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On Regularity of Solutions of Nonlinear Parabolic Systems

JINDŘICH NEČAS - VLADIMÍR ŠVERÁK

1. - Introduction

It is well-known that the full regularity of the elliptic systems

$$D_\alpha A_i^\alpha(Du) = 0$$

in two dimensions can (under standard assumptions) be proved by using $W^{1,2+\delta}$-estimates for linear elliptic systems with L^∞ coefficients. (See, for example, M. Giaquinta [2]). The purpose of this paper is to show that a similar method can be used when dealing with nonlinear parabolic systems

$$\frac{\partial u_i}{\partial t} = D_\alpha A_i^\alpha(Du).$$

The idea is to show that $\frac{\partial u}{\partial t}$ is bounded in $L^\infty(-T, 0; L^{2+\delta}(\Omega))$ and then apply the theory of elliptic systems. The required estimate is obtained by using estimates for solutions of linear parabolic systems with L^∞-coefficients. (See Lemma 1). In the two-dimensional case we get full regularity.

2. - Preliminaries

Let $n \geq 2$, $N \geq 1$. We shall be dealing with open sets $Q = \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$, where Ω is a bounded domain in $\mathbb{R}^n$ and $T > 0$. A typical point of $\mathbb{R}^{n+1}$ is denoted by $z = (x, t), x \in \mathbb{R}^n, t \in \mathbb{R}$.

For $\delta > 0$ we let

$$\Omega_\delta = \{x \in \Omega,\ \text{dist}\ (x, \partial\Omega) > \delta\}$$

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and

$$Q_\delta = \Omega_\delta \times (-T + \delta, 0).$$

For $x \in \mathbb{R}^n$ and $\rho > 0$ we define

$$B_{x,\rho} = \{y \in \mathbb{R}^n, |x - y| < \rho\}.$$

If $a, b \in \mathbb{R}$, we denote by $a \wedge b$ the minimum of the two numbers.

The Sobolev spaces W_p^k, $\overset{\circ}{W}{}_p^k$ are defined in the standard way.

The space $L^2(-T, 0; W_2^1(\Omega))$ is denoted by $W_2^{1,0}(Q)$. The norm $[\cdot]_{2,Q}$ on $W_2^{1,0}(Q)$ is defined by

$$[u]_{2,Q} = \{\int_Q (|u|^2 + \sum_{i=1}^n |D_i u|^2)\}^{\frac{1}{2}}.$$

The spaces $L^\infty(-T, 0; L^p(\Omega))$, $p \geq 1$ will be denoted by $L^{p,\infty}(Q)$ and the corresponding norm is denoted by $\|\cdot\|_{p,\infty,Q}$.

The usual L^p-norm is denoted by $\|\cdot\|_{p,Q}$.

Let us consider the nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} - D_\alpha A_i^\alpha(Du) = 0. \qquad (i = 1, 2, \ldots, N) \tag{1}$$

where $u = (u_1, \ldots, u_N), Du = (D_\alpha u_i)_{1 \leq i \leq N, 1 \leq \alpha \leq n} = (\frac{\partial u_i}{\partial x_\alpha})_{1 \leq i \leq N, 1 \leq \alpha \leq n}$ is the gradient matrix of u and the summation over repeated indexes is understood.

We shall suppose that the functions A_i^α have continuous derivatives satisfying

$$\left\{\sum_{i,j} \sum_{\alpha,\beta} |\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi)|^2\right\}^{\frac{1}{2}} \leq M \tag{2}$$

and

$$\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi)\pi_\alpha^i \pi_\beta^j \geq \nu|\pi|^2, \qquad \nu > 0,$$

for every $\xi, \pi \in \mathbb{R}^{nN}$.

(Of course, for higher regularity results we have to assume higher smoothness of A_i^α).

By a weak solution of (1) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$\int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - A_i^\alpha(Du) D_\alpha \varphi_i\right) dz = 0$$

for every $\varphi \in \overset{\circ}{W}{}_2^1(Q)$.

