \rightharpoonup y' & \text{weakly * in } L^\infty(0, T; L^2(\Omega))
\end{cases}
\]

where \( y = y(x, t) \) is the unique solution of

\[
(6.20) \quad \begin{cases} y'' - \Delta y + \mu y = h & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = z, \ y'(0) = u_0^1 & \text{in } \Omega \\ y \in C^0([0, T]; V) \cap C^1([0, T]; L^2(\Omega)).
\end{cases}
\]
Since $h$ belongs to $C^0([0, T]; L^2(\Omega))$ and since $-\Delta + \mu$ is an isomorphism from $V$ into $V'$, $y'' = \Delta y - \mu y + h$ belongs to $C^0([0, T]; V')$. Defining $u$ by $u = y'$, we deduce from (6.20) that

$$u'(0) = y''(0) = \Delta z - \mu z = u^1.$$ 

Thus $u$ is the solution of (6.13). Since $u_\varepsilon = y'_\varepsilon$, we have in particular

$$\tilde{u}_\varepsilon = \tilde{y}'_\varepsilon \rightharpoonup y' = u \quad \text{weakly * in } L^\infty(0, T; L^2(\Omega)),$$

and Theorem 6.2 is proved. □

If assumptions (6.9)-(6.11) of Theorem 6.1 on the data $f_\varepsilon$ and $u_\varepsilon^0$ are replaced by stronger ones, the strong convergence of $u_\varepsilon$ follows; indeed we have the following result, which is in some sense the analogue of Theorem 4.1.

**THEOREM 6.3.** Assume that (2.1) holds true and consider a sequence of data which satisfy

(6.21) $f_\varepsilon \in L^1(0, T; L^2(\Omega_e))$ and $\tilde{f}_\varepsilon \rightarrow f$ strongly in $L^1(0, T; L^2(\Omega))$

(6.22) $u_\varepsilon^0 \in L^2(\Omega_e)$ and $\tilde{u}_\varepsilon^0 \rightarrow u^0$ strongly in $L^2(\Omega)$

(6.23) $u_\varepsilon^1 \in H^{-1}(\Omega)$ and $u_\varepsilon^1 \rightarrow u^1$ strongly in $H^{-1}(\Omega)$.

Then

(6.24) $\tilde{u}_\varepsilon \rightarrow u$ strongly in $C^0([0, T]; L^2(\Omega))$

where $u = u(x, t)$ is the solution of (6.13).

**PROOF.** Proceed as in the proof of Theorem 6.2 and observe that the sequence of solutions $y_\varepsilon$ of (6.19) now satisfies the hypotheses of Theorem 4.1. Thus, in particular

$$\tilde{u}_\varepsilon = \tilde{y}'_\varepsilon \rightharpoonup y' = u \quad \text{strongly in } C^0([0, T]; L^2(\Omega))$$

which is the desired result. □

**REMARK 6.4.** There is no contradiction between hypotheses (6.11) and (6.23). Indeed when $u_\varepsilon^1 \in H^{-1}(\Omega)$ and $u_\varepsilon^1$ tends strongly to $u^1$ in $H^{-1}(\Omega)$, it follows from the definition (2.48) of the operator $P_\varepsilon = R^*_\varepsilon$ that

$$P_\varepsilon u_\varepsilon^1 \rightharpoonup u^1 \quad \text{weakly in } W^{-1,q}(\Omega) \quad \text{with} \quad 1 < q < n/(n-1).$$
Note also that assumption (6.9) of Theorem 6.2 (respectively assumption (6.21) in the statement of Theorem 6.3) can be replaced by the weaker one

\[(6.25) \quad h(x, t) = \int_0^t \tilde{f}(x, s) ds \rightarrow h(x, t) = \int_0^t f(x, s) ds \quad \text{weakly in } L^1(0, T; L^2(\Omega))\]

(respectively strongly in \(L^1(0, T; L^2(\Omega))\)).

We now give a convergence result concerning the case of non-smooth data (see (6.1)) and holes smaller than the critical size (i.e. satisfying (5.1)). If we further assume that \(u^1_e\) belongs to \(H^{-1}(\Omega)\) (and not only to \(H^{-1}(\Omega_e)\)), we can here replace hypothesis (6.11) by a simpler one which formally corresponds to the choice \(w_e = 1\).

