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An Elementary Treatment of a General Diophantine Problem

NIGEL WATT

1. - Introduction

This paper is concerned with the order of magnitude of $I_8(H, \Delta)$, the number of integer solutions $(h_1, \dots, h_4, k_1, \dots, k_4)$ of

$$(1) \quad \sum_{j=1}^4 (h_j^2 - k_j^2) = \sum_{j=1}^4 (h_j - k_j) = 0$$

and

$$(2) \quad \left| \sum_{j=1}^4 \left[g\left(\frac{h_j}{H}\right) - g\left(\frac{k_j}{H}\right) \right] \right| < \Delta,$$

with

$$(3) \quad H \leq h_j, k_j < 2H, \text{ for } j = 1, \dots, 4,$$

where H is a positive integer, $\Delta > 0$ and g is a fixed complex-valued function defined on the interval $[1, 2]$. The following result is obtained.

THEOREM. *If g is analytic on some open subset V of \mathbb{C} containing the interval $[1, 2]$ and*

$$(4) \quad g^{(3)}(x)g^{(5)}(x) \neq (g^{(4)}(x))^2, \text{ for } 1 \leq x \leq 2,$$

then

$$(5) \quad I_8(H, \Delta) \ll H^4 + \Delta H^5 \log^3 H.$$

Interest in $I_8(H, \Delta)$ goes back to 1985, when E. Bombieri and H. Iwaniec showed, in [2], that if $g^{(1)}(x) = x^\beta$, for $1 \leq x \leq 2$, where β is some real

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constant other than 0 or 1, then

$$(6) \quad I_8(H, \Delta) \ll_{\epsilon} (1 + \Delta H) H^{4+\epsilon}, \text{ for } \epsilon > 0.$$

Later I obtained the bound (5) for $\beta = \frac{1}{2}$ (see [5, Theorem 1]).

The bounds for $\beta = \frac{1}{2}$ are important in applications to exponential sums. In [1] Bombieri and Iwaniec used the bound (6), for $\beta = \frac{1}{2}$, to show that $\zeta\left(\frac{1}{2} + it\right) \ll_{\epsilon} t^{\epsilon+9/56}$, for $t > \epsilon > 0$. The bound (5), for $\beta = \frac{1}{2}$, has been applied in [3] and [4].

To prove (6) Bombieri and Iwaniec treated $I_8(H, \Delta)$ as a mean-value of an exponential sum and used an ingenious argument involving Poisson summation. In this paper the more elementary techniques of [5] are adapted to generalize [5, Theorem 1]. The resulting Theorem (above) does not generalize the bound (6) of Bombieri and Iwaniec, as the condition (4) is not satisfied for $\beta = 2$.

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2. - Preliminary results and definitions

The positive number $(2^{\frac{1}{4}} - 1)^{-1}$, which we shall henceforth refer to as α , occurs naturally in the next Lemma. It satisfies the equation

$$\left(1 + \frac{1}{\alpha}\right)^4 = 2.$$

Now

$$\left(1 + \frac{1}{6}\right)^4 < 2 < \left(1 + \frac{4}{21}\right)^4,$$

so

$$(7) \quad \frac{21}{4} < \alpha < 6.$$

Also

$$\alpha^4 = 4\alpha^3 + 6\alpha^2 + 4\alpha + 1,$$

so the sequence $\{\alpha^n\}$ is generated by the recurrence formula,

$$(8) \quad \alpha^r = 4\alpha^{r-1} + 6\alpha^{r-2} + 4\alpha^{r-3} + \alpha^{r-4}.$$

Starting from the lower bounds $0, \frac{1}{6}, 1$ and $\frac{21}{4}$ for $\alpha^{-2}, \alpha^{-1}, \alpha^0$ and α (respec-

tively), we use the recurrence formula to produce the lower bounds

$$(9) \quad \frac{83}{3}, \frac{439}{3}, \frac{2320}{3}, \frac{16349}{4}, 21602, \frac{685027}{6}, 603419 \text{ and } \frac{12756793}{4}$$

for $\alpha^2, \dots, \alpha^9$ (respectively).