We shall also be dealing with linear strongly parabolic systems

$$\frac{\partial u_i}{\partial t} - D_\alpha a_{ij}^{\alpha\beta} D_\beta u_j = 0 \qquad (i = 1, \dots, N) \tag{4}$$

where $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(z)$ are L^∞-functions in Q satisfying for almost every $z \in \Omega$ the conditions

$$\left\{ \sum_{i,j} \sum_{\alpha,\beta} |a_{ij}^{\alpha\beta}|^2 \right\}^{\frac{1}{2}} \leq M \tag{2'}$$

and

$$a_{ij}^{\alpha\beta} \xi_\beta^j \xi_\alpha^i \geq \nu |\xi|^2 \tag{3'}$$

for every $\xi \in \mathbb{R}^{nN}$. By a weak solution of (4) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$\int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - a_{ij}^{\alpha\beta} D_\beta u_j D_\alpha \varphi_i \right) dz = 0 \tag{$*$}$$

for every $\varphi \in \overset{\circ}{W}{}_2^1(Q)$.

We shall use the following well-known results.

(i) If u is a weak solution of (1) or (4), then u is continuous in time with respect to the L^2-norm. More precisely, if $\Omega' \subset\subset \Omega$, then the map $t \to u(\cdot, t)$ from $(-T, 0)$ into $L^2(\Omega')$ is continuous. (See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3, Lemma 4.3).

(ii) We have the imbedding

$$L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)$$

where

$$q_0 = \begin{cases} \dfrac{2(n+2)}{n}, & \text{if } n > 2 \\ \text{is any number } \in [1, 4) & \text{if } n = 2. \end{cases}$$

(See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 2).

We denote by c_i various constants. The value of these constants can depend on ν, M, Ω, T, n and N. The dependence on additional parameters will be indicated.

3. - L^p-estimates

The first statement of the following Lemma is well known (see, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3).

The second statement will be used for the $L^\infty(-T,0;L^{2+\delta}(\Omega))$-estimate mentioned in the introduction.

LEMMA 1. *Let u be a weak solution of the linear system (4). Then, for any $\delta > 0$,*

(i)
$$u \in L^{2,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$$

and
$$\|u\|_{2,\infty,Q_\delta} + [u]_{2,Q_\delta} \le c_1(\delta)\|u\|_{2,Q}.$$

(ii) *For every $p \in [2,(2+\frac{\nu}{NM}) \wedge q_0)$ the function u belongs to $L^{p,\infty}(Q_\delta)$ and*

$$\|u\|_{p,\infty,Q_\delta} \le c_2(\delta,p)\|u\|_{2,Q}.$$

PROOF. Let $\gamma \ge 1$ and let $k > 0$ be such that $meas\{z \in Q, |u(z)| = k\} = 0$. Define $g_k : [0,\infty) \to \mathbb{R}$ by

$$g_k(t) = \begin{cases} t^\gamma, & \text{if } 0 \le t \le k, \\ k^\gamma + \gamma k^{\gamma-1}(t-k), & \text{if } t \ge k. \end{cases}$$

Clearly $g_k'(t) = \gamma(t \wedge k)^{\gamma-1}$ and

$$g_k''(t) = \begin{cases} \gamma(\gamma-1)t^{\gamma-2}, & \text{if } 0 < t < k, \\ 0, & \text{if } t > k. \end{cases}$$

Define also the function $\eta^k : \mathbb{R}^N \to \mathbb{R}$ by

$$\eta^k(u) = g_k(|u|^2)$$

We have

$$\eta^k_{u_i}(u) = \frac{\partial \eta^k}{\partial u_i}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}u_i$$

$$\eta^k_{u_i u_j}(u) = \frac{\partial^2 \eta^k}{\partial u_i \partial u_j}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}(\delta_{ij} + 2(\gamma-1)d_{ij}(u)).$$

In the second formula we assume $|u|^2 \ne k$ and

$$d_{ij}(u) = \begin{cases} 0, & \text{if } |u|^2 > k, \\ \dfrac{u_i u_j}{|u|^2}, & \text{if } |u|^2 < k. \end{cases}$$

