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Stochastic Viability and Invariance

JEAN-PIERRE AUBIN - GIUSEPPE DA PRATO

Introduction

The main aim of this paper is to extend to the stochastic case Nagumo's Theorem on viability and/or invariance properties of closed subsets with respect to a differential equation.

Let us consider a closed convex subset K of $X := \mathbf{R}^n$ and a stochastic differential equation

$$d\xi = f(\xi(t))dt + g(\xi(t))dW(t)$$

the solution of which is given by the formula

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

where f and g are lipschitzean functions defined on K.

We want to characterize the (*stochastic*) *viability property* of K with respect to the pair (f, g): for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is *viable in* K, in the sense that

$$\forall\, t \in [0, T], \text{ for almost all } \omega \in \Omega, \ \ \xi_\omega(t) \in K_\omega.$$

To that purpose, we adapt to the stochastic case the concept of contingent cone to a subset. Let us consider a $\mathcal{F}_t$-random variable $x \in K$ (see Section 1.1 below).

We define the *stochastic contingent set* $\mathcal{T}_K(t, x)$ to K at x (with respect to $\mathcal{F}_t$) as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of $\mathcal{F}_{t+h_n}$-

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measurable random variables a^n and b^n such that

$$\begin{cases} \text{i)} & \mathbf{E}(\|a^n\|^2) \to 0 \\ \text{ii)} & \mathbf{E}(\|b^n\|^2) \to 0 \\ \text{iii)} & \mathbf{E}(b^n) = 0 \\ \text{iv)} & b^n \text{ is independent of } \mathcal{F}_t \end{cases}$$

and satisfying, for almost all $\omega \in \Omega$,

$$\forall\, n \geq 0,\ \ x_\omega + v_\omega(W_\omega(t+h_n) - W(t)) + h_n\gamma_\omega + h_n a_\omega^n + \sqrt{h_n} b_\omega^n \in K_\omega.$$

Then we shall prove essentially that the following conditions are equivalent:

1. – The subset K enjoys the viability property with respect to the pair (f,g).

2. – for every $\mathcal{F}_t$-random variable x viable in K,

$$(f(x), g(x)) \in \mathcal{T}_K(t,x).$$

This condition means that for every $\mathcal{F}_t$-random variable x viable in K

•
$$f(x) \in K \ \&\ g(x) \in K$$

when K is a vector subspace,

•
$$\langle x, g(x)\rangle = 0 \ \&\ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 = 0$$

when K is the unit sphere

•
$$\langle x, g(x)\rangle = 0 \ \&\ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 \leq 0$$

when K is the unit ball.

One can, for instance, deduce that a vector subspace K of the state space X is (*stochastically*) *viable or controlled invariant* by a linear stochastic control system

$$dx(t) = (Ax(t) + Bu(t))dt + gdW(t)$$

(in the sense that for any initial process $\xi_0 \in K$, there exists a solution $\xi(\cdot)$ which is viable in K) if and only if

$$A(K) \subset K + \mathrm{Im}(B) \ \&\ g \in K.$$

(The first of these conditions is the necessary and sufficient condition of controlled invariance for linear systems. See [9] for instance).

We shall devote the first section to the definition of the stochastic contingent and tangent sets to a random set-valued variable and we shall generalize the result we mentioned by giving necessary conditions for viability and sufficient conditions for invariance. The other sections will be devoted to an elementary calculus of stochastic tangent sets to direct images, inverse images and intersections of closed subsets.

1. - The Main Theorem

1.1. - *Stochastic Tangent Sets*

Let us consider a complete probability space $(\Omega, \mathcal{F}, P)$, an increasing family of σ-sub-algebras $\mathcal{F}_t \subset \mathcal{F}$ and a finite dimensional vector-space $X := \mathbf{R}^n$.

The constraints are defined by closed subsets $K_\omega \subset X$, where the set-valued map

$$K : \omega \in \Omega \hookrightarrow K_\omega \subset X$$

is assumed to be $\mathcal{F}_0$-measurable (which can be regarded as a *random set-valued variable*).

We denote by $\mathcal{K}$ the subset

$$\mathcal{K} := \{u \in L^2(\Omega, \mathcal{F}, P) \,|\, \text{for almost all } \omega \in \Omega,\ u_\omega \in K_\omega\}.$$

For simplicity, we restrict ourselves to scalar $\mathcal{F}_t$-Wiener processes $W(t)$.

DEFINITION 1.1 (Stochastic Contingent Set). *Let us consider a $\mathcal{F}_t$-random variable $x \in \mathcal{K}$ (i.e., a $\mathcal{F}_t$-measurable selection of K).*

We define the stochastic contingent set $\mathcal{T}_K(t, x)$ *to K at x (with respect to $\mathcal{F}_t$) as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of $\mathcal{F}_{t+h_n}$-random variables a^n and b^n such that*

$$(1) \qquad \begin{cases} \text{i)} & \mathbf{E}(\|a^n\|^2) \to 0 \\ \text{ii)} & \mathbf{E}(\|b^n\|^2) \to 0 \\ \text{iii)} & \mathbf{E}(b^n) = 0 \\ \text{iv)} & b^n \text{ is independent of } \mathcal{F}_t \end{cases}$$

and satisfying

$$(2) \qquad \forall n \geq 0,\ x + v(W(t+h_n) - W(t)) + h_n\gamma + h_n a^n + \sqrt{h_n} b^n \in \mathcal{K}.$$

