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Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the \( n \)-laplacian

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1. - Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with smooth boundary. We are looking for a solution of the following problem:

Let $1 < p \leq n$, find $u \in W^{1,p}_0(\Omega) \setminus \{0\}$ such that

$$
\Delta_p u = f(x, u)|u|^{p-2} \quad \text{in} \quad \Omega
$$

$$
u \geq 0,
$$

where $\Delta_p u = \text{div}(\nabla |u|^{p-2} \nabla u)$ is the $p$-Laplacian and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$-function with $f(x, 0) = 0$, $f(x, t) \geq 0$ for $t \geq 0$ and of critical growth.

For $p = 2$ and $n \geq 3$, Brézis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when $f$ has critical growth of the form $u^{(n+2)/(n-2)} + \lambda u$. A generalization of this result, on the same lines, for the $p$-Laplacian with $p \leq n$ and $p^2 \leq n$, has been studied by Garcia Azorero-Peral Alonso [7]. When $p = n$, in view of the Trudinger [13] imbedding, a critical growth function $f(x, u)$ behaves like $\exp(b|u|^{n/(n-1)})$ for some $b > 0$. In this context, when $p = n = 2$ and $\Omega$ is a ball in $\mathbb{R}^2$, existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brézis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let $f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)})$ be a function of critical growth and $F(x, t)$ be its primitive (see definition (2.1)). For $u \in W^{1,n}_0(\Omega)$, let

$$
J(u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \int_{\Omega} F(x, u) \, dx
$$

THEOREM Let $f(x, t) = h(x, t) \exp(b|t|^{\alpha/(n-1)})$ be a function of critical growth on $\Omega$. Then

1) $J : W^{1,n}_0(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale Condition on the interval $(-\infty, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1})$;

2) Let $f'(x, t) = \frac{\partial}{\partial t} f(x, t)$ and further assume that

$$\sup_{x \in \Omega} f'(x, 0) < \lambda_1(\Omega)$$

$$\lim_{t \to \infty} \inf_{x \in \Omega} h(x, t)t^{n-1} = \infty,$$

then there exists some $u_0 \in W^{1,n}_0(\Omega) \setminus \{0\}$ such that

$$\Delta_n u_0 = f(x, u_0)u_0^{n-2} \quad \text{in} \ \Omega$$

$$u_0 \geq 0$$

$$u_0 = 0 \quad \text{on} \ \partial\Omega.$$

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adapt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

$$a(\Omega, f)^n = \inf \left\{ J(u); \left. \int_{\Omega} |\nabla u|^n \, dx = \int_{\Omega} f(x, u)u^{n-1}\, dx \right| u \neq 0 \right\},$$

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that $\alpha_n$ is the best constant appearing in Moser’s [10] result about the Trudinger’s imbedding of $W^{1,n}_0(\Omega)$. In view of this, one expects that $J$ should satisfy the Palais-Smale Condition on $(-\infty, \frac{1}{n} \left(\frac{\alpha_n}{b}\right)^{n-1})$. Therefore, in order to get a minimizer of (1.8), the question remains to show that

$$a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$$
and this has been achieved by showing the following relation

\begin{equation}
\sup_{u} \int_{\Omega} f(x, a(\Omega, f)) w^{n-1} \, dx \leq a(\Omega, f).
\end{equation}

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when \( n = 2 \).

2. - Preliminaries

Let \( \Omega \) be a bounded domain with smooth boundary. In view of the Trudinger-Moser \([13, 10]\) imbedding, we have the following definition of functions of critical growth.

**DEFINITION 2.1.** Let \( h: \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \)-function and \( b > 0 \). Let \( f(x, t) = h(x, t) \exp \left( b |t|^{n/(n-1)} \right) \). We say that \( f \) is a function of critical growth on \( \Omega \) if the following holds:

There exist constants \( M > 0, \alpha \in [0, 1) \) such that, for every \( \epsilon > 0 \), and for every \( (x, t) \in \overline{\Omega} \times (0, \infty) \),

- \((H_1)\) \( f(x, 0) = 0, \ f(x, t) > 0, \ f(x, t)t^{n-1} = f(x, -t)(-t)^{n-1} \);
- \((H_2)\) \( f'(x, t) > \frac{f(x, t)}{t^\alpha} \), where \( f'(x, t) = \frac{\partial f}{\partial t}(x, t) \);
- \((H_3)\) \( F(x, t) \leq M(1 + f(x, t)t^{n-2+\alpha}) \), where

\[ F(x, t) = \int_{0}^{t} f(x, s)s^{n-2} \, ds \]

is the primitive of \( f \);

- \((H_4)\) \( \limsup_{t \to \infty} \sup_{x \in \Omega} h(x, t) \exp \left( -\epsilon t^{n/(n-1)} \right) = 0 \),

\[ \liminf_{t \to \infty} h(x, t) \exp \left( \epsilon t^{n/(n-1)} \right) = \infty. \]

Let \( A(\Omega) \) denote the set of all functions of critical growth on \( \Omega \).