Let ω_ϵ be a family of symmetric mollifying functions satisfying

$\omega_\epsilon \in \mathcal{D}(\mathbb{R}),\ \omega_\epsilon(t) \geq 0,$

$\omega_\epsilon(t) = \omega_\epsilon(-t),$

support $\omega_\epsilon \subset (\epsilon, -\epsilon),$

$\int_{\mathbb{R}} \omega_\epsilon = 1.$

For $f \in L^1(Q)$ let us denote by $(f)_\epsilon$ the function defined a.e. in Q by

$$(f)_\epsilon(x,t) = \int_{\mathbb{R}} f(x, t-s)ds.$$

(We extend f by zero outside Q). Let $\psi \in \overset{\circ}{W}{}_2^1(\Omega \times (-T+\epsilon, -\epsilon))$. Following E. Giusti, M. Giaquinta [3] we set $\varphi = (\psi)_\epsilon$ in $(*)$ and we see that

$$\int_Q \frac{\partial (u_i)_\epsilon}{\partial t} \psi_i dz = -\int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon D_\alpha \psi_i dz. \tag{5}$$

Let $\theta \in \mathcal{D}(\Omega)$ with $0 \leq \theta \leq 1$ and $\theta = 1$ on Ω_δ and let $\rho \in \mathcal{D}(-T,0)$, $\rho \geq 0$. For ϵ sufficiently small we can use (5) with

$$\psi_i = \eta_{u_i}^k(u_\epsilon)\theta^2\rho^2$$

to get

$$\begin{aligned}\int_Q \left(\frac{\partial}{\partial t}\eta^k(u_\epsilon)\right)\theta^2\rho^2 dz = &-\int_Q \left(a_{ij}^{\alpha\beta} D_\beta u_j\right)_\epsilon D_\alpha(\eta_{u_i}^k(u_\epsilon))\theta^2\rho^2 dz \\ &-\int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon \eta_{u_i}^k(u_\epsilon) 2\theta D_\alpha\theta\rho^2 dz.\end{aligned}$$

Integrating by parts on the left-hand side, letting $\epsilon \to 0$ and then using the chain rule for the derivative $D_\alpha(\eta_{u_i}^k(u))$ (which is legal) we see that

$$\begin{aligned}\int_Q \eta^k(u)\theta^2(\rho^2)' dz = &-\int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i u_l}^k(u) D_\alpha u_l \theta^2\rho^2 dz \\ &-\int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i}^k(u) 2\theta D_\alpha\theta\rho^2 dz.\end{aligned} \tag{6}$$

Since $|d_{ij}| \leq 1$ we see that if $0 \leq 2(\gamma-1) < \frac{\nu}{NM}$ then the matrix $\tilde{a}_{lj}^{\alpha\beta} = a_{ij}^{\alpha\beta}(\delta_{il} + 2(\gamma-1)d_{il}(u))$ satisfies the condition (2') with ν replaced by $\nu_1 = \nu - 2(\gamma-1)NM$.

We can estimate the right-hand side of (6) by

$$
\begin{aligned}
\nu_1 \int_Q & 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \\
& + 2M \left\{ \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \right\}^{\frac{1}{2}} \\
& \times \left\{ \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz \right\}^{\frac{1}{2}} \\
\leq & -\frac{\nu_1}{2} \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 dz \\
& + \frac{4M^2}{\nu_1} \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz.
\end{aligned}
$$

Let $t_1 \in (-T + \delta, 0)$. As we have remarked in Section 2, under our assumptions the function $t \to u(\cdot, t)$ is continuous mapping of $(-T, 0)$ into $L^2(\Omega_\delta)$. Hence we can use (6) with ρ defined by

$$
\rho^2(t) = \begin{cases} 0 & \text{if } t \in (-T, -T + \frac{\delta}{2}) \\ \frac{2}{\delta}(t + T - \frac{\delta}{2}) & \text{if } t \in (-T + \frac{\delta}{2}, -T + \delta) \\ 1 & \text{if } t \in (-T + \delta, t_1) \\ 0 & \text{if } t \in (t_1, 0). \end{cases}
$$

We get

$$
\begin{aligned}
\int_Q & \eta^k(u(x, t_1))\theta(x) dx + \frac{\nu_1}{2} \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \\
& \leq c_3 \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 dz + \int_{\Omega \times (-T, t_1)} (\rho^2)' \eta^k(u) \theta^2 dz.
\end{aligned}
$$

Letting $\gamma = 1$ we get (i).