The stochastic tangent set $\mathcal{S}_K(t, x)$ *to K at x (with respect to $\mathcal{F}_t$) is defined as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: There exist adapted continuous processes $a(s)$ and $b(s)$ converging to 0 when $s \to t$ such that, for h small enough,*

$$(3) \qquad x + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s) \in \mathcal{K}.$$

It follows readily that

$$\mathcal{S}_K(t,x) \subset \mathcal{T}_K(t,x).$$

Indeed, we set

$$a^h := \frac{1}{h}\int_t^{t+h} a(s)ds$$

and

$$b^h := \frac{1}{\sqrt{h}}\int_t^{t+h} b(s)dW(s)$$

and we observe that

$$\begin{cases} \mathbf{E}\left(\|a^h\|^2\right) = \frac{1}{h^2}\mathbf{E}\left(\left\|\int_t^{t+h} a(s)ds\right\|^2\right) \\ \leq \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|(a(s)\|^2)\right) ds \end{cases}$$

converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

In the same way,

$$\begin{cases} \mathbf{E}\left(\|b^h\|^2\right) = \frac{1}{h}\mathbf{E}\left(\left\|\int_t^{t+h} b(s)dW(s)\right\|^2\right) \\ = \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|b(s)\|^2\right) ds \end{cases}$$

converges also to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

The expectation of b^h is obviously equal to 0, and b^h is independent of $\mathcal{F}_t$. □

We see also that

$$\text{if } (v,\gamma) \in \mathcal{T}_K(t,x), \text{ then } \mathbf{E}(\gamma) \in T_{\mathbf{E}(K)}(\mathbf{E}(x))$$

where $T_{\mathbf{E}(K)}(\mathbf{E}(x))$ denotes the *contingent cone* to the integral of K at $\mathbf{E}(x)$, because, by taking the expectation in both sides of formula (3), we infer that

$$\mathbf{E}(x) + h_n\mathbf{E}(\gamma) + h_n\mathbf{E}(a^n) \in \mathbf{E}(K)$$

where $\mathbf{E}(K)$ denotes the subset of expectations $\mathbf{E}(K)$ of random variables $x \in \mathcal{K}$.

Let us denote by $K_t := \mathbf{E}(K|\mathcal{F}_t)$ the conditional expectation of the set-valued map random variable K (i.e., the projection of $\mathcal{K}$ onto $L^2(\Omega,\mathcal{F}_t,P)$). By taking the conditional expectation in both sides of formula (3), we deduce that

$$\text{if } (v,\gamma) \in \mathcal{T}_K(t,x), \text{ then } \gamma \in T_{K_t}(x)$$

because $\mathbf{E}(x) = x$ for any $\mathcal{F}_t$-measurable random variable.

1.2. - *Stochastic Invariance*

We consider the stochastic differential equation

$$d\xi = f(\xi(t))dt + g(\xi(t))dW(t) \tag{4}$$

where f and g are lipschitzean.

We say that a stochastic process $\xi(t)$ defined by

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s) \tag{5}$$

is a solution to the stochastic differential equation (4) if functions f and g satisfy:

$$\text{for almost all } \omega \in \Omega, \ f(\xi(\cdot)) \in L^1(0,T;X) \text{ and } g(\xi(\cdot)) \in L^2(0,T;X).$$

We refer to [7] for instance for more general sufficient conditions on f and g implying the existence of such solutions.

DEFINITION 1.2. *We shall say that a stochastic process $x(\cdot)$ is* viable in K *if and only if*

$$\forall t \in [0,T], \ x(t) \in \mathcal{K} \tag{6}$$

i.e., if and only if

$$\forall \ t \in [0,T], \text{ for almost all } \omega \in \Omega, \ \xi_\omega(t) \in K_\omega.$$

The random set-valued variable K is said to be (stochastically) invariant *by the pair (f,g) if every solution ξ to the stochastic differential equation starting at a random variable $x \in K$ is* viable in K.

When K is a subset of X (i.e., a constant set-valued random variable) and when the maps (f,g) are defined on K, we shall say that K enjoys the (stochastic) viability property *with respect to the pair (f,g) if, for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is* viable in K.

Since K_ω and $\xi_\omega(0)$ are $\mathcal{F}_0$ measurable, the projection map $\Pi_{K_\omega}(\xi_\omega(0))$ is also $\mathcal{F}_0$-measurable (see [2, Theorem 8.2.13, p. 317]). Then there exists a $\mathcal{F}_0$-measurable selection $y_\omega \in \Pi_{K_\omega}(\xi_\omega(0))$, which we call a *projection of the random variable $\xi(0)$ onto the random set-valued variable K.*

THEOREM 1.3 (Stochastic Viability). *Let K be a closed convex subset of X and (f,g) maps be defined on K satisfying the assumptions of the* *existence theorem of a solution to the stochastic differential equation* (4). *Then the following conditions are equivalent*:

1. – *The subset* K *enjoys the viability property with respect to the pair* (f,g)

2. – *for every* $\mathcal{F}_t$*-random variable* x *viable in* K,

$$(f(x), g(x)) \in \mathcal{T}_K(t,x) \tag{7}$$

3. – *for every* $\mathcal{F}_t$*-random variable* x *viable in* K,

$$(f(x), g(x)) \in \mathcal{S}_K(t,x). \tag{8}$$

We shall deduce this theorem from more general Theorems 1.4 and 1.5 dealing with set-valued random variables instead of closed convex subsets.

