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1. - Introduction

It is well-known that the full regularity of the elliptic systems

$$D_a A_2^u(Du) = 0$$

in two dimensions can (under standard assumptions) be proved by using $W^{1,2+d}$-estimates for linear elliptic systems with $L^\infty$ coefficients. (See, for example, M. Giaquinta [2]). The purpose of this paper is to show that a similar method can be used when dealing with nonlinear parabolic systems

$$\frac{\partial u_i}{\partial t} = D_a A_2^u(Du).$$

The idea is to show that $\frac{\partial u}{\partial t}$ is bounded in $L^\infty(-T, 0; L^{2+d}(\Omega))$ and then apply the theory of elliptic systems. The required estimate is obtained by using estimates for solutions of linear parabolic systems with $L^\infty$-coefficients. (See Lemma 1). In the two-dimensional case we get full regularity.

2. - Preliminaries

Let $n \geq 2$, $N \geq 1$. We shall be dealing with open sets $Q = \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $T > 0$. A typical point of $\mathbb{R}^{n+1}$ is denoted by $z = (x, t), x \in \mathbb{R}^n, t \in \mathbb{R}$.

For $\delta > 0$ we let

$$\Omega_\delta = \{ x \in \Omega, \ \text{dist} (x, \partial \Omega) > \delta \}$$

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and
\[ Q_\delta = \Omega_\delta \times (-T + \delta, 0). \]
For \( x \in \mathbb{R}^n \) and \( \rho > 0 \) we define
\[ B_{x,\rho} = \{ y \in \mathbb{R}^n, |x - y| < \rho \}. \]
If \( a, b \in \mathbb{R} \), we denote by \( a \wedge b \) the minimum of the two numbers.

The Sobolev spaces \( W^k_p, W^k_p^k \) are defined in the standard way.

The space \( L^2(-T, 0; W^1_2(\Omega)) \) is denoted by \( W^{1,0}_2(Q) \). The norm \([ \cdot ]_{2,Q} \) on \( W^{1,0}_2(Q) \) is defined by
\[
[u]_{2,Q} = \left\{ \int_Q \left( |u|^2 + \sum_{i=1}^n |D_i u|^2 \right) \right\}^{\frac{1}{2}}.
\]

The spaces \( L^\infty(-T, 0; L^p(\Omega)) \), \( p \geq 1 \) will be denoted by \( L^{p,\infty}(Q) \) and the corresponding norm is denoted by \( \| \cdot \|_{p,\infty,Q} \).

The usual \( L^p \)-norm is denoted by \( \| \cdot \|_{p,Q} \).

Let us consider the nonlinear parabolic system
\[
\frac{\partial u_i}{\partial t} - D_\alpha A^\alpha_i(Du) = 0 \quad (i = 1, 2, \ldots, N)
\]
where \( u = (u_1, \ldots, u_N), Du = (D_\alpha u_i)_{1 \leq i \leq N, 1 \leq \alpha \leq n} = (\frac{\partial u_i}{\partial x_\alpha})_{1 \leq i \leq N, 1 \leq \alpha \leq n} \) is the gradient matrix of \( u \) and the summation over repeated indexes is understood.

We shall suppose that the functions \( A^\alpha_i \) have continuous derivatives satisfying
\[
\left\{ \sum_{i,j} \sum_{\alpha,\beta} \left| \frac{\partial A^\alpha_i}{\partial \xi_\beta^j}(\xi) \right|^2 \right\}^{\frac{1}{2}} \leq M
\]
and
\[
\frac{\partial A^\alpha_i}{\partial \xi_\beta^j}(\xi) \pi_\alpha^i \pi_\beta^j \geq \nu |\pi|^2, \quad \nu > 0,
\]
for every \( \xi, \pi \in \mathbb{R}^{nN} \).

Of course, for higher regularity results we have to assume higher smoothness of \( A^\alpha_i \).