We can use (i) and the imbedding

$$
L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)
$$

to infer that $\|u\|_{q_0, Q_\delta} \leq c_4(\delta) \|u\|_{2,Q}$. Using this and letting $k \to \infty$ in (7) we get (ii) with $p = 2\gamma$.

LEMMA 2. *Let u be a weak solution of the nonlinear system* (1). *Then $u \in W_2^1(Q_\delta)$, the derivatives $D_i u$, $i = 1, \dots, n$ and $D_{n+1} u = \frac{\partial u}{\partial t}$ belong to the space $L^{p,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$ and for each $i = 1, \dots, n, n+1$*

$$\|D_i u\|_{p,\infty,Q_\delta} + [D_i u]_{2,Q_\delta} \leq [u]_{2,Q}.$$

PROOF. As above, we denote by Du the vector $(D_1 u, \dots, D_n u) \in \mathbb{R}^{nN}$. let us fix an index r, $1 \leq r \leq n+1$ and let $e_r \in \mathbb{R}^n \times \mathbb{R}$ be the r-th vector of the canonical basis. Let $\delta' > 0$. For $0 < h < \delta'$ let

$$u_h(z) = h^{-1}[u(z) - u(z - he_r)].$$

Define the functions $a_{hij}^{\alpha\beta} \in L^\infty(Q_{\delta'})$ for a.e. $z \in Q_{\delta'}$ by

$$a_{hij}^{\alpha\beta}(z) = \int_0^1 A^\alpha_{i,\xi^j_\beta}(Du(z) - hDu_h(z) + \tau h Du_h(z))d\tau.$$

It is not difficult to see that u_h is the weak solution of the linear system

$$\frac{\partial u_{hi}}{\partial t} - D_\alpha a_{hij}^{\alpha\beta} D_\beta u_{hj} = 0$$

in $Q_{\delta'}$. The functions $a_{hij}^{\alpha\beta}$ clearly satisfy the conditions (2') and (3'). Hence, by Lemma 1

$$\begin{aligned} &\|u_h\|_{p,\infty,Q_{2\delta'}} \leq c_6(\delta', p)\|u_h\|_{2,Q_{\delta'}} \\ &\|Du_h\|_{2,Q_{2\delta'}} \leq c_7(\delta')\|u_h\|_{2,Q_{\delta'}}. \end{aligned} \tag{8}$$

Suppose first $1 \leq r \leq n$. In this case the difference is taken in the direction of the space variables. Since $u \in W_2^{1,0}(Q)$, we have

$$\|u_h\|_{2,Q_{\delta'}} \leq \|D_r u\|_{2,Q}. \tag{9}$$

Using Nirenberg's Lemma we see from (8) that $Du \in W_2^{1,0}(Q_{\delta'})$ and

$$\begin{aligned} &\|D_r u\|_{p,\infty,Q_{2\delta'}} \leq c_6(\delta', p)\|D_r u\|_{2,Q_{\delta'}} \\ &\|DD_r u\|_{2,Q_{2\delta'}} \leq c_7(\delta')\|D_r u\|_{2,Q_{\delta'}} \end{aligned} \tag{10}$$

for every $1 \leq r \leq n$. Now let $r = n+1$. Following S. Campanato [1] we notice that we can use equation (1) and the L^2-estimate of $D_\alpha D_\beta u$ obtained above to infer that $\frac{\partial u}{\partial t} \in L^2(Q_{2\delta'})$ and

$$\|\frac{\partial u}{\partial t}\|_{2,Q_{2\delta'}} \leq c_8(\delta')\|Du\|_{2,Q}. \tag{11}$$

Now we can use (8) with Q replaced by $Q_{2\delta'}$ and using (11) we get by the same argument as above

$$\|\frac{\partial u}{\partial t}\|_{p,\infty,Q_{4\delta'}} \leq c_9(\delta',p)\|Du\|_{2,Q}$$

$$\|D\frac{\partial u}{\partial t}\|_{2,Q_{4\delta'}} \leq c_{9'}(\delta')\|Du\|_{2,Q}.$$

The proof is finished.