1.3. - *Necessary Conditions*

Let K be a set-valued random variable.

THEOREM 1.4. *We posit the assumptions of the existence theorem of a solution to the stochastic differential equation* (4). *If the random set-valued variable* K *is invariant by the pair* (f,g), *then for every* $\mathcal{F}_t$*-random variable* x *viable in* K,

$$(f(x), g(x)) \in \mathcal{S}_K(t,x). \tag{9}$$

PROOF. We consider the viable stochastic process $\xi(t)$

$$\xi(h) = x + \int_0^h f(\xi(s))ds + \int_0^h g(\xi(s))dW(s) \tag{10}$$

which is a solution to the stochastic differential equation (4) starting at x.

We can write it in the form

$$\xi(t) = \xi(0) + hf(\xi(0)) + g(\xi(0))W(h) + \int_0^h a(s)ds + \int_0^h b(s)dW(s)$$

where

$$\begin{cases} a(s) = f(\xi(s)) - f(\xi(0)) \\ b(s) = g(\xi(s)) - g(\xi(0)) \end{cases}$$

converge to 0 with s.

Since $\xi(h)_\omega$ belongs to K_ω for almost all ω, we derive that the pair $(f(x), g(x))$ belongs to the stochastic set:

$$(f(x), g(x)) \in \mathcal{S}_K(t,y)$$

and thus, that

$$(f(x), g(x)) \in \mathcal{T}_K(t, y).$$

Hence the Theorem ensues. □

1.4. - *Sufficient Conditions*

THEOREM 1.5 (Stochastic Invariance). *We posit the assumptions of the existence theorem of a solution to the stochastic differential equation* (4).

Assume that for every $\mathcal{F}_t$-random variable x, there exists a $\mathcal{F}_t$-measurable projection $y \in \Pi_K(x)$ such that

$$(f(x), g(x)) \in \mathcal{T}_K(t, y). \tag{11}$$

Then the set-valued random variable K is invariant by (f, g).

REMARK. Observe that the sufficient condition of invariance requires the verification of the "stochastic tangential condition" (11) for every stochatic process y, including stochastic processes which are not viable in K. □

In order to prove Theorem 1.5, we need the following:

LEMMA 1.6. *Let K be a random set-valued variable, $\xi(0)$ a $\mathcal{F}_0$-adapted stochastic process.*

We define

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

and we choose a $\mathcal{F}_0$-measurable projection $y \in \Pi_K(\xi(0))$.

Then, for any pair of $\mathcal{F}_0$-random variable (γ, v) in the stochastic contingent set $\mathcal{T}_K(0, y)$, the following estimate

$$\begin{cases} \liminf_{t_n \to 0} \left(\mathbf{E}(d_K^2(\xi(t_n)) - \mathbf{E}(d_K^2(\xi(0))\right) / t_n \\ \leq 2\mathbf{E}\left(\langle \xi(0) - y, f(\xi(0)) - \gamma \rangle\right) + \mathbf{E}(\|g(\xi(0)) - v\|^2) \end{cases}$$

holds true.

PROOF. Let us set $x = \xi(0)$, choose a projection $y \in \Pi_K(x)$ and take (γ, v) in the stochastic contingent set $\mathcal{T}_K(0, y)$. This means that there exist sequences $t_n > 0$ converging to 0 and $\mathcal{F}_{t_n}$-measurable a^n and b^n satisfying the assumptions (1) and

$$\forall n \geq 0, \text{ for almost all } \omega \in \Omega, \; y_\omega + v_\omega W_\omega(t_n) + \gamma_\omega t_n + t_n a_\omega^n + \sqrt{t_n} b_\omega^n \in K_\omega.$$

Therefore

$$\begin{cases} d_K^2(\xi(t_n)) - d_K^2(\xi(0)) \\ \leq \left\| x + \int_0^{t_n} f(\xi(s))ds + \int_0^{t_n} g(\xi(s))dW(s) \right. \\ \left. - y - vW(t_n) - \gamma t_n - t_n a^n - \sqrt{t_n} b^n \right\|^2 - \|x-y\|^2 \\ = \left\| (x-y) + \int_0^{t_n} (f(\xi(s)) - \gamma)ds + \int_0^{t_n} (g(\xi(s)) - v)dW(s) \right. \\ \left. - t_n a^n - \sqrt{t_n} b^n \right\|^2 - \|x-y\|^2 \\ =: I \end{cases}$$