By a weak solution of (1) we mean a function \( u \in W^{1,0}_2(Q) \) satisfying
\[
\int_Q \left( u_i \frac{\partial \varphi_i}{\partial t} - A^\alpha_i(Du) D_\alpha \varphi_i \right) dz = 0
\]
for every \( \varphi \in W^{1,0}_2(Q) \).
We shall also be dealing with linear strongly parabolic systems

\[
\frac{\partial u_i}{\partial t} - D_\alpha a^\alpha_{ij} D_\beta u_j = 0 \quad (i = 1, \ldots, N)
\]

where \(a^\alpha_{ij} = a^\alpha_{ij}(z)\) are \(L^\infty\)-functions in \(Q\) satisfying for almost every \(z \in \Omega\) the conditions

\[
\left\{ \sum_{i,j} \sum_{\alpha,\beta} |a^\alpha_{ij}|^2 \right\}^{\frac{1}{2}} \leq M
\]

and

\[
a^\alpha_{ij} \xi^i \xi^j \geq \nu |\xi|^2
\]

for every \(\xi \in \mathbb{R}^{nN}\). By a weak solution of (4) we mean a function \(u \in W^{1,0}_2(Q)\) satisfying

\[
\int_Q \left( u_i \frac{\partial \varphi_i}{\partial t} - a^\alpha_{ij} D_\beta u_j D_\alpha \varphi_i \right) dz = 0
\]

for every \(\varphi \in W^{1,0}_2(Q)\).

We shall use the following well-known results.

(i) If \(u\) is a weak solution of (1) or (4), then \(u\) is continuous in time with respect to the \(L^2\)-norm. More precisely, if \(\Omega' \subset \subset \Omega\), then the map \(t \mapsto u(\cdot, t)\) from \((-T, 0)\) into \(L^2(\Omega')\) is continuous. (See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4], Chap. 3, Lemma 4.3).

(ii) We have the imbedding

\[
L^{2,\infty}(Q) \cap W^{1,0}_2(Q) \hookrightarrow L^\Phi(Q)
\]

where

\[
q_0 = \begin{cases} 
\frac{2(n+2)}{n}, & \text{if } n > 2 \\
\text{is any number } \in [1,4) & \text{if } n = 2.
\end{cases}
\]

(See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4], Chap. 2).

We denote by \(c_i\) various constants. The value of these constants can depend on \(\nu, M, \Omega, T, n\) and \(N\). The dependence on additional parameters will be indicated.
3. - $L^p$-estimates

The first statement of the following Lemma is well known (see, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3). The second statement will be used for the $L^\infty(-T,0;L^{2p}(\Omega))$-estimate mentioned in the introduction.

**LEMMA 1.** Let $u$ be a weak solution of the linear system (4). Then, for any $\delta > 0$,

(i) $u \in L^{2,\infty}(Q_\delta) \cap W^{1,0}_2(Q_\delta)$

and

(ii) For every $p \in [2, (2 + N/M) \wedge q_0)$ the function $u$ belongs to $L^{p,\infty}(Q_\delta)$ and

$$||u||_{p,\infty,Q_\delta} \leq c_2(\delta, p)||u||_{2,Q}.$$  

**PROOF.** Let $\gamma \geq 1$ and let $k > 0$ be such that $\text{meas}\{z \in Q, |u(z)| = k\} = 0$. Define $g_k : [0, \infty) \to \mathbb{R}$ by

$$g_k(t) = \begin{cases} t^\gamma, & \text{if } 0 \leq t \leq k, \\ k^\gamma + \gamma k^{\gamma-1}(t-k), & \text{if } t \geq k. \end{cases}$$

Clearly $g'_k(t) = \gamma(t \wedge k)^{\gamma-1}$ and

$$g''_k(t) = \begin{cases} \gamma(\gamma - 1)t^{\gamma-2}, & \text{if } 0 < t < k, \\ 0, & \text{if } t > k. \end{cases}$$

Define also the function $\eta^k : \mathbb{R}^N \to \mathbb{R}$ by

$$\eta^k(u) = g_k(|u|^2)$$

We have

$$\eta^k_{uu}(u) = \frac{\partial \eta^k}{\partial u_i}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}u_i$$

$$\eta^k_{u_iu_j}(u) = \frac{\partial^2 \eta^k}{\partial u_i \partial u_j}(u) = 2\gamma(|u|^2 \wedge k)^{\gamma-1}(\delta_{ij} + 2(\gamma - 1)d_{ij}(u)).$$