**EXAMPLES.** In view of \((H_1)\), it is enough to define \( f \) on \( \overline{\Omega} \times (0, \infty) \).

1) For \( m > 1, \ b > 0, \ \beta \geq 0 \) and \( 0 \leq \alpha < \frac{n}{n-1}, \ f(x, t) = t^m \exp(\beta t^\alpha) \exp \left( bt^{n/(n-1)} \right) \) is in \( A(\Omega) \).

2) \( f(x, t) = t^2 e^{-t} \exp \left( t^{n/(n-1)} \right) \) is in \( A(\Omega) \).

3) Let \( f(x, t) = h(x, t) \exp \left( bt^{n/(n-1)} \right), \) satisfying \((H_1)\) and \((H_4)\).
Further assume that \( h'(x, t) \geq \frac{h(x, t)}{t} \) for \((x, t) \in \overline{\Omega} \times (0, \infty)\). Then \( f \) is in \( A(\Omega) \).

For
\[
\frac{f'(x, t)}{f(x, t)} = \frac{h'(x, t)}{h(x, t)} + \frac{nb}{n - 1} t^{1/(n-1)} > \frac{1}{t},
\]
and hence \( f \) satisfy (H2).

Let \( \varepsilon > 0 \), and \( \sigma = \frac{1}{n - 1} \)

\[
F(x, t) - F(x, \varepsilon) = \frac{n - 1}{n b} \int_{\varepsilon}^{t} h(x, s) s^{n-1-\sigma} \frac{d}{ds} \exp \left( bs^{n/(n-1)} \right) ds
\]

\[
\leq \frac{n - 1}{n b} \left[ f(x, t) t^{n-2-\sigma} - f(x, \varepsilon) \varepsilon^{n-2-\sigma} \right].
\]

This implies that there exists a constant \( M > 0 \) such that \( F(x, t) \leq M[1 + f(x, t) t^{n-2-\sigma}] \) for \((x, t) \in \overline{\Omega} \times (0, \infty)\). This shows that \( f \) satisfy (H3) and hence \( f \in A(\Omega) \).

Let \( W_{0}^{1,n}(\Omega) \) be the usual Sobolev space and \( f(x, t) = h(x, t) \exp \left( b t^{n/(n-1)} \right) \) be in \( A(\Omega) \). For \( u \in W_{0}^{1,n}(\Omega) \), define

\[
\| u \|^n = \int_{\Omega} |\nabla u|^n dx
\]

(2.1)

\[
J(u) = \frac{1}{n} \| u \|^n - \int_{\Omega} F(x, u) dx
\]

(2.2)

\[
I(u) = \frac{1}{n} \int_{\Omega} f(x, u) u^{n-1} dx - \int_{\Omega} F(x, u) dx
\]

(2.3)

\[
\partial B(\Omega, f) = \left\{ u \in W_{0}^{1,n}(\Omega) \setminus \{0\}; \| u \|^n = \int_{\Omega} f(x, u) u^{n-1} dx \right\}
\]

(2.4)

\[
\frac{a(\Omega, f)^n}{n} = \inf \{ J(u); u \in \partial B(\Omega, f) \}
\]

(2.5)

\[
\lambda_1(\Omega) = \inf \left\{ \| u \|^n; \int_{\Omega} |u|^n dx = 1 \right\}
\]

\[
\alpha_n = n \omega_n^{1/(n-1)}, \text{ where } \omega_n \text{ = Volume of } S^{n-1}.
\]
DEFINITION OF MOSER FUNCTIONS. Let $x_0 \in \Omega$ and $R \leq d(x_0, \partial \Omega)$, where $d$ denotes the distance from $x_0$ to $\partial \Omega$. For $0 < \ell < R$, define

$$m_{\ell, R}(x, x_0) = \begin{cases} 
\left( \log \frac{R}{\ell} \right)^{1 - \frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\
\frac{-\log a}{\left( \log \frac{R}{\ell} \right)^{1/n}} & \text{if } \ell \leq r = |x - x_0| \leq R \\
0 & \text{if } |x - x_0| \geq R.
\end{cases}$$

Then it is easy to see that $m_{\ell, R} \in W^{1,1}_0(\Omega)$ and $\|m_{\ell, R}\| = 1$.