The latter term can be split in the following way:

$$\begin{cases} I = 2\left\langle x - y, \int_0^{t_n} (g(\xi(s)) - v)dW(s) \right\rangle & I_1 \\ +2\left\langle x - y, \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right\rangle & I_2 \\ +\| \int_0^{t_n} (g(\xi(s)) - v)dW(s)\|^2 & I_3 \\ +\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2 & I_4 \\ +2\left\langle \int_0^{t_n} (g(\xi(s)) - v)dW(s), \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right\rangle & I_5 \\ -2\left\langle x - y + \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right. & \\ \left. + \int_0^{t_n} (g(\xi(s)) - v)dW(s), t_n a^n \right\rangle & I_6 \\ -2\left\langle x - y + \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right. & \\ \left. + \int_0^{t_n} (g(\xi(s)) - v)dW(s), \sqrt{t_n} b^n \right\rangle & I_7 \\ +\|t_n a^n + \sqrt{t_n} b^n\|^2. & I_8 \end{cases}$$

We take the expectation of this inequality in both sides and estimate each term of the right-hand side. First, we observe that

$$\mathbf{E}\left(\left\langle x - y, \int_0^{t_n} (g(\xi(s)) - v)dW(s) \right\rangle\right) = 0$$

so that the expectation of the first term I_1 of the right-hand side of the above inequality vanishes.

The second term I_2 is estimated by $2t_n\alpha_n$ where

$$\alpha_n := \mathbf{E}\left(\left\langle x - y, \frac{1}{t_n}\int_0^{t_n} (f(\xi(s)) - \gamma)ds \right\rangle\right)$$

converges to

$$\alpha := \mathbf{E}\left(\langle x - y, f(\xi(0)) - \gamma\rangle\right).$$

The third term I_3 is estimated by $t_n\beta_n$ where

$$\begin{cases} \beta_n := \dfrac{1}{t_n}\mathbf{E}\left(\| \displaystyle\int_0^{t_n} (g(\xi(s)) - v)dW(s))\|^2\right) \\ = \dfrac{1}{t_n}\displaystyle\int_0^{t_n} \mathbf{E}\left(\|g(\xi(s)) - v\|^2\right) ds \end{cases}$$

converges to

$$\beta := \mathbf{E}(\|g(\xi(0)) - v\|^2)$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$. The fourth term I_4 is easily estimated by $t_n\delta_n$ where

$$\begin{cases} \delta_n := \dfrac{1}{t_n}\mathbf{E}\left(\| \displaystyle\int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2\right) \\ \leq \displaystyle\int_0^{t_n} \mathbf{E}\left(\|f(\xi(s)) - \gamma\|^2\right) ds \leq ct_n \end{cases}$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

By the Cauchy-Schwarz inequality, the term I_5 is estimated by $2t_n\eta_n$ where

$$\begin{cases} \eta_n := \dfrac{1}{t_n}\mathbf{E}\left(\left\langle \displaystyle\int_0^{t_n} (g(\xi(s)) - v)dW(s), \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right\rangle\right) \\ \leq \dfrac{1}{t_n}\mathbf{E}(\| \displaystyle\int_0^{t_n} (g(\xi(s)) - v)dW(s)\|^2)^{1/2}\,\mathbf{E}(\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2)^{1/2} \\ = \dfrac{1}{t_n}\left(\displaystyle\int_0^{t_n} \mathbf{E}(\|g(\xi(s)) - v\|^2 ds)\right)^{1/2} (\mathbf{E}(\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2)^{1/2} \\ \leq \sqrt{t_n}\left(\dfrac{1}{t_n}\displaystyle\int_0^{t_n} \mathbf{E}(\|g(\xi(s)) - v\|^2)\right)^{1/2} \\ \left(t_n\displaystyle\int_0^{t_n} (\mathbf{E}(\|f(\xi(s)) - \gamma\|^2)\right)^{1/2} \leq ct_n^{\frac{1}{2}} \end{cases}$$

We now estimate the three latter terms involving the errors a^n and b^n. We begin with I_6. First,

$$\mathbf{E}\left(\langle x - y, a^n\rangle\right) \leq \mathbf{E}\left(\|x - y\|^2\right)^{\frac{1}{2}} \left(\mathbf{E}(\|a^n\|^2)\right)^{\frac{1}{2}}$$

which converges to 0 by assumption (1)i).

Then, Cauchy-Schwarz inequality implies that

$$\begin{cases} \mathbf{E}\left(\left\langle\int_0^{t_n}(f(\xi(s))-\gamma)ds, a^n\right\rangle\right) \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(f(\xi(s))-\gamma)ds\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}}. \end{cases}$$

Eventually the stochastic term is estimated in the following way:

$$\begin{cases} \mathbf{E}\left(\left\langle\int_0^{t_n}(g(\xi(s))-v)dW(s), a^n\right\rangle\right) \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(g(\xi(s))-v)dW(s)\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \\ = \left(\int_0^{t_n}\mathbf{E}\left(\|g(\xi(s))-v)\|^2 ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \end{cases}$$

which obviously converges to 0.

We continue with I_7. We have

$$\mathbf{E}\left\langle x-y, \frac{1}{\sqrt{t_n}}b^n\right\rangle = 0$$

since b^n is independent of $x-y$ and $\mathbf{E}(b^n)=0$.