LEMMA 1. Let $\delta \geq 0$ and $K > 0$. For $r = 1, \dots, 4$, let

$$|x_r| \leq K, |y_r| \leq K \text{ and } \left| \sum_{j=1}^4 (x_j^r - y_j^r) \right| \leq \delta K^r.$$

Then, for $i = 1, \dots, 4$,

$$(10) \quad \left| \prod_{j=1}^4 (x_i - y_j) \right| \leq \frac{147}{4} \delta K^4$$

and, for $r = 5, 6, 7, \dots$,

$$(11) \quad \left| \sum_{j=1}^4 (x_j^r - y_j^r) \right| \leq \frac{1}{25} \delta (\alpha K)^r.$$

PROOF. It is sufficient to prove these results for $\delta > 0$, since the case $\delta = 0$ then follows by letting δ tend to zero from above.

For $r = 1, 2, 3, \dots$, let

$$S_r = \sum_{j=1}^4 x_j^r \quad \text{and} \quad T_r = \sum_{j=1}^4 y_j^r.$$

Let E_0, \dots, E_4 be given by the polynomial expansion,

$$(X - x_1) \cdots (X - x_4) = E_0 X^4 + E_1 X^3 + \cdots + E_4.$$

Define F_0, \dots, F_4 similarly, but in terms of y_1, \dots, y_4 . Then

$$S_r + S_{r-1} E_1 + \cdots + S_1 E_{r-1} + r E_r = 0 = T_r + T_{r-1} F_1 + \cdots + T_1 F_{r-1} + r F_r,$$

for $r = 1, \dots, 4$. Therefore, for $R = 1, \dots, 4$,

$$\begin{aligned} R|E_R - F_R| &\leq \sum_{r=1}^R |S_r E_{R-r} - T_r F_{R-r}| \\ &\leq \sum_{r=1}^R [|S_r (E_{R-r} - F_{R-r})| + |F_{R-r} (S_r - T_r)|] \\ &\leq \sum_{r=1}^R \left[4K^r |E_{R-r} - F_{R-r}| + \binom{4}{R-r} K^{R-r} \delta K^r \right]. \end{aligned}$$

Starting from the fact that $E_0 = F_0 = 1$, we deduce that

$$\begin{aligned} |E_1 - F_1| &\leq \delta K, \\ |E_2 - F_2| &\leq \frac{9}{2} \delta K^2, \\ |E_3 - F_3| &\leq 11 \delta K^3 \end{aligned}$$

and

$$|E_4 - F_4| \leq \frac{81}{4} \delta K^4.$$

Let i be an integer with $1 \leq i \leq 4$. Then

$$(x_i - y_1) \cdots (x_i - y_4) = x_i^4 + F_1 x_i^3 + \cdots + F_4$$

and

$$0 = (x_i - x_1) \cdots (x_i - x_4) = x_i^4 + E_1 x_i^3 + \cdots + E_4.$$

Therefore

$$\begin{aligned} |(x_i - y_1) \cdots (x_i - y_4)| &\leq |F_1 - E_1| K^3 + \cdots + |F_4 - E_4| \\ &\leq \left(1 + \frac{9}{2} + 11 + \frac{81}{4}\right) \delta K^4 = \frac{147}{4} \delta K^4, \end{aligned}$$

which is the result (10).

For $r = 5, 6, 7, \dots$,

$$S_r + S_{r-1} E_1 + \cdots + S_{r-4} E_4 = 0 = T_r + T_{r-1} F_1 + \cdots + T_{r-4} F_4.$$

Therefore, for $R = 5, 6, 7, \dots$,

$$\begin{aligned} |S_R - T_R| &\leq \sum_{r=1}^4 (|S_{R-r}(E_r - F_r)| + |F_r(S_{R-r} - T_{R-r})|) \\ &\leq 147 \delta K^R + \sum_{r=1}^4 \binom{4}{r} K^r |S_{R-r} - T_{R-r}|. \end{aligned}$$