In the second formula we assume $|u|^2 \neq k$ and

$$d_{ij}(u) = \begin{cases} 0, & \text{if } |u|^2 > k, \\ \frac{u_iu_j}{|u|^2}, & \text{if } |u|^2 < k. \end{cases}$$
Let \( \omega_\epsilon \) be a family of symmetric mollifying functions satisfying
\[
\omega_\epsilon \in \mathcal{D}(\mathbb{R}), \quad \omega_\epsilon(t) \geq 0,
\]
\[
\omega_\epsilon(t) = \omega_\epsilon(-t),
\]
support \( \omega_\epsilon \subset (\epsilon, -\epsilon) \),
\[
\int_{\mathbb{R}} \omega_\epsilon = 1.
\]

For \( f \in L^1(Q) \) let us denote by \( (f)_\epsilon \) the function defined a.e. in \( Q \) by
\[
(f)_\epsilon(x, t) = \int_{\mathbb{R}} f(x, t - s) ds.
\]
(We extend \( f \) by zero outside \( Q \)). Let \( \psi \in W^1_2(\Omega \times (-T + \epsilon, -\epsilon)) \). Following E. Giusti, M. Giaquinta [3] we set \( \varphi = (\psi)_\epsilon \) in \( (*) \) and we see that
\[
\int_{Q} \frac{\partial (u_\epsilon)}{\partial t} \psi_i dz = -\int_{Q} (a_{ij}^{\alpha \beta} \partial_i \eta(u_\epsilon)) \partial_\alpha \psi_j dz.
\]

Let \( \theta \in \mathcal{D}(\Omega) \) with \( 0 \leq \theta \leq 1 \) and \( \theta = 1 \) on \( \Omega_\delta \) and let \( \rho \in \mathcal{D}(-T, 0) \), \( \rho \geq 0 \). For \( \epsilon \) sufficiently small we can use (5) with
\[
\psi_i = \eta^k(u_\epsilon) \theta^2 \rho^2
\]
to get
\[
\int_{Q} \left( \frac{\partial}{\partial t} \eta^k(u_\epsilon) \right) \theta^2 \rho^2 dz = -\int_{Q} (a_{ij}^{\alpha \beta} \partial_i \eta(u_\epsilon)) \partial_\alpha \rho^2 dz
\]
\[
- \int_{Q} (a_{ij}^{\alpha \beta} \partial_i \eta^k(u_\epsilon)) 2\theta \partial_\alpha \theta \rho^2 dz.
\]
Integrating by parts on the left-hand side, letting \( \epsilon \to 0 \) and then using the chain rule for the derivative \( D_\alpha (\eta^k(u_\epsilon)) \) (which is legal) we see that
\[
\int_{Q} \eta^k(u) \theta^2 (\rho^2)' dz = -\int_{Q} a_{ij}^{\alpha \beta} D_\beta u_j \eta^k(u) D_\alpha u_i \theta^2 \rho^2 dz
\]
\[
- \int_{Q} a_{ij}^{\alpha \beta} D_\beta u_j \eta^k(u) 2\theta D_\alpha \theta \rho^2 dz.
\]
Since \( |d_{ij}| \leq 1 \) we see that if \( 0 \leq 2(\gamma - 1) < \frac{\nu}{NM} \) then the matrix
\[
a_{ij}^{\alpha \beta} = a_{ij}^{\alpha \beta} (d_{ij} + 2(\gamma - 1)d_{ij}(u))
\]
satisfies the condition \( (2') \) with \( \nu \) replaced by
\[
\nu_1 = \nu - 2(\gamma - 1)NM.
\]
We can estimate the right-hand side of (6) by

\[
\nu_1 \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 \, dz \\
+ 2M \left\{ \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} \, dz \right\}^{\frac{1}{\gamma}} \\
\times \left\{ \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 \, dz \right\}^{\frac{1}{\gamma}} \\
\leq - \frac{\nu_1}{2} \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1} |Du|^2 \theta^2 \rho^2 \, dz \\
+ \frac{4M^2}{\nu_1} \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 \, dz.
\]