Cauchy-Schwarz inequality implies that

$$\begin{cases} \mathbf{E}\left(\left\langle\int_0^{t_n}(f(\xi(s))-\gamma)ds, \frac{1}{\sqrt{t_n}}b^n\right\rangle\right) \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(f(\xi(s))-\gamma)ds\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|\frac{1}{\sqrt{t_n}}b^n\|^2\right)^{\frac{1}{2}} \\ = \frac{1}{\sqrt{t_n}}\left(t_n\int_0^{t_n}\mathbf{E}\left(\|(f(\xi(s))-\gamma)\|^2\right)ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}}. \end{cases}$$

Finally, the worst term of I_7 is estimated in the following way:

$$\begin{cases} \mathbf{E}\left(\left\langle \int_0^{t_n} (g(\xi(s)) - v)dW(s), \frac{1}{\sqrt{t_n}} b^n \right\rangle\right) \\ \leq \frac{1}{\sqrt{t_n}} \sqrt{\mathbf{E}\left(\left\| \int_0^{t_n} (g(\xi(s)) - v)dW(s) \right\|^2\right)} \sqrt{\mathbf{E}\left(\|b^n\|^2\right)} \\ = \left(\frac{1}{t_n} \int_0^{t_n} \mathbf{E}(\|g(\xi(s)) - v)\|^2 ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}} \end{cases}$$

which converges to 0 by assumption (1)ii).

It remains to estimate the last term of I_8. There is no difficulty because

$$\frac{1}{t_n}\mathbf{E}\left(\left\|t_n a^n + \sqrt{t_n} b^n\right\|^2\right) = \mathbf{E}\left(\left\|\sqrt{t_n} a^n + b^n\right\|^2\right)$$

converges to 0.

Taking all these statements into account, we deduce the inequality of the lemma. □

REMARK. If we denote by $\mathrm{Var}(K)$ the subset of variances $\mathrm{Var}(x)$ when x ranges over $\mathcal{K}$, we deduce from the proof of the above Lemma that

$$\mathrm{cov}(x, \gamma) + \|v\|^2 \in T_{\mathrm{Var}(K)}(\mathrm{Var}(x)) \ \square$$

PROOF OF THEOREM 1.5. Since the solution to the stochastic differential equation for any $h \geq 0$ can be written as

$$\xi(t + h) = \xi(t) + \int_t^{t+h} f(\xi(s))ds + \int_t^{t+h} g(\xi(s))dW(s)$$

we deduce from Lemma 1.6 that

$$\begin{cases} \liminf_{h \to 0+} \left(\mathbf{E}\left(d_K^2(\xi(t + h)) - \mathbf{E}\left(d_K^2(\xi(t))\right)\right)\right)/h \\ \leq 2\mathbf{E}\left(\langle \xi(t) - y(t), g(\xi(t)) - \gamma\rangle\right) + \mathbf{E}\left(\|g(\xi(t)) - v\|^2\right) \end{cases}$$

for any $\mathcal{F}_t$-measurable selection $y(t)$ of $\Pi_K(\xi(t))$ and any $(v(t), \gamma(t)) \in \mathcal{T}_K(t, y(t))$.

Since there exists a selection $y(t)$ of $\Pi_K(\xi(t))$ such that we can take $v(t) := g(\xi(t))$ and $\gamma(t) := f(\xi(t))$ by assumption, we infer that setting

$$\varphi(t) := \mathbf{E}\left(d_K^2(\xi(t))\right)$$

the contingent epiderivative $D_\uparrow\varphi(t)(1)$ is non positive.

This implies that $\varphi(t) \leq 0$ for all $t \in [0, T]$. If not, there would exist $T > 0$ such that $\varphi(T) > 0$. Since φ is continuous, there exists $\eta \in]0, T[$ such that

$$\forall t \in]T - \eta, T], \ \ \varphi(t) > 0.$$

Let us introduce the subset

$$A := \{s \in [0, T] \,|\, \forall t \in]s, t], \ \ \varphi(t) > 0\}$$

and $t_0 := \inf A$.

We observe that for any $t \in]t_0, T], \phi(t) > 0$ and that $\varphi(t_0) = 0$. Indeed, if $\varphi(t_0) > 0$, there would exist $t_1 \in]t_1, t_0[$ such that $\varphi(t) > 0$ for all $t \in]t_1, t_0]$, i.e., $t_1 \in A$, so that t_0 would not be an infimum.

Therefore, $D_{\uparrow}\varphi(t)(1) \leq 0$ for any $t \in]t_0, T]$ and thus, we obtain the contradiction

$$0 < \varphi(T) = \varphi(T) - \varphi(t_0) \leq 0$$

by [1, Chapter 6].