THEOREM 1. *Let u be a weak solution of the system (1) and let p be the exponent from Lemma 1. The for each $\delta > 0$*

$$\frac{\partial u}{\partial t} \in L^{p,\infty}(Q_\delta)$$

and

$$u \in L^\infty(-T+\delta, 0; W^2_q(\Omega_\delta))$$

for some $q = q(\nu, M, p, \delta)$ with $2 < q < p$. Moreover

$$\|u\|_{L^\infty(-T+\delta,0;W^2_q(\Omega_\delta))} + \|\frac{\partial u}{\partial t}\|_{p,\infty,Q_\delta} \leq c_{10}(\delta,p,q)\|u\|_{2,Q}.$$

PROOF. Let $\delta' > 0$. We notice that u can be considered as a weak solution of the linear system (4) with

$$a^{\alpha\beta}_{ij}(z) = \int_0^1 A^\alpha_{i,\xi^j_\beta}(\tau Du(z))d\tau.$$

(See, for example S. Campanato [1]). Using this and Lemma 1 we get estimates for the norms $\|u\|_{p,\infty,Q_{\delta'}}$ and $[u]_{2,Q_{\delta'}}$. Now we can use Lemma 2 to get estimates of the norms $\|Du\|_{p,\infty,Q_{2\delta'}}$, $\|\frac{\partial u}{\partial t}\|_{p,\infty,Q_{2\delta'}}$. Lemma 2 also implies $D_\alpha D_\beta u \in L^2(Q_{2\delta'})$, $(0 \leq \alpha,\ \beta \leq n)$. We see that equation (1) is satisfied pointwise almost everywhere in $Q_{2\delta'}$ and that for almost every $t \in (-T+2\delta', 0)$ the function $u(\cdot,t)$ belongs to $W^2_2(\Omega_{2\delta'})$ and is the weak solution of the elliptic system

$$D_\alpha A^\alpha_i(Dv) = \frac{\partial u_i}{\partial t}$$

in $\Omega_{2\delta'}$. We can now use well-known L^p-estimates for elliptic systems (see Lemma 3 below). The proof is finished.

LEMMA 3. *Let $p > 2$ and let $g \in L^p(\Omega)$. Let $u \in W^1_2(\Omega)$ be a weak solution of the elliptic system*

$$D_\alpha A^\alpha_i(Du) = g_i \qquad i = 1, \dots, n \tag{12}$$

Then there exists $q = q(\nu, M, p) > 2$ *such that* $u \in W^2_{q,loc}(\Omega)$. *Moreover, for every* $\delta > 0$

$$\|u\|_{W^2_q(\Omega_\delta)} \leq c_{11}(\nu, M, p, q, \delta)(\|u\|_{W^1_2(\Omega)} + \|g\|_{p,\Omega}).$$

PROOF. Using the standard difference quotient technique, it is not difficult to verify that the following computations are legal.

Let $1 \leq s \leq n$. We let $v = D_s u$ and take the s-th derivative of (12). We get

$$D_\alpha a^{\alpha\beta}_{ij} D_\beta v_j = D_s g_i \tag{13}$$

where $a^{\alpha\beta}_{ij}(z) = A^\alpha_{i,\xi^j_\beta}(Du(x))$.

This implies

$$\frac{\nu}{2}\int_\Omega \varsigma^2 |Dv|^2 dx \leq c_{12}(\nu, M)\int_\Omega (|v|^2|D\varsigma|^2 + |g|^2)dx \tag{14}$$

for every $\varsigma \in \mathcal{D}(\Omega)$ (Cacciopoli's inequality). The required estimate can now be obtained by using the technique of reverse Hölder inequalities. (See, for example, M. Giaquinta [2], Chap. 5, Theorem 2.2). The proof is finished.