Now, for $r = 1, 2, 3, \dots$, let U_r be given by

$$|S_r - T_r| = \left(U_r - \frac{21}{2}\right) \delta K^r.$$

Then, for $R = 5, 6, 7, \dots$,

$$U_R \leq \sum_{r=1}^4 \binom{4}{r} U_{R-r}.$$

Starting from the upper bound of $\frac{23}{2}$ for U_1, U_2, U_3 and U_4 , we apply this inequality to produce the upper bounds

$$\frac{345}{2}, \frac{1633}{2}, \frac{8717}{2}, \frac{46069}{2} \text{ and } \frac{243455}{2}$$

for U_5, \dots, U_9 (respectively). Using (9) we can easily check that

$$26U_r \leq \alpha^r, \text{ for } r = 6, \dots, 9.$$

Now suppose that R is an integer greater than 9 and such that

$$U_r \leq \frac{1}{26}\alpha^r, \text{ for } r = 6, \dots, R - 1.$$

Then

$$U_R \leq \frac{1}{26} \sum_{r=1}^4 \binom{4}{r} \alpha^{R-r} = \frac{1}{26} \alpha^R,$$

by (8). The result (11), for $r > 5$, follows by induction. To complete the proof note that $25 \left(U_5 - \frac{21}{2} \right) \leq 25 \times 162 < \alpha^5$, by (9).

Let (h_1, \dots, k_4) be an integer solution of (1). Then so is (h_1+t, \dots, k_4+t) , for any integer t . As in [5] we call this set of integer solutions of (1) a family f (say). The integer solution (h_1, \dots, k_4) of (1) is called trivial if and only if (h_1, \dots, h_4) is a permutation of (k_1, \dots, k_4) . The family f is called trivial if and only if

$$(h_i - k_1) \cdots (h_i - k_4) = (k_i - h_1) \cdots (k_i - h_4) = 0,$$

for $i = 1, \dots, 4$.

LEMMA 2. *An integer solution of (1) is trivial if and only if it is a member of a trivial family. The number of trivial integer solutions of (1) and (3) is*

$$A(H) = 24H^4 - 72H^3 + 82H^2 - 33H.$$

PROOF. This Lemma is almost a restatement of [5, Lemma 4], since there is a one-to-one correspondence between the trivial integer solutions of (1) and (3) and the trivial integer solutions of (1) with $0 \leq h_i, k_i < H$, for $i = 1, \dots, 4$. The correspondence is given by

$$(h_1, \dots, k_4) \longrightarrow (h_1 - H, \dots, k_4 - H).$$

LEMMA 3. *The integer solutions of (1) and (3) fall into $O(H^4)$ families.*

PROOF. This Lemma follows directly from [5, Lemma 6].

LEMMA 4. Let $\delta > 0$ and $K > 0$. Let F be the number of non-trivial families which contain a member $(a_1, \dots, a_4, b_1, \dots, b_4)$ with

$$\left| \sum_{j=1}^4 (a_j^r - b_j^r) \right| \leq \delta K^r, \text{ for } r = 3, 4,$$

and

$$|a_i| \leq K, |b_i| \leq K, \text{ for } i = 1, \dots, 4.$$

Then

$$F \ll \delta K^4 \log^2 K.$$

PROOF. If $K < 2$, then

$$a_i \in \{-1, 0, 1\} \text{ and } b_i \in \{-1, 0, 1\}, \text{ for } i = 1, \dots, 4,$$

and it follows from (1) that there are no such non-trivial families. The rest of the proof is almost the same as that of [5, Lemma 9], except that the appeals to [5, Lemma 8] are made with $r = 1$, $x_j = a_j$ and $y_j = b_j$, for $j = 1, \dots, 4$.

For $J \subset [H, 2H)$, let $S(J)$ be the set of integer solutions of (1) and (2) with $h_i \in J$ and $k_i \in J$, for $i = 1, \dots, 4$.

LEMMA 5. If $[H, 2H) = J_1 \cup \dots \cup J_Q$, then

$$I_8(H, \Delta) \ll Q^7 \sum_{q=1}^Q |S(J_q)|.$$

PROOF. The condition (2) implies that

$$\left| \sum_{j=1}^4 \left[\operatorname{Re} g \left(\frac{h_j}{H} \right) - \operatorname{Re} g \left(\frac{k_j}{H} \right) \right] \right| < \Delta$$

and

$$\left| \sum_{j=1}^4 \left[\operatorname{Im} g \left(\frac{h_j}{H} \right) - \operatorname{Im} g \left(\frac{k_j}{H} \right) \right] \right| < \Delta.$$

Conversely we note that if $-\Delta < x, y < \Delta$, then $|x + iy| < \sqrt{2}\Delta$. Therefore the Lemma follows by [4, Lemma 2.2 and Lemma 2.3].