Consequently, for every $t \in [0, T]$, we have

$$\mathbf{E}\left(d_K^2(\xi(t))\right) = \int_\Omega d_{K_\omega}^2(\xi_\omega(t))dP(\omega) = 0$$

Since the integrand is nonnegative, we infer that, almost surely, $d_{K_\omega}(\xi_\omega(t)) = 0$, i.e., that the stochastic process ξ is viable in K. □

Proof of Theorem 1.3. The necessary condition following obviously from Theorem 1.4, it remains to prove that it is sufficient. To that purpose, we extend the maps f and g defined on K by the maps $\tilde{f}$ and $\tilde{g}$ defined on the whole space by

$$\tilde{f}(x) := f(\pi_K(x)) \ \& \ \tilde{g}(x) := g(\pi_K(x)).$$

Then the pair $(\tilde{f}, \tilde{g})$ satisifies obviously condition

$$(\tilde{f}(x), \tilde{g}(x)) \in \mathcal{T}_K(t, \pi_K(x))$$

so that K is invariant by $(\tilde{f}, \tilde{g})$ thanks to Theorem 1.5. Since these maps do coincide on K, we infer that K is a viability domain of (f, g). □

2. - Stochastic Tangent Sets to Direct Images

PROPOSITION 2.1. *Let us consider a* random set-valued variable K, *i.e., a* $\mathcal{F}_0$*-measurable set-valued map*

$$K : \omega \in \Omega \hookrightarrow K_\omega \subset X.$$

Let φ *be a* $C^{(2)}$*-map from* X *to a finite dimensional vector-space* Y. *If*

$$(\gamma, v) \in \mathcal{S}_K(t, x)$$

then

$$\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v), \varphi'(x)v\right) \in \mathcal{S}_{\varphi(K)}(t, \varphi(x)).$$

This result follows from the following consequence of the Itô's formula:

LEMMA 2.2. *Let* φ *be a* $C^{(2)}$*-map from* X *to a finite dimensional vector-space* Y. *Consider two continuous processes* $a(s)$ *and* $b(s)$ *converging to* 0 *when* $s \to t$. *Then there exist two continuous processes* $a_1(s)$ *and* $b_1(s)$ *converging to* 0 *when* $s \to t$ *such that*

$$\begin{cases} \varphi\left(x + \displaystyle\int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s)\right) \\ = \varphi(x) + \displaystyle\int_t^{t+h} \left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v) + a_1(s)\right) ds \\ + \displaystyle\int_t^{t+h} \varphi'(x)(v + b_1(s)) dW(s) \end{cases}$$

where

$$\begin{cases} a_1(s) := \varphi'(\xi(s))a(s) + (\varphi'(\xi(s)) - \varphi'(x))\gamma \\ + \dfrac{1}{2}(\varphi''(\xi(s)) - \varphi''(x))(v, v) \\ + \dfrac{1}{2}\varphi''(\xi(s))(b(s), b(s)) + \varphi''(\xi(s))(v, b(s)) \end{cases}$$

and

$$b_1(s) := \varphi'(\xi(s))b(s) + (\varphi'(\xi(s)) - \varphi'(x))v.$$

PROOF. Let us take $t = 0$ and set $\xi(h) := x + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s)$. By Itô's formula, we have

$$\begin{cases} \varphi(\xi(h)) - \varphi(x) \\ = \int_0^h \left(\varphi'(\xi(s))(\gamma + a(s)) + \frac{1}{2}\varphi''(\xi(s))(v + b(s), v + b(s))\right) ds \\ + \int_0^h \left(\varphi'(\xi(s))(v + b(s))\right) dW(s) \\ h\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v,v)\right) + W(h)\left(\varphi'(x)v\right) \\ + \int_0^h \left((\varphi'(\xi(s)) - \varphi'(x))\gamma + \varphi'(\xi(s))a(s)\right) ds \\ + \int_0^h \left(\frac{1}{2}(\varphi''(\xi(s)) - \varphi''(x))(v,v)\right. \\ \left. + \frac{1}{2}\varphi''(\xi(s))(b(s), b(s)) + \varphi''(\xi(s))(v, b(s))\right) ds \\ + \int_0^h \left((\varphi'(\xi(s)) - \varphi'(x))v + \varphi'(\xi(s))b(s)\right) dW(s) \end{cases}$$

So, the proof of the Lemma ensues. □

PROOF OF PROPOSITION 2.1. If (γ, v) belongs to $S_K(t,x)$, then there exist adapted stochastic processes $a(s)$ and $b(s)$ such that

$$x + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s) \in K$$

and thus, thanks to the preceding lemma,

$$\begin{cases} \varphi\left(x + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s)\right) \\ = \varphi(x) + \int_t^{t+h} \left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v,v) + a_1(s)\right) ds \\ + \int_t^{t+h} \left(\varphi'(x)(v + b_1(s))\right) dW(s) \in \varphi(K) \end{cases}$$

where $a_1(s)$ and $b_1(s)$ converge to 0 when $s \to t$. This means that

$$\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v,v), \varphi'(x)v\right) \in S_{\varphi(K)}(t, \varphi(x)) \quad \square$$

3. - Stochastic Contingent Sets to Inverse Images

Let φ be a $\mathcal{C}^{(2)}$-differentiable map from X to Y and $M : X \mapsto Y$ be a random set-valued variable. We deduce from Proposition 2.1 that if

$$(\gamma, v) \in \mathcal{S}_{\varphi^{-1}(M)}(t, x)$$

then

$$\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v), \varphi'(x)v\right) \in \mathcal{S}_M(t, \varphi(x)).$$

We shall prove the converse inclusion under further assumptions.