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

**Theorem 2.1 (Moser).** 1) Let $u \in W^{1,1}_0(\Omega)$, and $p < \infty$, then

$$\exp \left( |u|^{n/(n-1)} \right) \in L^p(\Omega).$$

2) \(\left( \frac{\alpha}{b} \right)^{n-1} = \max \left\{ e^n; \sup_{|w| \leq 1} \int_{\Omega} \exp \left( \frac{b}{n} |w|^{n/(n-1)} \right) dx < \infty \right\}.\)

**Theorem 2.2 (P.L. Lions).** Let \(\{u_k; \|u_k\| = 1\}\) be a sequence in \(W^{1,1}_0(\Omega)\) converging weakly to a non-zero function $u$. Then, for every $p < (1 - \|u\|^{n/(n-1)})^{-1}$,

$$\sup_k \int_{\Omega} \exp \left( p \frac{\alpha}{b} |u_k|^{n/(n-1)} \right) dx < \infty.$$

3. - Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

**Lemma 3.1.** Let $f \in A(\Omega)$. Then we have

1) If $u \in W^{1,1}_0(\Omega)$, then $f(x, u) \in L^p(\Omega)$ for all $p \geq 0$.

2) \(\left( \frac{\alpha}{b} \right)^{n-1} = \sup \left\{ e^n; \sup_{|w| \leq 1} \int_{\Omega} f(x, cw) w^{n-1} dx < \infty \right\}.\)

3) Let \(\{u_k\}\) and \(\{v_k\}\) be bounded sequences in \(W^{1,1}_0(\Omega)\) converging weakly and for almost every $x$ in $\Omega$ to $u$ and $v$ respectively. Further assume that

$$\lim_{k \to \infty} \|u_k\|^n < \left( \frac{\alpha}{b} \right)^{n-1}.$$

Then, for every integer $\ell \geq 0$,

$$\lim_{k \to \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k dx = \int_{\Omega} \frac{f(x, u)}{u} v dx.$$
4) Let \( \{u_k\} \) be a sequence in \( W^{1,n}_0(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to \( u \), such that
\[
\sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty.
\]
Then, for any \( 0 \leq \tau < 1 \),
\[
\lim_{k \to \infty} \int_{\Omega} f(x, |u_k|) |u_k|^{n-2+\tau} \, dx = \int_{\Omega} f(x, |u|) |u|^{n-2+\tau} \, dx,
\]
\[
\lim_{k \to \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.
\]

5) \( I(u) \geq 0 \) for all \( u \) and \( I(u) = 0 \) iff \( u \equiv 0 \). Further, there exists a constant \( M_1 > 0 \) such that, for all \( u \in W^{1,n}_0(\Omega) \),
\[
\int_{\Omega} f(x, u) u^{n-1} \, dx \leq M_1(1 + I(u)).
\]

**LEMMA 3.2.** Let \( f = h \exp \left( b|t|^{n/(n-1)} \right) \in A(\Omega) \) and define
\[
h_0(t) = \inf_{x \in \Omega} h(x, t), \quad M_0 = \sup_{t \geq 0} h_0(t)t^{n-1}, \quad R_0 = \sup_{x \in \partial \Omega} d(x, \partial \Omega),
\]
and
\[
k_0 = \begin{cases} \left( \frac{n}{M_0} \right)^{n/(n-1)} M_0^{1/(n-1)} & \text{if } M_0 < \infty \\ 0 & \text{if } M_0 = \infty \end{cases}
\]

Let \( a \geq 0 \) be such that
\[
\sup_{\|w\| \leq 1} \int_{\Omega} f(x, aw)w^{n-1} \, dx \leq a.
\]

If \( \frac{k_0}{b} < 1 \), then \( a^n < \left( \frac{a_n}{b} \right)^{n-1} \).

**PROOF.** From 2) of lemma 3.1, we have \( a^n \leq \left( \frac{a_n}{b} \right)^{n-1} \). Suppose \( a^n = \left( \frac{a_n}{b} \right)^{n-1} \). Let \( x_0 \in \Omega \) such that \( d(x_0, \partial \Omega) = R_0 \) and \( 0 < t < R_0 \). Let
\[
m_{t}(x) = m_{t,R_0}(x, x_0).
\]
be the Moser functions and
\[ t = a \omega_n^{-1/n} \left( \log \frac{R_0}{\ell} \right)^{(n-1)/n}. \]
then from (3.1) we have
\[ a \geq \int_{\Omega} f(x, am_{\ell}) m_{\ell}^{n-1} \, dx \]
\[ \geq \int_{B(x_0, \ell)} h_0(am_{\ell}) m_{\ell}^{n-1} \exp \left( ba^{n/(n-1)} m_{\ell}^{n/(n-1)} \right) \, dx \]
\[ = \frac{h_0(t)t^{n-1} \omega_n R_0^n}{na^{n-1}}. \]
This implies that
\[ \left( \frac{\alpha_n}{b} \right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1} \omega_n R_0^n}{n}. \]
That is, for all \( t \in (0, \infty) \),
\[ b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)} \]
and hence
\[ b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \inf_{t \geq 0} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)} \leq k_0 \]
which contradicts the hypothesis \( b > k_0 \). Hence \( a^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \) and this proves the lemma.