3. - Proof of the Theorem

For $c \in \mathbb{C}$ and $r \geq 0$, let

$$D(c, r) = \{z \in \mathbb{C} \mid |z - c| \leq r\}.$$

For $1 \leq x \leq 2$, let

$$R(x) = \sup\{r \in [0, 1] \mid D(x, r) \subset V\}.$$

Then R is a continuous positive-valued function on $[1, 2]$, so there exists a positive constant ρ such that

$$(12) \quad R(x) > \rho, \text{ for } 1 \leq x \leq 2.$$

Let A be the union of the sets $D(x, \rho)$ with $1 \leq x \leq 2$. Then A is a compact subset of \mathbb{C} . Let B be the maximum value attained by $|g(z)|$ for $z \in A$. By (4), $B > 0$. If $1 \leq x \leq 2$, then by Cauchy's Integral Formulae,

$$(13) \quad |g^{(n)}(x)| \leq n! \rho^{-n} B, \text{ for } n = 0, 1, 2, \dots.$$

For $1 \leq x \leq 2$, let

$$E(x) = |g^{(3)}(x)g^{(5)}(x) - (g^{(4)}(x))^2|.$$

By (4), E is a positive-valued continuous function on $[1, 2]$, so there exists a positive constant ε such that

$$(14) \quad E(x) \geq \frac{3!4!B^2}{\rho^8} \varepsilon, \text{ for } 1 \leq x \leq 2.$$

By (13),

$$(15) \quad \varepsilon \leq 5 + 4 = 9.$$

Let Q be the least integer with

$$(16) \quad \varepsilon \rho Q \geq \alpha^5.$$

We divide $[H, 2H]$ into disjoint intervals J_1, \dots, J_Q of equal length. Let J be one of these intervals. Let $(h_1, \dots, k_4) \in f \cap S(J)$, where f is a family. Let (a_1, \dots, b_4) be the member of f with $b_4 = 0$. Then, for $i = 1, \dots, 4$,

$$a_i = h_i - k_4 \text{ and } b_i = k_i - k_4,$$

so that

$$(17) \quad |a_i| \leq \frac{H}{Q} \text{ and } |b_i| \leq \frac{H}{Q}.$$

For $r = 0, 1, 2, \dots$, let

$$d_r = \sum_{j=1}^4 (a_j^r - b_j^r).$$

Let

$$(18) \quad \delta = \max_{r=3,4} \left[\frac{Q}{H} \right]^r |d_r|.$$

By (1), (17) and Lemma 1 (11),

$$(19) \quad d_0 = d_1 = d_2 = 0$$

and

$$(20) \quad |d_r| \leq \frac{1}{25} \delta \left[\frac{\alpha H}{Q} \right]^r, \text{ for } r = 5, 6, 7, \dots.$$

To estimate $|f \cap S(J)|$ we consider the function

$$G(z) = \sum_{j=1}^4 \left[g \left(\frac{h_j}{H} + z \right) - g \left(\frac{k_j}{H} + z \right) \right],$$

which (by (12)) is certainly analytic for $|z| \leq \rho$. The members of $f \cap S(J)$ are in one-to-one correspondence with those integer solutions t of

$$(21) \quad \left| G \left(\frac{t}{H} \right) \right| < \Delta$$

which have

$$(22) \quad h_j + t\epsilon J \text{ and } k_j + t\epsilon J, \text{ for } j = 1, \dots, 4.$$

Now, by (7), (15) and (16),

$$(23) \quad \rho Q \geq \frac{\alpha^5}{9} > \left(\frac{25}{3} \right)^2 \alpha > 64\alpha > 256,$$

so that $\frac{h_j}{H} \in D \left(\frac{k_4}{H}, \rho \right)$, for $j = 1, \dots, 4$. Therefore, for $n = 0, 1, 2, \dots$,

$$g^{(n)} \left(\frac{h_j}{H} \right) = \sum_{r=0}^{\infty} \frac{1}{r!} c_{n+r} \left(\frac{a_i}{H} \right)^r,$$

where $c_r = g^{(r)}\left(\frac{k_4}{H}\right)$, for $r = 0, 1, 2, \dots$. The various $g^{(n)}\left(\frac{k_j}{H}\right)$ have similar Taylor series. Combining these results we find that

$$G^{(n)}(0) = \sum_{r=0}^{\infty} \frac{1}{r!} c_{n+r} d_r H^{-r}, \text{ for } n = 0, 1, 2, \dots.$$

