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Dynamical Systems with Newtonian Type Potentials

MARCO DEGIOVANNI - FABIO GIANNONI

1. - Introduction

The study of periodic solutions to a conservative dynamical system

$$q'' + \text{grad } V(q) = 0 \tag{1.1}$$

has been faced by several authors in the last years, mostly when the function V is defined in all $\mathbb{R}^n$. We refer the reader to the survey paper [16] and references therein.

The case of potentials of Newtonian type (namely $V(x) \approx -1/|x|$) has brought to consider also the case in which V is defined only in an open subset Ω of $\mathbb{R}^n$ and goes to ∞ at the points of $\partial\Omega$ (case of singular potentials).

As known, also in this case problem (1.1) has a variational structure. The solutions can be regarded as critical points of a functional f (the Lagrangian integral) defined in an open subset of a functional space. Further details are contained in the next section.

Singular potentials have been mainly studied under the so called strong force assumption (introduced in [10]), which is verified, for instance, by $V(x) = -1/|x|^\alpha$ if and only if $\alpha \geq 2$. This hypothesis implies that $f(q)$ goes to infinity whenever q approaches $\partial\Omega$. In [6, 10], because of the assumptions on Ω and V, solutions are found by a minimization technique. In [12] hypotheses are made in order to apply a linking argument. Minimax techniques are involved also in [2, 7, 13].

Another type of singular potential, in which strong force assumption does not occur, is the so called potential well [1, 3]. In this case $f(q)$ may remain bounded, even if q approaches $\partial\Omega$. Nevertheless it is possible to find solutions with values in Ω, because of the assumptions on V and grad V.

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Our aim is to study the case in which V is defined in a neighbourhood of the origin and goes to $-\infty$ at the origin. However we do not make strong force assumption, as we want to obtain results including the case $V(x) = -1/|x|^\alpha$ for every $\alpha \geq 1$. Therefore it may happen that $f(q)$ remains bounded, when q approaches the origin. Nevertheless we look for solutions which do not cross the origin.

Roughly speaking, we assume that V is subjected to a double estimate of the form

$$\psi_0\left(\frac{1}{|x|}\right) \leq -V(x) \leq \psi_1\left(\frac{1}{|x|}\right),$$

for some functions ψ_0 and ψ_1. Given $T > 0$, it is possible to define (see (2.7) and (2.8)) two extended real numbers $\vartheta_0(\psi_0, T)$ and $\vartheta_1(\psi_1, T)$. The first one gives a lower estimate of the Lagrangian integral on the curves which meet the singularity. The second one gives an upper estimate of the Lagrangian integral on the circular trajectories of minimal period T and speed of constant modulus, which lie on a suitable sphere centred at the origin.

In the main theorems ((2.14), (2.19) and (2.28)) we prove that there exists a T-periodic solution of (1.1) which does not cross the singularity, provided that $\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T)$. Several properties of ϑ_0 and ϑ_1 are given, which allow to verify the condition $\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T)$ in some perturbative situations ((2.25), (2.26), (2.31) and (2.32)).

In the last section we prove some sharper results under the assumption that V is even.

Some results of the present paper were announced in [8, 9].

2. - Periodic solutions for some singular potentials

Throughout this section

$$V : \Omega \to]-\infty, 0]$$

will denote a function of class C^1 on the open subset Ω of $\mathbb{R}^n (n \geq 2)$, and T a strictly positive real number. Our aim is to study the existence of T-periodic curves $q : \mathbb{R} \to \Omega$ of class C^2 such that:

$$\text{for every } t \in \mathbb{R}, \quad q''(t) + \operatorname{grad} V(q(t)) = 0. \tag{2.1}$$

As well known, this problem has a variational structure. First of all, by the change of variable $\tilde{q}(t) = q(Tt)$, we are reduced to the study of 1-periodic curves $q : \mathbb{R} \to \Omega$ of class C^2 such that:

$$\text{for every } t \in \mathbb{R}, \quad q''(t) + T^2 \operatorname{grad} V(q(t)) = 0. \tag{2.2}$$

Let

$$H = \{q \in H^1([0,1], \mathbb{R}^n) : q(0) = q(1)\}$$
$$\Lambda = \{q \in H : q(t) \in \Omega, \ \forall t \in [0,1]\}$$

and let $f : \Lambda \to \mathbb{R}$ be defined by

$$f(q) = \frac{1}{2} \int_0^1 |q'|^2 \mathrm{d}t - T^2 \int_0^1 V(q) \mathrm{d}t.$$

Then Λ is an open subset of the real Hilbert space H, the Lagrangian integral f is a functional of class C^1 on Λ and, if $q \in \Lambda$ we have that grad $f(q) = 0$ if and only if q is the restriction to [0,1] of a solution to (2.2).