COROLLARY. *Let the assumptions of Theorem 1 be satisfied.*

(i) *If* $n \leq 4$, *then* u *is Hölder continuous in* Q.

(ii) *If* $n \leq 2$, *then* Du *is Hölder continuous in* Q.

(iii) *If* $n \leq 2$ *and the functions* A^α_i *are smooth, then the solution* u *is smooth.*

REMARK. If $n \geq 3$, then Du may not be continuous. Examples are provided by nonregular solutions of elliptic systems. These can be found in J. Nečas [5].

PROOF OF THE COROLLARY. Let $\delta > 0$.

(i) Since $W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{0,\alpha}(\Omega_\delta)$ with $\alpha = (2 - \frac{n}{q}) \wedge 1$, we have $u \in L^\infty(-T+\delta, 0; C^{0,\alpha}(\Omega_\delta))$.

Since we have also $\frac{\partial u}{\partial t} \in L^2(Q_\delta)$, u is Hölder continuous by Lemma 4 below.

(ii) In this case we have $W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{1,\beta}(\Omega_\delta)$ $\beta = 1 - \frac{n}{q}$.

Hence $Du \in L^\infty(-T+\delta, 0; C^{0,\beta}(\Omega_\delta))$. Using the Hölder continuity of u it is easy to see that in fact $Du(\cdot, t) \in C^{0,\beta}(\Omega_\delta)$ for every $t \in (-T+\delta, 0)$, the $C^{0,\beta}$-norm being bounded independently of t.

Now we can use Lemma 3.1, Chap. 2 from O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4] to infer that Du is Hölder continuous in Q_δ.

(iii) The higher regularity follows in the standard way from the theory of linear equations.

LEMMA 4. *Let $\alpha > 0$, $q > 1$, $\delta > 0$ and suppose*

$$u \in L^\infty(-T, 0; C^{0,\alpha}(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^q(Q).$$

Denote $K_1 = \|u\|_{L^\infty(-T,0;C^{0,\alpha}(\Omega))}$, $K_2 = \|\frac{\partial u}{\partial t}\|_{q,Q}$. Then there exists $K = K(K_1, K_2, \delta)$ such that

$$|\tilde{u}(x, t_1) - \tilde{u}(x, t_2)| \leq K|t_1 - t_2|^\beta$$

for every $x \in \Omega_\delta$ and every t_1, $t_2 \in (-T, 0)$, where $\beta = \dfrac{\alpha/q'}{\alpha + n/q}$, $q' = \dfrac{q}{q-1}$ and $\tilde{u}$ is a suitable representative of u.

PROOF. Suppose first that u is continuous. Let $x \in \Omega_\delta$ and let $0 < \rho < \delta$. Define

$$w_\rho(t) = \frac{1}{|B_{x,\rho}|} \int_{B_{x,\rho}} u(y, t) dy.$$

It is easy to see that w'_ρ is bounded in $L^q(-T, 0)$ by $c_{13}\rho^{-\frac{n}{q}} K_2$.

Let $t_1, t_2 \in (-T, 0)$. We can write

$$\begin{aligned}
&|u(x, t_1) - u(x, t_2)| \\
&\qquad \leq |u(x, t_1) - w_\rho(t_1) + w_\rho(t_1) - w_\rho(t_2) + w_\rho(t_2) - u(x, t_2)| \\
&\qquad\qquad \leq 2K_1\rho^\alpha + c_{12}K_2\rho^{-\frac{n}{q}}|t_1 - t_2|^{\frac{1}{q'}}.
\end{aligned}$$

The proof is easily finished by using this inequality with $\rho = |t_1 - t_2|^{\frac{\beta}{\alpha}}$.

REMARK. It is not difficult to see that if the boundary of Ω is sufficiently regular (say, lipshitzian), then K can be chosen independent of δ.

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