Let \( t_1 \in (-T + \delta, 0) \). As we have remarked in Section 2, under our assumptions the function \( t \to u(\cdot, t) \) is continuous mapping of \((-T, 0)\) into \(L^2(\Omega_0)\). Hence we can use (6) with \( \rho \) defined by

\[
\rho^2(t) = \begin{cases} 
0 & \text{if } t \in (-T, -T + \frac{\delta}{2}) \\
\frac{2}{\delta}(t + T - \frac{\delta}{2}) & \text{if } t \in (-T + \frac{\delta}{2}, -T + \delta) \\
1 & \text{if } t \in (-T + \delta, t_1) \\
0 & \text{if } t \in (t_1, 0).
\end{cases}
\]

We get

\[
\int_Q \eta^k(u(x, t_1))\theta(x) \, dx + \frac{\nu_1}{2} \int_Q |Du|^2 \theta^2 \rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} \, dz \\
\leq c_3 \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} |D\theta|^2 \rho^2 \, dz + \int_{\Omega \times (-T, t_1)} (\rho^2)^{\gamma} \eta^k(u)\theta^2 \, dz.
\]

Letting \( \gamma = 1 \) we get (i).

We can use (i) and the imbedding

\[
L^{2,\infty}(Q) \cap W^{1,0}_2(Q) \hookrightarrow L^0(Q)
\]

to infer that \( \|u\|_{\infty, Q_1} \leq c_4(\delta)\|u\|_{L_2(Q)}. \) Using this and letting \( k \to \infty \) in (7) we get (ii) with \( p = 2\gamma \).
LEMMA 2. Let $u$ be a weak solution of the nonlinear system (1). Then $u \in W^1_2(Q_\delta)$, the derivatives $D_i u$, $i = 1, \ldots, n$ and $D_{n+1} u = \frac{\partial u}{\partial t}$ belong to the space $L^{p, \infty}_2(Q_\delta) \cap W^{1, 0}_2(Q_\delta)$ and for each $i = 1, \ldots, n, n + 1$

$$
\|D_i u\|_{p, \infty, Q_\delta} + [D_i u]_{2, Q_\delta} \leq [u]_{2, Q_\delta}.
$$

PROOF. As above, we denote by $D u$ the vector $(D_1 u, \ldots, D_n u) \in \mathbb{R}^{nN}$. Let us fix an index $r$, $1 \leq r \leq n + 1$ and let $e_r \in \mathbb{R}^n \times \mathbb{R}$ be the $r$-th vector of the canonical basis. Let $\delta' > 0$. For $0 < h < \delta'$ let

$$
u_h(z) = h^{-1} [u(z) - u(z - he_r)].$$

Define the functions $a^{\alpha \beta}_{hi} \in L^\infty(Q_\theta)$ for a.e. $z \in Q_\theta$ by

$$
a^{\alpha \beta}_{hi}(z) = \int_0^1 A^{\alpha \beta}_{hi} (D u(z) - h D u_h(z) + \tau h D u_h(z)) d\tau.
$$

It is not difficult to see that $u_h$ is the weak solution of the linear system

$$
\frac{\partial u_h}{\partial t} - D_\alpha a^{\alpha \beta}_{hij} D_\beta u_h = 0
$$

in $Q_\theta$. The functions $a^{\alpha \beta}_{hij}$ clearly satisfy the conditions (2') and (3'). Hence, by Lemma 1

$$
\|u_h\|_{p, \infty, Q_{2\delta'}} \leq c_6(\delta', p) \|u_h\|_{2, Q_\theta},
$$

$$
\|D u_h\|_{2, Q_{2\delta'}} \leq c_7(\delta') \|u_h\|_{2, Q_\theta}.
$$

Suppose first $1 \leq r \leq n$. In this case the difference is taken in the direction of the space variables. Since $u \in W^{1, 0}_2(Q)$, we have

$$
\|u_h\|_{2, Q_\theta} \leq \|D r u\|_{2, Q_\theta}.
$$

Using Nirenberg's Lemma we see from (8) that $D u \in W^{1, 0}_2(Q_\theta)$ and

$$
\|D r u\|_{p, \infty, Q_{2\delta'}} \leq c_6(\delta', p) \|D r u\|_{2, Q_\theta},
$$

$$
\|D D r u\|_{2, Q_{2\delta'}} \leq c_7(\delta') \|D r u\|_{2, Q_\theta},
$$

for every $1 \leq r \leq n$. Now let $r = n + 1$. Following S. Campanato [1] we notice that we can use equation (1) and the $L^2$-estimate of $D_\alpha D_\beta u$ obtained above to infer that $\frac{\partial u}{\partial t} \in L^2(Q_{2\delta'})$ and