**Theorem 6.4.** Assume that (5.1) holds true and consider a sequence of data which satisfy

\[(6.26) \quad f_e \in L^1(0, T; L^2(\Omega_e)) \quad \text{and} \quad \tilde{f}_e \rightharpoonup f \quad \text{weakly in } L^1(0, T; L^2(\Omega))\]

\[(6.27) \quad u^0_e \in L^2(\Omega_e) \quad \text{and} \quad u^0_e \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega)\]

\[(6.28) \quad u^1_e \in H^{-1}(\Omega) \quad \text{and} \quad u^1_e \rightharpoonup u^1 \quad \text{weakly in } H^{-1}(\Omega).\]

The solution \(u_e\) of (6.2) then satisfies

\[(6.29) \quad \tilde{u}_e \rightharpoonup u \quad \text{weakly * in } L^\infty(0, T; L^2(\Omega))\]

where \(u = u(x, t)\) is the unique solution (in the transposition sense) of

\[
\begin{cases}
  u'' - \Delta u = f & \text{in } Q \\
  u = 0 & \text{on } \Sigma \\
  u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega \\
  u \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)).
\end{cases}
\]

**Proof.** To prove Theorem 6.4, it is sufficient to repeat the proof of Theorem 6.2, just observing that when (5.1) and (6.29) hold true, the solution \(z^e\) of (6.17) now satisfies

\[\tilde{z}_e \rightharpoonup z \quad \text{weakly in } H^1_0(\Omega)\]

where \(z\) is the solution of

\[
\begin{cases}
  -\Delta z = -u^1 & \text{in } D'(\Omega) \\
  z \in H^1_0(\Omega).
\end{cases}
\]

\[\square\]
REMARK 6.5. In the setting of Theorem 6.4, Theorem 6.3 applies without any modification: if (5.1), (6.21), (6.22) and (6.23) hold true, then \( \tilde{u}_\varepsilon \) tends strongly in \( C^0([0,T];L^2(\Omega)) \) to the solution \( u \) of (6.30) (see (6.24)).  

Appendix: \( \mathcal{D}(\Omega) \) is dense in \( V \)

For the sake of completeness, we present in this Appendix the proof of the following result.

THEOREM A.1. Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \) and let \( \mu \in H^{-1}(\Omega) \) be a positive and finite Radon measure on \( \Omega \). Then \( \mathcal{D}(\Omega) \) is dense in the space \( V = H^1_0(\Omega) \cap L^2(\Omega;d\mu) \) endowed with its natural norm.

PROOF. We proceed in three steps.

First step. We first prove that \( V \cap L^\infty(\Omega) \cap L^\infty(\Omega;d\mu) \) is dense in \( V \).

For \( v \in V \) and for any \( k \geq 0 \) define

\[
T_k v = \begin{cases} 
  k & \text{if } v \geq k \\
  v & \text{if } |v| \leq k \\
  -k & \text{if } v \leq -k. 
\end{cases}
\]

It is well known that \( T_k v \in H^1_0(\Omega) \cap L^\infty(\Omega) \) for every \( k \geq 0 \) and that

\[
(A.1) \quad T_k v \rightharpoonup v \quad \text{strongly in } H^1_0(\Omega) \quad \text{as } k \to \infty.
\]

On the other hand (see e.g. H. Lewy and G. Stampacchia [13, Appendix]) we know that \( |T_k(x)| \leq k \) for all \( x \in \Omega \), except on a set of zero capacity. Since \( \mu \) is a Radon measure and belongs to \( H^{-1}(\Omega) \), M. Grun-Rehomme [10, Lemme 3] asserts that every set of zero capacity is \( \mu \)-measurable and is of zero \( \mu \)-measure. We have therefore \( |T_k v(x)| \leq k \) for all \( x \in \Omega \), except on a set of \( \mu \)-measure zero and thus

\[
T_k v \in L^\infty(\Omega;d\mu) \subset L^2(\Omega;d\mu).
\]

Hence \( T_k v \) belongs to \( V \).