PROPOSITION 3.1. *Let φ be a $\mathcal{C}^{(2)}$-differentiable map from X to Y and $M : X \mapsto Y$ be a random set-valued variable. Assume that for almost all $\omega \in \Omega$ and for every $x \in \partial K_\omega$, the map $\varphi'(x)$ is surjective. Then $(\gamma, v) \in \mathcal{S}_{\varphi^{-1}(M)}(t, x)$ if and only if*

$$\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v), \varphi'(x)v\right) \in \mathcal{S}_M(t, \varphi(x)).$$

PROOF. It remains to assume that

$$\left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v), \varphi'(x)v\right) \in \mathcal{S}_M(t, \varphi(x))$$

and to infer that $(\gamma, v) \in \mathcal{S}_{\varphi^{-1}(M)}(t, x)$. Let us take $t = 0$. We know that there exist continuous processes $a_1(s)$ and $b_1(s)$ converging to 0 with s such that

$$\begin{cases} \varphi(x) + \displaystyle\int_t^{t+h} \left(\varphi'(x)\gamma + \frac{1}{2}\varphi''(x)(v, v) + a_1(s)\right) ds \\ + \displaystyle\int_t^{t+h} \left(\varphi'(x)(v + b_1(s))\right) dW(s) \in \mathcal{M}. \end{cases}$$

We observe that if $u(s)$ is a $\mathcal{F}_s$-measurable random variable, so is

$$\varphi'(\xi(s))^+ u(s)$$

where $\varphi'(\xi(s))^+$ denotes the orthogonal right-inverse of $\varphi'(\xi(s))$. Indeed, by [2, Theorem 8.2.9, p. 315], the random set-valued random variable $\varphi'(\xi(s))^{-1}(u(s))$ is $\mathcal{F}_s$-measurable. Then [2, Theorem 8.2.11, p. 316] implies that the Banach constant

$$\inf_{\varphi'(\xi(s))v = u(s)} \|v\|$$

is measurable and that $\varphi'(\xi(s))^+(u(s))$ is also measurable.

This being said, we associate with $a_1(s)$ and $b_1(s)$ the $\mathcal{F}_s$-measurable processes defined successively by

$$b(s) := -\varphi'(\xi(s))^+ \left(b_1(s) + (\varphi'(\xi(s)) - \varphi'(x))v\right)$$

and

$$\begin{cases} a(s) := -\varphi'(\xi(s))^+ \big(a_1(s) - (\varphi'(\xi(s)) - \varphi'(x))\gamma \\ + \dfrac{1}{2} (\varphi''(\xi(s)) - \varphi''(x))(v, v) \\ + \dfrac{1}{2} \varphi''(\xi(s))(b(s), b(s)) + \varphi''(\xi(s))(v, b(s))\big). \end{cases}$$

By Lemma 2.2, we infer that

$$\begin{cases} \varphi \left(x + \displaystyle\int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s) \right) \\ = \varphi(x) + \displaystyle\int_t^{t+h} \left(\varphi'(x)\gamma + \frac{1}{2} \varphi''(x)(v, v) + a_1(s) \right) ds \\ + \displaystyle\int_t^{t+h} \left(\varphi'(x)(v + b_1(s))\right) dW(s) \in \mathcal{M} \end{cases}$$

and thus, that

$$x + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s) \in \varphi^{-1}(\mathcal{M}) \quad \square$$

COROLLARY 3.2. *We list here examples of stochastic tangent sets*:

- *When K is a vector subspace, then*

$$S_K(t, x) = K \times K$$

- *When K is the unit sphere, then, for all $x \in K$, $(v, \gamma) \in S_K(t, x)$ if and only if*

$$\langle x, v \rangle = 0 \ \& \ \langle x, \gamma \rangle + \frac{1}{2} \|v\|^2 = 0$$

- *When K is the unit ball, then, for all $x \in K$ and for almost all ω such that $\|x_\omega\| = 1$, $(v, \gamma) \in S_K(t, x)$ if and only if*

$$\begin{cases} \text{for almost all } \omega \textit{ such that } \|x_\omega\| = 1 \\ \langle x, v \rangle = 0 \ \& \ \langle x, \gamma \rangle + \dfrac{1}{2} \|v\|^2 \leq 0. \end{cases}$$

EXAMPLE: *Viable or Controlled Invariant Linear Control Systems.* Let us consider a stochastic control system

$$dx(t) = (Ax(t) + Bu(t))dt + gdW(t)$$

and a vector subspace K of the state space X. Then K is (*stochastically*) *viable or controlled invariant* (in the sense that for any initial process $\xi_0 \in K$, there exists a solution $\xi(\cdot)$ which is viable in K) if and only if

$$A(K) \subset K + \mathrm{Im}(B) \ \& \ g \in K$$

(The first of these conditions is the necessary and sufficient condition of controlled invariance for linear systems.).