$$
\|\frac{\partial u}{\partial t}\|_{2, Q_{2\delta'}} \leq c_8(\delta') \|D u\|_{2, Q_\theta}.
$$
Now we can use (8) with $Q$ replaced by $Q_{2\delta'}$ and using (11) we get by the same argument as above

$$\|\frac{\partial u}{\partial t}\|_{L^{p,\infty}(Q_{\delta'})} \leq c_9(\delta', p)\|Du\|_{L^2,Q}$$

$$\|D\frac{\partial u}{\partial t}\|_{L^{2,\infty}(Q_{\delta'})} \leq c_9(\delta')\|Du\|_{L^2,Q}.$$ 

The proof is finished.

**THEOREM 1.** Let $u$ be a weak solution of the system (1) and let $p$ be the exponent from Lemma 1. Then for each $\delta > 0$

$$\frac{\partial u}{\partial t} \in L^{p,\infty}(Q_{\delta})$$

and

$$u \in L^{\infty}(-T + \delta, 0; W^{2,q}_{q}(Q_{\delta}))$$

for some $q = q(\nu, M, p, \delta)$ with $2 < q < p$. Moreover

$$\|u\|_{L^{\infty}(-T + \delta, 0; W^{2,q}_{q}(Q_{\delta}))} + \|\frac{\partial u}{\partial t}\|_{L^{p,\infty}(Q_{\delta})} \leq c_{10}(\delta, p, q)\|u\|_{L^2,Q}.$$ 

**PROOF.** Let $\delta' > 0$. We notice that $u$ can be considered as a weak solution of the linear system (4) with

$$(\text{See, for example S. Campanato [1]). Using this and Lemma 1 we get estimates for the norms } \|u\|_{L^{p,\infty}(Q_{\delta'})} \text{ and } \|u\|_{L^{p,\infty}(Q_{\delta'})}. \text{ Now we can use Lemma 2 to get estimates of the norms } \|Du\|_{L^{p,\infty}(Q_{2\delta'})}. \text{ Lemma 2 also implies } D_\alpha D_\beta u \in L^{2}(Q_{2\delta'}) \text{, (0} \leq \alpha, \beta \leq n). \text{ We see that equation (1) is satisfied pointwise almost everywhere in } Q_{2\delta'} \text{ and that for almost every } t \in (-T + 2\delta', 0) \text{ the function } u(\cdot, t) \text{ belongs to } W^{2,q}_{q}(Q_{2\delta'}) \text{ and is the weak solution of the elliptic system}$

$$D_\alpha A^\alpha(Du) = \frac{\partial u}{\partial t}$$

in $Q_{2\delta'}$. We can now use well-known $L^p$-estimates for elliptic systems (see Lemma 3 below). The proof is finished.

**LEMMA 3.** Let $p > 2$ and let $g \in L^p(\Omega)$. Let $u \in W^{1}_p(\Omega)$ be a weak solution of the elliptic system

$$(12) \quad D_\alpha A^\alpha(Du) = g_i \quad i = 1, \ldots, n$$
Then there exists \( q = q(\nu, M, p) > 2 \) such that \( u \in W^2_q(\Omega) \). Moreover, for every \( \delta > 0 \)

\[
\|u\|_{W^2_q(\Omega)} \leq c_{11}(\nu, M, p, q, \delta)(\|u\|_{W^2_q(\Omega)} + \|g\|_{p, \Omega}).
\]

**Proof.** Using the standard difference quotient technique, it is not difficult to verify that the following computations are legal.

Let \( 1 \leq s \leq n \). We let \( v = D^s u \) and take the \( s \)-th derivative of (12). We get

\[
(13) \quad D_\alpha a_{ij}^\alpha D_\beta v_j = D_\delta g_i,
\]

where \( a_{ij}^\alpha(z) = A_{i,j}^\alpha(Du(z)) \).