Hence, by (13), (19), (20) and (23),

$$\left| G^{(n)}(0) - \sum_{r=3}^4 \frac{c_{n+r} d_r}{r! H^r} \right| \leq \frac{n! B \delta}{25 \rho^n} \sum_{r=5}^{\infty} \binom{n+r}{n} \left[\frac{\alpha}{\rho Q} \right]^r, \text{ for } n = 0, 1, 2, \dots.$$

Now, if n and s are non-negative integers and $0 \leq x < 1$, then by Taylor's Theorem there exists $\eta \in (0, 1)$ with

$$(24) \quad \sum_{r=s}^{\infty} \binom{n+r}{n} x^r = \binom{n+s}{n} x^s (1 - \eta x)^{-(n+s+1)}.$$

Therefore, if we let $\tau = \frac{1}{1 - \frac{\alpha}{\rho Q}}$, then, for $n = 0, 1, 2, \dots$,

$$(25) \quad \begin{aligned} \left| G^{(n)}(0) - \sum_{r=3}^4 \frac{c_{n+r} d_r}{r! H^r} \right| &\leq \frac{n! B \delta}{25 \rho^n} \binom{n+5}{5} \left[\frac{\alpha}{\rho Q} \right]^5 \tau^{n+6} \\ &\leq \frac{(n+5)!}{5!} \left[\frac{\tau}{\rho} \right]^n \frac{\tau^6 B \epsilon \delta}{25 (\rho Q)^4}, \end{aligned}$$

by (16).

Now, by (23), $1 < \tau < \frac{64}{64-1}$,

so

$$(26) \quad 1 < \tau^7 < \frac{64}{64-7} < \frac{25}{22}.$$

Therefore, since

$$\begin{bmatrix} c_5 & -c_4 \\ -c_4 & c_3 \end{bmatrix} \begin{bmatrix} c_3 & c_4 \\ c_4 & c_5 \end{bmatrix} = E \left(\frac{k_4}{H} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

it follows from (25) that, for $r = 3, 4$,

$$\left| \frac{E\left(\frac{k_4}{H}\right) d_r}{r! H^r} + \sum_{n=0}^1 (-1)^{n+r} c_{8-n-r} G^{(n)}(0) \right| < \frac{1}{22} \left[\sum_{n=0}^1 |c_{8-n-r}| \frac{(n+5)!}{5! \rho^n} \right] \frac{B \epsilon \delta}{(\rho Q)^4}.$$

Hence, by (13) and (14),

$$\begin{aligned} \left[\frac{Q}{H}\right]^r |d_r| &\leq \frac{(Q\rho)^r}{B\varepsilon} [(8-r)|G(0)| + \rho|G^{(1)}(0)|] + \frac{(14-r)}{2} (\rho Q)^{r-4} \delta \\ &\leq \frac{(Q\rho)^r}{B\varepsilon} \Gamma + \frac{1}{2} (\rho Q)^{r-4} \delta, \text{ for } r = 3, 4, \end{aligned}$$

where $\Gamma = 5|G(0)| + \rho|G^{(1)}(0)|$.

By (23), $\rho Q > 1$, so it now follows from (18) that

$$(27) \quad \left[\frac{Q}{H}\right]^4 |d_4| \leq \delta \leq \frac{2(Q\rho)^4}{B\varepsilon} \Gamma$$

and

$$(28) \quad \left[\frac{Q}{H}\right]^3 |d_3| \leq \frac{2(Q\rho)^3}{B\varepsilon} \Gamma.$$

We now return to (25) and apply the results (13), (15), (26), (27) and (28) to show that, for $n = 0, 1, 2, \dots$,

$$\begin{aligned} (29) \quad |G^{(n)}(0)| &\leq 2 \frac{(n+3)! \Gamma}{3! \rho^n \varepsilon} + 2 \frac{(n+4)! \Gamma}{4! \rho^n \varepsilon} + \frac{1}{11} \frac{(n+5)! \tau^n \Gamma}{5! \rho^n} \\ &\leq \frac{5\Gamma}{\varepsilon} \frac{(n+5)!}{5!} \left(\frac{\tau}{\rho}\right)^n. \end{aligned}$$