Indeed, the proof of Theorem 1.4 shows that such condition is necessary. To prove that it is sufficient, we consider the regulation map R_K defined by

$$R_K(x) : 0 \ \{u \in U \mid Bu \in K - Ax\}$$

(which has nonempty values by assumption) and the feedback control R defined by

$$\|Rx\| = \inf_{u \in U \mid Bu \in K - Ax} \|u\|$$

Therefore it is clear that

$$(Ax + BRx, g) \in K \times K = \mathcal{S}_K(t, x)$$

so that Theorem 1.3 implies that K enjoys the viability property with respect to the pair $(A + BR, g)$, and thus, that K is controlled invariant. □

3.1. - *Stochastic Contingent Sets to an Intersection*

We shall prove that another class of stochastic tangent sets is stable by intersection.

We introduce the subsets $\mathcal{T}_K^{\natural}(t, x)$ as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: For all sequences $h_n > 0$ converging to 0, there exists a $\mathcal{F}_{t+h_n}$-random variable a^n such that

$$\mathbf{E}(\|a^n\|^2) \to 0$$

and

$$\forall n \geq 0, \ \ x + v(W(t + h_n) - W(t)) + h_n \gamma + h_n a^n \in \mathcal{K}.$$

It follows readily that

$$\mathcal{T}_K^{\natural}(t, x) \subset \mathcal{T}_K(t, x)$$

THEOREM 3.3. *Let X and Y be finite dimensional vector-spaces and A be a linear operator from X to Y. Let $L \subset X$ and M be closed subsets and define $K := L \cap A^{-1}(M)$. Assume that the transversality condition*

$$\text{for almost all } \omega \in \Omega, \ A\left(C_{L_\omega}(x_\omega)\right) \bigcap C_{M_\omega}(Ax_\omega) = Y$$

holds true. Therefore

$$\begin{cases} (\gamma, v) \in \mathcal{T}_L^{\natural}(t,x) \ \& \ (A\gamma, Av) \in \mathcal{T}_M^{\natural}(t, Ax) \\ \text{if and only if } (\gamma, v) \in \mathcal{T}_K^{\natural}(t,x). \end{cases}$$

PROOF. The first condition following obviously from the second one, let us take any $(\gamma, v) \in \mathcal{T}_L^{\natural}(0,x)$ such that $(A\gamma, Av) \in \mathcal{T}_M^{\natural}(Ax)$. Hence, for any sequence $h_n > 0$ converging to 0, there exist sequences γ_n and δ_n converging to γ and $A\gamma$ respectively such that, for all $n \geq 0$,

$$x + vW(h_n) + h_n\gamma_n \in \mathcal{L} \ \& \ Ax + Av_n + h_n\delta_n \in \mathcal{M}.$$

We now apply [2, Theorem 4.3.1, p. 146] to the subsets $L_\omega \times M_\omega$ of $X \times Y$ and the continuous map $A \ominus \mathbf{1}$ associating to any (x,y) the element $Ax - y$. It is obvious that the transversality condition

$$A\left(C_{L_\omega}(x_\omega)\right) \cap C_{M_\omega}(Ax_\omega) = Y$$

implies the surjectivity assumption of [2, Theorem 4.3.1, p. 146]. The pair

$$(x_\omega + v_\omega W_\omega(h_n) + h_n\gamma_{n_\omega}, Ax_\omega + Av_\omega W_\omega(h_n) + h_n\delta_{n_\omega})$$

belongs to $L_\omega \times M_\omega$ and

$$\begin{aligned} (A \ominus \mathbf{1})((x_\omega + v_\omega W_\omega(h_n) + h_n\gamma_{n_\omega}, Ax_\omega + Av_\omega W_\omega(h_n) + h_n\delta_{n_\omega})) \\ = h_n(A\gamma_{n_\omega} - \delta_{n_\omega}). \end{aligned}$$

Therefore, by [2, Theorem 4.3.1, p. 146] and the measurable selection Theorem (see [2, Theorem 8.1.3, p. 308]), there exist $l > 0$ and a $\mathcal{F}_t$-measurable solution $(\widehat{x}_{n_\omega}, \widehat{y}_{n_\omega}) \in L_\omega \times M_\omega$ to the equation

$$(A \ominus \mathbf{1})(\widehat{x}_{n_\omega}, \widehat{y}_{n_\omega}) = 0$$

(i.e., $\widehat{y}_{n_\omega} = A\widehat{x}_{n_\omega}$) such that

$$\begin{cases} \|x_\omega + v_\omega W_\omega(h_n) + h_n\gamma_{n_\omega} - \widehat{x}_{n_\omega}\| \\ + \|Ax_\omega + Av_\omega W_\omega(h_n) + h_n\delta_{n_\omega} - \widehat{y}_{n_\omega}\| \\ \leq lh_n\|A\gamma_{n_\omega} - \delta_{n_\omega}\|. \end{cases}$$

Hence $\widehat{u}_{n_\omega} := (\widehat{x}_n - x_\omega - v_\omega W_\omega(h_n))/h_n$ converges to γ, and for all $n \geq 0$, we know that $x_\omega + v_\omega W_\omega(h_n) + h_n\widehat{u}_n$ belongs to $L_\omega \cap A^{-1}(M_\omega)$ because $x_\omega - v_\omega W_\omega(h_n)) + h_n\widehat{u}_n = \widehat{x}_n$ and $A(x_\omega - v_\omega W_\omega(h_n)) + h_n\widehat{u}_n) = \widehat{y}_n$. □

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