**Lemma 3.3. (Compactness Lemma).** Let \( f \) be in \( A(\Omega) \) and \( \{u_k\} \) be a sequence in \( W_0^{1,n}(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to a non-zero function \( u \). Further, assume that

(i) \( \text{There exists } C \in \left( 0, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \right] \text{ such that } \lim_{k \to \infty} J(u_k) = C; \)

(ii) \( \|u\|^n \geq \int_{\Omega} f(x, u)u^{n-1} \, dx; \)

(iii) \( \sup_{k} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx < \infty; \)

then
\[ \lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx = \int_{\Omega} f(x, u)u^{n-1} \, dx. \]
PROOF. From 5) of lemma 3.1, \( I(u) > 0 \). Therefore, from (ii) we have \( J(u) \geq I(u) > 0 \) and \( J(u) \leq \lim_{k \to \infty} J(u_k) = C \). Hence we can choose an \( \varepsilon > 0 \) such that

\[
(C - J(u)) (1 + \varepsilon)^{n-1} < \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

Let \( \beta = \int F(x, u) \, dx \). Then, from (iii) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
= n(C + \beta).
\]

From (3.2) and (3.3) we can choose a \( k_0 > 0 \) such that, for all \( k \geq k_0 \),

\[
(1 + \varepsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n < \frac{C + \beta}{C - J(u)} = \left( 1 - \frac{\|u\|}{n(C + \beta)} \right)^{-1}.
\]

Now choose \( p \) such that

\[
(1 + \varepsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n \leq p^{n-1} < \frac{C + \beta}{C - J(u)}.
\]

Applying theorem 2.2 to the sequence \( \frac{u_k}{\|u_k\|} \) and using (3.3) and (3.5), we have

\[
\sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

From (3.5) and (3.6), we have

\[
\sup_k \int_{\Omega} \exp \left( (1 + \varepsilon)^{n-1} b\|u_k\|^{n/(n-1)} \right) \, dx
\leq \sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

Let

\[
M_1 = \sup_{(x, t) \in \Omega \times \mathbb{R}} |h(x, t)t^{n-1}| \exp \left( -\frac{b}{2} |t|^{n/(n-1)} \right).
\]
and \( N > 0 \). Then from (3.7) we have

\[
\int_{|u_k| \geq N} f(x, u_k) u_k^{n-1} \, dx = \int_{|u_k| \geq N} h(x, u_k) u_k^{n-1} \exp \left( b|u_k|^{\eta/(n-1)} \right) \, dx \\
\leq M_1 \int_{|u_k| \geq N} \exp \left( -\frac{b}{2}|u_k|^{\eta/(n-1)} \right) \exp \left[ (1 + \epsilon)b|u_k|^{\eta/(n-1)} \right] \, dx \\
= O \left( \exp \left( -\frac{b}{2} N^{\eta/(n-1)} \right) \right). 
\]

Hence

\[
\int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = \int_{|u_k| \leq N} f(x, u_k) u_k^{n-1} \, dx + O \left( \exp \left( -\frac{b}{2} N^{\eta/(n-1)} \right) \right). 
\]

Now letting \( k \to \infty \), and \( N \to \infty \) in the above equation, we obtain

\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = \int_{\Omega} f(x, u) u^{n-1} \, dx.
\]

This proves the lemma.

**LEMMA 3.4.** Let \( f \in A(\Omega) \) and assume that

(i) \( \lim_{t \to \infty} h_0(t) t^{n-1} = \infty \),

where \( h_0(t) = \inf_{x \in \Omega} h(x,t) \);

(ii) \( \sup_{x \in \Omega} f'(x,0) < \lambda_1(\Omega) \);

then

\[
0 < a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1}. 
\]

**PROOF.** The lemma is proved in several steps.

**STEP 1.** \( a(\Omega, f) > 0 \).

Suppose \( a(\Omega, f) = 0 \). Then there exists a sequence \( \{u_k\} \) in \( \partial B(\Omega, f) \) such that \( J(u_k) \to 0 \) as \( k \to \infty \). Since \( J(u_k) = I(u_k) \), hence from (5) of lemma 3.1

\[
\sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty \\
\sup_k \|u_k\|^n < \infty.
\]
Then, by extracting a subsequence, we can assume that \( \{u_k\} \) converges weakly and for almost every \( x \) in \( \Omega \) to a function \( u \). Now by Fatou's lemma,

\[
0 \leq I(u) \leq \lim_{k \to \infty} I(u_k) = \lim_{k \to \infty} J(u_k) = 0.
\]

Hence \( u \equiv 0 \). From (3.9) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

Let \( v_k = \frac{u_k}{\|u_k\|} \) and converging weakly to \( v \). Using \( u_k \in \partial B(\Omega, f) \), (3.12), 3) of lemma 3.1 and (ii), we have

\[
1 = \lim_{k \to \infty} \int_{\Omega} \frac{f(x, u_k)}{u_k} v_k^n \, dx
\]

\[
= \int_{\Omega} f'(x, 0) u^n \, dx < \lambda_1(\Omega) \int_{\Omega} v^n \, dx \leq 1,
\]

which is a contradiction. This proves step 1.