Since we are mainly interested in the case $\Omega = \mathbb{R}^n \backslash \{0\}$, let us set also

$$X_0 = \{q \in H : \exists \ t \in [0,1] \text{ with } q(t) = 0\}.$$

First of all it is useful to premise some considerations concerning the case of radial potentials and, in particular, the case $V(x) = -\frac{1}{|x|}$.

For the following result the reader is referred to [11].

(2.3) LEMMA. *Let $T > 0$ and let $\gamma :]0,1[\to]0,+\infty[$ be the unique solution of class C^2 of the problem*

$$\begin{cases} \gamma'' + T^2 \dfrac{1}{\gamma^2} = 0 \\ \gamma'\left(\dfrac{1}{2}\right) = 0 \\ \lim\limits_{t \to 0} \gamma(t) = \lim\limits_{t \to 1} \gamma(t) = 0. \end{cases}$$

Then we have

$$\frac{1}{2} \int_0^1 (\gamma')^2 \mathrm{d}t = \frac{1}{2} \ (2\pi T^2)^{2/3},$$

$$T^2 \int_0^1 \frac{1}{\gamma} \ \mathrm{d}t = (2\pi T^2)^{2/3},$$

$$\|\gamma\|_{L^\infty(0,1)} = \left(\frac{2T^2}{\pi^2}\right)^{1/3}.$$

(2.4) PROPOSITION. *Let $\psi :]0,+\infty[\to [0,+\infty[$ be a non-decreasing, non-constant, convex function. Then we have*

$$\min \left\{ \frac{1}{2} \int_0^1 (\gamma')^2 \mathrm{d}t + \psi\left(\int_0^1 \frac{\mathrm{d}t}{\gamma}\right) : \gamma \in H_0^1(0,1), \ \gamma \geq 0 \right\}$$
$$= \min \left\{ 2\pi^2 R^2 + \psi\left(\frac{1}{R}\right) : R > 0 \right\}.$$

PROOF. Let R be the number minimizing the expression in the right hand side.

Since the expression in the left hand side is finite for some γ (for instance $\gamma(t) = (t(1-t))^{2/3}$) and ψ is non-decreasing, by Rellich's theorem and Fatou's lemma we get the existence of some minimizing γ. Such a minimum γ is unique, because $\frac{1}{2}\int_0^1 (\eta')^2 dt$ is strictly convex on $H_0^1(0,1)$ and $\psi\left(\int_0^1 \frac{dt}{\eta}\right)$ is convex on $\{\eta \in H_0^1(0,1) : \eta \geq 0\}$.

We claim that $\gamma(t) > 0$ for all $t \in]0,1[$. Indeed, by contradiction, let us suppose that $\gamma(t) = 0$ for some t in $]0,1[$. Let $a \in]0,t[$ and $b \in]t,1[$ be such that $\gamma(a) = \gamma(b) > 0$. Then the function η, defined by $\eta(t) = \max\{\gamma(t), \gamma(a)\}$ in the interval $[a,b]$ and $\eta(t) = \gamma(t)$ elsewhere, is another minimum which is different from γ.

Now let us suppose ψ of class C^1. Then

$$4\pi^2 R^3 = \psi'(1/R), \tag{2.5}$$

$\gamma \in C^2(]0,1[)$ and $\gamma'' + \frac{\psi'(x)}{\gamma^2} = 0$ in $]0,1[$, where $x = \int_0^1 \frac{dt}{\gamma}$. Moreover, by uniqueness of γ we have also $\gamma'\left(\frac{1}{2}\right) = 0$.

Then, by (2.3) we get

$$\frac{1}{2}\int_0^1 |\gamma'|^2 dt = \frac{1}{2}\,(2\pi\psi'(x))^{2/3},$$

$$\psi'(x)\int_0^1 \frac{dt}{\gamma} = (2\pi\psi'(x))^{2/3}.$$

Therefore $x\psi'(x) = (2\pi\psi'(x))^{2/3}$, namely $\psi'(x) = \frac{4\pi^2}{x^3}$, which implies by (2.5) that $x = 1/R$.

Finally we obtain

$$\frac{1}{2}\int_0^1 (\gamma')^2 dt + \psi\left(\int_0^1 \frac{dt}{\gamma}\right) = \frac{1}{2}\,(2\pi\psi'(x))^{2/3} + \psi(x)$$
$$= \frac{2\pi^2}{x^2} + \psi(x) = 2\pi^2 R^2 + \psi\left(\frac{1}{R}\right).$$

In the general case ψ can be approximated in a non-decreasing way by a sequence of functions ψ_h which are also of class C^1 (see, for instance, [4]). By the previous step the thesis is true for ψ_h.