From (A.1) we deduce that for a suitable subsequence (still denoted by \( k \)), \( T_k v(x) \) tends to \( v(x) \) for all \( x \in \Omega \), except on a set of zero capacity (see e.g. J. Frehse [9, Theorem 2.3]). Applying M. Grun-Rehomme [10, Lemme 3] once again, we deduce that, for the same subsequence, \( T_k v(x) \) tends to \( v(x) \) for all \( x \in \Omega \) except on a set of \( \mu \)-measure zero. Finally the same argument as above
ensures that $|T_k v(x)| \leq |v(x)|$ except on a set of $\mu$-measure zero. Therefore, applying Lebesgue’s Theorem, we conclude that

$$T_k v \to v \quad \text{strongly in } L^2(\Omega; d\mu) \text{ as } k \to \infty.$$ 

We have thus shown that $V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$ is dense in $V$.

Second step. We now prove that $V \cap C^0_\infty(\Omega)$ is dense in $V$, where $C^0_\infty(\Omega)$ is the space of continuous functions with compact support in $\Omega$.

Consider $v \in V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$ and let $v_k$ be a sequence such that

$$\begin{cases} v_k \in D(\Omega) & \text{for each } k \text{ fixed} \\ v_k \to v & \text{strongly in } H^1_0(\Omega) \text{ as } k \to \infty. \end{cases}$$

Define

$$M = \|v\|_{L^\infty(\Omega)} = \|v\|_{L^\infty(\Omega; d\mu)}$$

and consider the sequence

$$w_k = T_{M+1} v_k.$$ 

The function $w_k$ belongs to $C^0_\infty(\Omega) \cap V$.

Since the map $T_{M+1}$ is continuous from $H^1_0(\Omega)$ into itself, we have, in view of the definition of $M$

$$w_k = T_{M+1} v_k \to T_{M+1} v = v \quad \text{strongly in } H^1_0(\Omega) \text{ as } k \to \infty. \quad \text{(A.2)}$$

Extracting a suitable subsequence we deduce (use again J. Frehse [9] and M. Grun-Rehomme [10]) that $w_k(x)$ converges to $v(x)$ for all $x \in \Omega$ except on a set of $\mu$-measure zero. Since $|w_k(x)| \leq M + 1$ except on a set of $\mu$-measure zero, Lebesgue’s Theorem implies that

$$w_k \to v \quad \text{strongly in } L^2(\Omega; d\mu) \text{ as } k \to \infty. \quad \text{(A.3)}$$

We have thus proved that, when $v \in V \cap L^\infty(\Omega) \cap L^\infty(\Omega; d\mu)$,

$$\begin{cases} w_k \in V \cap C^0_\infty(\Omega) \\
 w_k \to v \text{ in } V \text{ as } k \to \infty \end{cases}$$

which implies, in view of the first step, that $V \cap C^0_\infty(\Omega)$ is dense in $V$.

Third step. In order to prove Theorem A.1, it is thus sufficient to approximate in $V$ any function of $V \cap C^0_\infty(\Omega)$ by functions of $D(\Omega)$.

Consider $v \in V \cap C^0_\infty(\Omega)$ and define

$$v_e = \rho_e \ast v$$
with $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho \left( \frac{x}{\varepsilon} \right)$, where $\rho \in \mathcal{D}(\mathbb{R}^n)$ is a non-negative function with support contained in the unit ball of $\mathbb{R}^n$ which satisfies $\int_{\mathbb{R}^n} \rho(x) dx = 1$.

For $\varepsilon > 0$ small enough, $v_\varepsilon$ belongs to $\mathcal{D}(\Omega)$. It is well known that

$$v_\varepsilon \to v \quad \text{strongly in } H^1_0(\Omega) \cap C(\overline{\Omega}) \quad \text{as } \varepsilon \to 0.$$  

Since $\mu$ is a finite Radon measure, the embedding $C(\overline{\Omega}) \subset L^2(\Omega; d\mu)$ is continuous and therefore

$$v_\varepsilon \to v \quad \text{strongly in } L^2(\Omega; d\mu) \quad \text{as } \varepsilon \to 0.$$  

The proof of Theorem A.1 is now complete. \hfill \Box

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