**Step 2.** For every \( u \in W^{1,n}_0(\Omega) \setminus \{0\} \), there exists a constant \( \gamma > 0 \) such that \( \gamma u \in \partial B(\Omega, f) \). Moreover, if

\[
\|u\|^n \leq \int_{\Omega} f(x, u) u^{n-1} \, dx,
\]

then \( \gamma \leq 1 \) and \( \gamma = 1 \) iff \( u \in \partial B(\Omega, f) \).

For \( \gamma > 0 \), define

\[
\psi(\gamma) = \frac{1}{\gamma} \int_{\Omega} f(x, \gamma u) u^{n-1} \, dx.
\]

Then, from 3) of lemma 3.1 and (ii), we have

\[
\lim_{\gamma \to 0} \psi(\gamma) = \int_{\Omega} f'(x, 0) u^n \, dx < \|u\|^n,
\]

\[
\lim_{\gamma \to \infty} \psi(\gamma) = \infty.
\]

Hence there exists \( \gamma > 0 \) such that \( \psi(\gamma) = \|u\|^n \); this implies that \( \gamma u \in \partial B(\Omega, f) \). From \((H_1)\) and \((H_2)\), it follows that \( \frac{f(x, u)}{t} u^{n-1} \) is an
increasing function for $t > 0$. Hence, if $u$ satisfies (3.13), it follows that $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$. This proves step 2.

**STEP 3.** $a(\Omega, f)^n < \left( \frac{\alpha_{n}}{b} \right)^{n-1}$.

Let $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$. From step 2, we can choose a $\gamma > 0$ such that $\gamma w \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma w) \leq \frac{n}{n} \|w\|^n = \frac{n}{n};$$

this implies that $a(\Omega, f) \leq \gamma$. Using again the fact that $\int \frac{f(x, tw)}{t} w^{n-1} \, dx$ is an increasing function of $t$ in $(0, \infty)$ and $\gamma w \in \partial B(\Omega, f)$, we have

$$\int_{\Omega} \frac{f(x, a(\Omega, f) w)}{a(\Omega, f)^n} w^{n-1} \, dx \leq \int_{\Omega} \frac{f(x, \gamma w)}{\gamma} w^{n-1} \, dx = 1.$$  

This implies that

$$(3.14) \quad \sup_{\|w\| \leq 1} \int_{\Omega} f(x, a(\Omega, f) w) w^{n-1} \, dx \leq a(\Omega, f).$$

Now from (i), (3.14) and lemma 3.2 we have $a(\Omega, f)^n < \left( \frac{\alpha_{n}}{b} \right)^{n-1}$. This proves the lemma.

**LEMMA 3.5.** Let $f \in A(\Omega)$ and $u_0 \in \partial B(\Omega, f)$ such that $J'(u_0) \neq 0$ ($J'(u)$ denote the derivative of $J$ at $u$). Then

$$J(u_0) > \inf\{J(u); u \in \partial B(\Omega, f)\}.$$  

**PROOF.** Choose $h_0 \in W_0^{1,n}(\Omega)$ such that $(J'(u_0), h_0) = 1$ and, for $\alpha, t \in \mathbb{R}$, define $\sigma_t(\alpha) = \alpha u_0 - t h_0$. Then

$$\lim_{t \to 0} \frac{d}{dt} J(\sigma_t(\alpha)) = -(J'(u_0), h_0) = -1$$

and hence we can choose $\epsilon > 0, \delta > 0$ such that, for all $\alpha \in [1 - \epsilon, 1 + \epsilon]$ and $0 < t \leq \delta$,

$$(3.15) \quad J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0).$$

Let $\rho_t(\alpha) = \|\sigma_t(\alpha)\|^n - \int_{\Omega} f(x, \sigma_t(\alpha)) \sigma_t(\alpha)^{n-1} \, dx$. 


Since \( f(x, \alpha u_0) u_0^{n-1} \) is an increasing function of \( \alpha \) and using \( u_0 \in \partial B(\Omega, f) \), by shrinking \( \epsilon \) and \( \delta \) if necessary, we have, for \( 0 < t \leq \delta \), \( \rho_t(1 - \epsilon) > 0 \) and \( \rho_t(1 + \epsilon) < 0 \). Hence there exists \( \alpha_t \) such that \( \rho_t(\alpha_t) = 0 \). Therefore \( \sigma_t(\alpha_t) \) is in \( \partial B(\Omega, f) \). Hence from (3.15) we have

\[
\inf\{J(u); \ u \in \partial B(\Omega, f)\} \leq J(\sigma_t(\alpha_t))
\]

\[
< J(\alpha_t u_0) \leq \sup_{t \in \mathbb{R}} J(tu_0) = J(u_0).
\]

This proves the lemma.