Now suppose that (21) and (22) hold. Then, by (23),

$$(30) \quad \left|\frac{t}{H}\right| \leq \frac{1}{Q} \leq \rho,$$

so

$$(31) \quad G\left(\frac{t}{H}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} G^{(n)}(0) \left(\frac{t}{H}\right)^n.$$

By (7), (16), (23), (24), (26), (29) and (30).

$$\begin{aligned} \left| \sum_{n=2}^{\infty} \frac{1}{n!} G^{(n)}(0) \left(\frac{t}{H}\right)^n \right| &\leq \frac{5\Gamma}{\varepsilon} \sum_{n=2}^{\infty} \binom{n+5}{5} \left|\frac{\tau t}{\rho H}\right|^n \leq \frac{5\Gamma}{\varepsilon} \binom{7}{5} \left|\frac{\tau t}{\rho H}\right|^2 \left[1 - \frac{\tau}{\rho H}\right]^{-8} \\ &\leq \left| \frac{105\tau^2 \Gamma t}{\varepsilon \rho^2 Q H} \left[1 - \frac{8 \times 2}{256}\right]^{-1} \right| < \left| \frac{256\Gamma t}{\alpha^5 \rho H} \left[1 - \frac{1}{2}\right]^{-1} \right| < \left| \frac{2^9 \Gamma t}{2^{10} \rho H} \right|. \end{aligned}$$

Therefore, by (31),

$$\left|G^{(1)}(0)\frac{t}{H}\right| \leq \left|G\left(\frac{t}{H}\right) - G(0)\right| + \frac{\Gamma|t|}{2\rho H}$$

and

$$\left|\Gamma\frac{t}{H}\right| \leq \left|5G(0)\frac{t}{H}\right| + \rho\left|G\left(\frac{t}{H}\right) - G(0)\right| + \left|\Gamma\frac{t}{2H}\right|.$$

Hence, by (2), (21), (23) and (30),

$$\left|\Gamma\frac{t}{H}\right| \leq \left|10G(0)\frac{t}{H}\right| + 2\rho\left|G\left(\frac{t}{H}\right) - G(0)\right| \leq 5\rho\Delta.$$

Therefore, by (27),

$$\left|\delta\frac{t}{H}\right| \leq \left(\frac{10Q^4\rho^5}{B\varepsilon}\right)\Delta,$$

so

$$(32) \quad |\delta t| \ll \Delta H.$$

We can now bound $|S(J)|$. First note that, by (17) and (18),

$$0 \leq \delta \leq 8.$$

Therefore, it is sufficient to consider the following three cases.

Case 1: $\delta = 0.$

By Lemma 1 (10), f is a trivial family. By Lemma 2, (h_1, \dots, k_4) is a trivial integer solution of (1) and $S(J)$ contains at most $24H^4$ such solutions.

Case 2: $\Delta H < \delta.$

By (32),

$$|f \cap S(J)| \ll 1.$$

Hence, by Lemma 3, at most $O(H^4)$ members of $S(J)$ fall into Case 2.

Case 3: $0 < \delta \leq \Delta H.$

By Lemma 2, f is not a trivial family. Since a_1, \dots, b_4 are integers and $Q \geq 1$, it follows from (17) and (18) that

$$\frac{1}{H^4} \leq \delta \leq 8.$$

We now consider the $O(\log H)$ non-empty subcases of the form,

$$\Delta_2 < \delta \leq 2\Delta_2,$$

where Δ_2 is an integer power of 2. By (32),

$$|f \cap S(J)| \ll \left(\frac{\Delta}{\Delta_2}\right) H.$$

By (17), (18) and Lemma 4, at most $O(\Delta_2 H^4 \log^2 H)$ families fall into any one such subcase. Hence, at most $O(\Delta H^5 \log^2 H)$ members of $S(J)$ fall into each subcase and the total number falling into Case 3 is therefore $O(\Delta H^5 \log^3 H)$.

Collecting the results from the three cases we find that

$$|S(J)| \ll H^4 + \Delta H^5 \log^3 H.$$

The Theorem now follows by Lemma 5.

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