This implies

\[
(14) \quad \frac{\nu}{2} \int_{\Omega} \zeta^2 |Dv|^2 dx \leq c_{12}(\nu, M) \int_{\Omega} (|v|^2 |D\zeta|^2 + |g|^2) dx
\]

for every \( \zeta \in \mathcal{D}(\Omega) \) (Cacciopoli’s inequality). The required estimate can now be obtained by using the technique of reverse Hölder inequalities. (See, for example, M. Giaquinta [2], Chap. 5, Theorem 2.2). The proof is finished.

**Corollary.** Let the assumptions of Theorem 1 be satisfied.

(i) If \( n \leq 4 \), then \( u \) is Hölder continuous in \( Q \).

(ii) If \( n \leq 2 \), then \( Du \) is Hölder continuous in \( Q \).

(iii) If \( n \leq 2 \) and the functions \( A^\alpha_q \) are smooth, then the solution \( u \) is smooth.

**Remark.** If \( n > 3 \), then \( Du \) may not be continuous. Examples are provided by nonregular solutions of elliptic systems. These can be found in J. Nečas [5].

**Proof of the Corollary.** Let \( \delta > 0 \).

(i) Since \( W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{0,\alpha}(\Omega_\delta) \) with \( \alpha = (2 - \frac{n}{q}) \wedge 1 \), we have \( u \in L^\infty(-T + \delta, 0; C^{0,\alpha}(\Omega_\delta)) \).

Since we have also \( \frac{\partial u}{\partial t} \in L^2(Q_\delta) \), \( u \) is Hölder continuous by Lemma 4 below.

(ii) In this case we have \( W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{1,\beta}(\Omega_\delta) \) \( \beta = 1 - \frac{n}{q} \).

Hence \( Du \in L^\infty(-T + \delta, 0; C^{0,\beta}(\Omega_\delta)) \). Using the Hölder continuity of \( u \) it is easy to see that in fact \( Du(\cdot, t) \in C^{0,\beta}(\Omega_\delta) \) for every \( t \in (-T + \delta, 0) \), the \( C^{0,\beta} \)-norm being bounded independently of \( t \).

Now we can use Lemma 3.1, Chap. 2 from O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva [4] to infer that \( Du \) is Hölder continuous in \( Q_\delta \).

(iii) The higher regularity follows in the standard way from the theory of linear equations.
LEMMA 4. Let $\alpha > 0$, $q > 1$, $\delta > 0$ and suppose

$$u \in L^\infty(-T, 0; C^{0,\alpha}(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^q(\Omega).$$

Denote $K_1 = \|u\|_{L^\infty(-T, 0; C^{0,\alpha}(\Omega))}$, $K_2 = \|\frac{\partial u}{\partial t}\|_{L^q(\Omega)}$. Then there exists $K = K(K_1, K_2, \delta)$ such that

$$|\bar{u}(x, t_1) - \bar{u}(x, t_2)| \leq K|t_1 - t_2|^\beta$$

for every $x \in \Omega_\delta$ and every $t_1, t_2 \in (-T, 0)$, where $\beta = \frac{\alpha/q'}{\alpha + n/q}$, $q' = \frac{q}{q - 1}$ and $\bar{u}$ is a suitable representative of $u$.

**PROOF.** Suppose first that $u$ is continuous. Let $x \in \Omega_\delta$ and let $0 < \rho < \delta$. Define

$$w_\rho(t) = \frac{1}{|B_{\rho}(x)|} \int_{B_{\rho}(x)} u(y, t)dy.$$

It is easy to see that $w_\rho$ is bounded in $L^q(-T, 0)$ by $c_{13}\rho^{-\frac{n}{q}}K_2$.

Let $t_1, t_2 \in (-T, 0)$. We can write

$$|u(x, t_1) - u(x, t_2)|$$

$$\leq |u(x, t_1) - w_\rho(t_1) + w_\rho(t_1) - w_\rho(t_2) + w_\rho(t_2) - u(x, t_2)|$$

$$\leq 2K_1\rho^n + c_{12}K_2\rho^{-\frac{n}{q}}|t_1 - t_2|^\gamma.$$

The proof is easily finished by using this inequality with $\rho = |t_1 - t_2|^\frac{\beta}{\alpha}$.

**REMARK.** It is not difficult to see that if the boundary of $\Omega$ is sufficiently regular (say, lipshitzian), then $K$ can be chosen independent of $\delta$.

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