**Proof of the Theorem.**

1) **Palais-Smale Condition.** Let \( C \in (-\infty, \frac{1}{n} \left( \frac{n}{a} \right)^{n-1}) \) and \( \{u_k\} \) be a sequence such that

\[
\lim_{k \to \infty} J(u_k) = C
\]

(3.16)

\[
\lim_{k \to \infty} J'(u_k) = 0.
\]

Let \( h \in W_0^{1,n}(\Omega) \), then we have

\[
(J'(u_k), h) = \int_\Omega |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla h \, dz - \int_\Omega f(x, u_k) u_k^{n-2} h \, dz.
\]

Hence we have

\[
J(u_k) - \frac{1}{n} (J'(u_k), u_k) = I(u_k).
\]

**CLAIM 1.**

\[
\sup_k \|u_k\| + \sup_k \int_\Omega f(x, u_k) u_k^{n-1} \, dx < \infty.
\]

Since \( \{J(u_k)\} \) and \( \{J'(u_k)\} \) are bounded and hence from (3.19), \( I(u_k) = O(\|u_k\|) \). Now from 5) of lemma 3.1, we have \( \int_\Omega f(x, u_k) u_k^{n-1} \, dx = O(\|u_k\|) \).

Now from \( (H_3) \) it follows that

\[
\int \Omega F(x, u_k) \, dx = O(\|u_k\|)
\]

and, by using the boundedness of \( J(u_k) \), we obtain \( \|u_k\|^n = O(\|u_k\|) \). This implies (3.20) and hence the claim.
By extracting a subsequence, we can assume that

\( u_k \to u_0 \) weakly and for almost all \( x \) in \( \Omega \).

(3.21)

**CASE (I).** \( C \leq 0 \).

From Fatou’s lemma and 5) of lemma 3.1, we have

\[
0 \leq I(u_0) \leq \lim_{k \to \infty} I(u_k)
\]

\[
= \lim_{k \to \infty} \left\{ J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle \right\}
\]

\[
= C.
\]

Hence \( u_0 \equiv 0 \). If \( C < 0 \), no Palais-Smale sequence exists. If \( C = 0 \), then from (3.20) and 4) of lemma 3.1 we have

\[
\lim_{k \to \infty} \| u_k \|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

This proves that \( u_k \to 0 \) strongly.

**CASE (II).** \( C \in \left( 0, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \right) \).

**CLAIM 2.** \( u_0 \neq 0 \) and \( u_0 \in \partial B(\Omega, f) \).

Suppose \( u_0 \equiv 0 \). Then, from (3.20) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \| u_k \|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
\]

(3.22)

\[
= nC < \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

Hence, from 3) of lemma 3.1 and (3.22), we have

\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx = \int_{\Omega} f(x, u_0)u_0^{n-1} \, dx = 0.
\]

This implies that \( \lim_{k \to \infty} I(u_k) = 0 \) and hence from (3.19)

\[
0 < C = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} = 0
\]
which is a contradiction. Hence \( u_0 \neq 0 \). From (3.20) and 4) of lemma 3.1, taking \( h \in C_0^\infty(\Omega) \) and letting \( k \to \infty \) in (3.19), we obtain
\[
\int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_\Omega f(x, u_0) u_0^{n-2} h \, dx.
\]

By density, the above equation holds for all \( h \in W_0^{1,n}(\Omega) \). Hence, by taking \( h = u_0 \), we obtain
\[
\|u_0\|^n = \int_\Omega f(x, u_0) u_0^{n-1} \, dx.
\]

Hence \( u_0 \in \partial B(\Omega, f) \) and this proves the claim.

Now from (3.20) and claim 2, \( \{u_k, u_0\} \) satisfy all the hypotheses of the compactness lemma 3.3. Hence we have
\[
\|u_0\|^n = \lim_{k \to \infty} \|u_k\|^n
\]

This implies that \( u_k \) converges to \( u_0 \) strongly. This proves the Palais-Smale condition.

2) Existence of Positive Solution. Since the critical points of \( J \) are the solutions of the equation (1.7) and \( J(u) = J(|u|) \) for all \( u \) in \( \partial B(\Omega, f) \) and hence in view of lemma 3.5, it is enough to prove that there exists \( u_0 \neq 0 \) such that
\[
\frac{a(\Omega, f)^n}{n} = J(u_0).
\]

Let \( u_k \in \partial B(\Omega, f) \) such that
\[
\lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.
\]
Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

\begin{equation}
\sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty,
\end{equation}

and

\begin{equation}
\sup_k \|u_k\| < \infty.
\end{equation}

Hence we can extract a subsequence such that

$u_k \rightharpoonup u_0$ weakly and for almost all $x$ in $\Omega$.

\textbf{CLAIM 3.} $u_0 \neq 0$ and

\begin{equation}
\|u_0\|^n \leq \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx.
\end{equation}

Suppose $u_0 \equiv 0$, then from (3.25) and 4) of lemma 3.1

\begin{equation}
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
= a(\Omega, f)^n.
\end{equation}

From lemma 3.4, we have $0 < a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

\[ \lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = 0. \]

This implies that

\[ 0 < \frac{a(\Omega, f)^n}{n} = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} I(u_k) = 0, \]

which is a contradiction. This proves $u_0 \neq 0$. Suppose (3.28) is false, then

\begin{equation}
\|u_0\|^n > \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx.
\end{equation}

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1}$, \{u_k, u_0\} satisfy all the hypotheses of lemma 3.3. Hence

\[ \lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx. \]
This implies that
\[ \|u_0\|_n^n \leq \lim_{k \to \infty} \|u_k\|_n^n = \lim_{k \to \infty} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx \]
\[ = \int_{\Omega} f(x, u_0) u_0^{n-1} \, dx. \]
contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists \(0 < \gamma \leq 1\) such that \(\gamma u_0 \in \partial B(\Omega, f)\). Hence
\[ \frac{a(\Omega, f)^n}{n} \leq J(\gamma u_0) = J(\gamma u_0) \]
\[ \leq J(u_0) \leq \lim_{k \to \infty} J(u_k) \]
\[ = \lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}. \]

This implies that \(\gamma = 1\) and \(u_0 \in \partial B(\Omega, f)\). Hence \(J(u_0) = \frac{a(\Omega, f)^n}{n}\) and this proves the Theorem.

4. Concluding Remarks

REMARK 4.1. (Regularity). From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in \(C^{1,\alpha}(\Omega)\) if \(n > 3\) and in \(C^{2,\alpha}(\Omega)\) if \(n = 2\).

REMARK 4.2. Let \(f \in A(\Omega)\) and \(f'(x, 0) < \lambda_1(\Omega)\) for all \(x \in \overline{\Omega}\). We prove the existence of a solution for (1.7) under the assumption that
\[ \lim_{t \to \infty} \inf_{x \in \overline{\Omega}} h(x, t)t^{n-1} = \infty. \]

The only place where it is used is to show that \(a(\Omega, f)^n < \left(\frac{a_n}{b}\right)^{n-1}\). But, from lemma 3.2, this inequality holds if
\[ \frac{k_0}{b} < 1. \]

Hence the theorem is true under the less restrictive condition (4.2).
Now the question is what happens if \( \lambda_1 \geq 1 \) or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions \( f \in A(\Omega) \) such that

\[
\liminf_{t \to \infty, x \in \Omega} h(x, t)t^{n-1} = 0
\]

for which no solution to problem (1.7) exists if \( \Omega \) is a ball of sufficiently small radius. In this context, we raise the following question:

Open Problem. Let \( \Omega \) be a ball and \( f \in A(\Omega) \) such that \( \sup_{x \in \Omega} f'(x, 0) \leq \lambda_1(\Omega) \). Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case \( n = 2 \), this question is related to the question of Brezis [3]: "where is the border line between the existence and non-existence of a solution of (1.7)?"

Remark 4.3. Let \( \lambda_1 > 0 \), then by using the norm

\[
\left( \int_{\Omega} |\nabla u|^n \, dx + \beta \int_{\Omega} |u|^n \, dx \right)^{1/n}
\]

in \( W_{0}^{1,n}(\Omega) \), the Theorem still holds if we replace \( -\Delta u \) by \( -\Delta u + \beta|u|^{n-2}u \) in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

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5. Appendix

Proof of the Lemma 3.1.

1) Let \( f(x, t) = h(x, t) \exp \left( b|t|^{n/(n-1)} \right) \in A(\Omega) \). From (H4), for every \( \epsilon > 0 \), there exists a \( C(\epsilon) > 0 \) such that

\[
|f(x, t)| \leq C(\epsilon) \exp \left( (b + \epsilon)|t|^{n/(n-1)} \right)
\]

and hence, from theorem 2.1, \( f(x, u) \in L^p(\Omega) \) for every \( p < \infty \).

2) From (H4), for every \( \epsilon > 0 \), there exist positive constants \( C_1(\epsilon) \) and \( C_2(\epsilon) \) such that

\[
|f(x, t)t^{n-1}| \leq C_1(\epsilon) \exp \left( b(1 + \epsilon)|t|^{n/(n-1)} \right)
\]
Hence, if \( c > 0 \) such that
\[
\sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1}\,dz < \infty,
\]

it implies that, for every \( \epsilon > 0 \),
\[
\sup_{\|w\| \leq 1} \int_{\Omega} \exp \left( b(1 - \epsilon)w^{n/(n-1)} \right) dz < \infty.
\]

Therefore, from Theorem 2.1, we have
\[
(1 - \epsilon)^{n-1} c^n \leq \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

This implies that
\[
\sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1}\,dz < \infty \right\} \leq \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

On the other hand, if \( c^n \leq \left( \frac{\alpha_n}{b} \right)^{n-1} \), then by choosing \( \epsilon > 0 \) such that
\[
(1 + \epsilon)^{2n-1} c^n < \left( \frac{\alpha_n}{b} \right)^{n-1},
\]
from Theorem 2.1 and from (5.1), we have
\[
\sup_{\|w\| \leq 1} \int_{\Omega} f(x, (1 + \epsilon)cw)w^{n-1}\,dz
\]
\[
\leq C_1(\epsilon) \sup_{\|w\| \leq 1} \int_{\Omega} \exp \left[ b \left( (1 + \epsilon)c|w| \right)^{n/(n-1)} \right] dz < \infty
\]
this proves
\[
\sup \left\{ c^n; \sup_{\|w\| \leq 1} \int_{\Omega} f(x, cw)w^{n-1}\,dz < \infty \right\} = \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

3) Since \( \lim_{k \to \infty} \|u_k\|^n < \left( \frac{\alpha_n}{b} \right)^{n-1}, \) from 2) we can choose a \( p > 1 \) such that
\[
c_1^p = \sup_{k} \int_{\Omega} |f(x, u_k)|^p\,dx < \infty.
\]
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and

Then, for any $N > 0$ and by Holder’s inequality,

$$
\left| \int_{|u_k|>N} f(x, u_k) v_k \, dx \right| \leq \frac{1}{N} \int_{\Omega} |f(x, u_k)| |v_k| \, dx \leq \frac{c_1 c_2}{N}.
$$

Hence

$$
\int_{\Omega} f(x, u_k) v_k \, dx = \int_{|u_k| \leq N} f(x, u_k) v_k \, dx + O(1/N).
$$

By dominated convergence theorem, letting $k \to \infty$ and then $N \to \infty$ in the above equation, it implies that

$$
\lim_{k \to \infty} \int_{\Omega} f(x, u_k) v_k \, dx = \int_{\Omega} f(x, u) v \, dx.
$$

4) Let $N > 0$, then

$$
\int_{|u|>N} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx \leq \frac{1}{N^{1-\tau}} \int_{\Omega} f(x, |u_k|)|u_k|^{n-1} \, dx
$$

$$
= \frac{1}{N^{1-\tau}} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = O \left( \frac{1}{N^{1-\tau}} \right).
$$

Hence

$$
\int_{\Omega} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx = \int_{|u| \leq N} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx + O \left( \frac{1}{N^{1-\tau}} \right).
$$

By dominated convergence theorem, letting $k \to \infty$ and $N \to \infty$ in the above equation, we obtain

$$
(5.3) \quad \lim_{k \to \infty} \int_{\Omega} f(x, |u_k|)|u_k|^{n-2+\tau} \, dx = \int_{\Omega} f(x, |u|)|u|^{n-2+\tau} \, dx.
$$

Now from $(H_3)$,

$$
|F(x, t)| \leq M(1 + |f(x, t)| |t|^{n-2+\tau}).
$$
for some $u \in [0, 1)$. Hence, from (5.3) and the dominated convergence theorem,

$$\lim_{k \to \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.$$  

5) From (H2) we have, for $t > 0,$

$$(5.4) \quad \frac{\partial}{\partial t} \left[ f(x, t) t^{n-1} - nF(x, t) \right] = \left[ f'(x, t) - \frac{f(x, t)}{t} \right] t^{n-1} > 0.$$  

Therefore from (H1) and (5.4), $f(x, t) t^{n-1} - nF(x, t)$ is an even positive function and increasing for $t > 0$. This implies that $I(u) \geq 0$ and $I(u) = 0$ iff $u \equiv 0$. From (H3) we have

$$nI(u) = \int_{\Omega} \left[ f(x, u) u^{n-1} - nF(x, u) \right] \, dx$$

$$\geq \int_{\Omega} \left[ f(x, u) u^{n-1} - nM(1 + |f(x, u)| |u|^{n-2+\epsilon}) \right] \, dx$$

$$\geq C_1 + \frac{1}{2} \int_{|u| \geq C_2} f(x, u) u^{n-1} \, dx$$

for some constants $C_1$ and $C_2 > 0$. This implies that there exists a constant $M_1 > 0$ such that

$$\int_{\Omega} f(x, u) u^{n-1} \, dx \leq M(1 + I(u)).$$

This proves the lemma 3.1.

REFERENCES


Di Benedetto, $C^{1, \alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Analysis - TMA, No. 8, 7 (1983), pp. 827-850.


Z. Nehari, On a class of non-linear second order differential equations, Trans AMS, 95 (1960), pp. 101-123.

P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, Jr diff. eqs, 51 (1984), pp. 126-150.


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