Let γ_h, R_h and γ, R be the minima corresponding to ψ_h and ψ. It is sufficient to prove that

$$\lim_{h\to\infty}\left[\frac{1}{2}\int_0^1(\gamma_h')^2\mathrm{d}t+\psi_h\left(\int_0^1\frac{\mathrm{d}t}{\gamma_h}\right)\right]=\frac{1}{2}\int_0^1(\gamma')^2\mathrm{d}t+\psi\left(\int_0^1\frac{\mathrm{d}t}{\gamma}\right),$$

$$\lim_{h\to\infty}\left[2\pi^2R_h^2+\psi_h\left(\frac{1}{R_h}\right)\right]=2\pi^2R^2+\psi\left(\frac{1}{R}\right).$$

We prove the first relation. The second one is similar. Since

$$\frac{1}{2}\int_0^1(\gamma_h')^2\mathrm{d}t+\psi_h\left(\int_0^1\frac{\mathrm{d}t}{\gamma_h}\right)\le\frac{1}{2}\int_0^1(\gamma')^2\mathrm{d}t+\psi_h\left(\int_0^1\frac{\mathrm{d}t}{\gamma}\right)$$
$$\le\frac{1}{2}\int_0^1(\gamma')^2\mathrm{d}t+\psi\left(\int_0^1\frac{\mathrm{d}t}{\gamma}\right),$$

we have to prove that

$$\frac{1}{2}\int_0^1(\gamma')^2\mathrm{d}t+\psi\left(\int_0^1\frac{\mathrm{d}t}{\gamma}\right)\le\liminf_{h\to\infty}\frac{1}{2}\int_0^1(\gamma_h')^2\mathrm{d}t+\psi_h\left(\int_0^1\frac{\mathrm{d}t}{\gamma_h}\right).$$

Since $\psi_h\ge 0$, up to a subsequence, (γ_h) converges to some η weakly in H_0^1. Let us remark that (ψ_h) converges to ψ uniformly on compact subsets of $]0,+\infty[$, because ψ and ψ_h are convex and non-decreasing. By Rellich's theorem and Fatou's lemma we get

$$\frac{1}{2}\int_0^1(\gamma')^2\mathrm{d}t+\psi\left(\int_0^1\frac{\mathrm{d}t}{\gamma}\right)\le\frac{1}{2}\int_0^1(\eta')^2\mathrm{d}t+\psi\left(\int_0^1\frac{\mathrm{d}t}{\eta}\right)$$
$$\le\liminf_{h\to\infty}\frac{1}{2}\int_0^1(\gamma_h')^2\mathrm{d}t+\psi_h\left(\int_0^1\frac{\mathrm{d}t}{\gamma_h}\right).\ \blacksquare$$

We point out that, in the particular case $\psi(s)=s$, (2.4) is contained also in [11].

Now we wish to give some comparison results concerning radial potentials of the form $V(x)=-\psi\left(\frac{1}{|x|}\right)$, with ψ satisfying the assumptions of (2.4).

(2.6) PROPOSITION. *Let* $T > 0$ *and* $\psi :]0, +\infty[\to [0, +\infty[$ *as in* (2.4). *Then we have*

$$\inf \left\{ \frac{1}{2} \int_0^1 (\gamma')^2 dt + T^2 \int_0^1 \psi\left(\frac{1}{\gamma}\right) dt : \gamma \in H_0^1(0,1),\ \gamma \geq 0 \right\}$$
$$= \inf \left\{ \frac{1}{2} \int_0^1 |q'|^2 dt + T^2 \int_0^1 \psi\left(\frac{1}{|q|}\right) dt : q \in X_0 \right\}$$

(*both sides may assume the value* $+\infty$).

PROOF. The inequality $\geq$ is obvious. On the other hand, since ψ does not depend explicitly on t, it does not matter to minimize in the right hand side for $q \in X_0$ or $q \in H_0^1(0,1;\mathbb{R}^n)$. In the second case we have also $|q| \in H_0^1(0,1)$, $|q| \geq 0$ and

$$\int_0^1 |q'|^2 dt \geq \int_0^1 (|q|')^2 dt$$

proving the opposite inequality. ∎

If $T > 0$ and $\psi :]0, +\infty[\to [0, +\infty[$ is non-decreasing, non-constant and convex, we set

$$\vartheta_0(\psi, T) = \inf \left\{ \frac{1}{2} \int_0^1 (\gamma')^2 dt + T^2 \int_0^1 \psi\left(\frac{1}{\gamma}\right) dt;\ \gamma \in H_0^1(0,1),\ \gamma \geq 0 \right\} \tag{2.7}$$

$$\vartheta_1(\psi, T) = \min \left\{ 2\pi^2 R^2 + T^2 \psi\left(\frac{1}{R}\right) : R > 0 \right\}, \tag{2.8}$$

with the convention $\psi(+\infty) = +\infty$ (ϑ_0 may assume also the value $+\infty$).

Let us point out that, if the Lagrangian integral f is associated with the potential $V(x) = -\psi\left(\frac{1}{|x|}\right)$, then

$$2\pi^2 R^2 + T^2 \psi\left(\frac{1}{R}\right)$$

is just the maximum of f among circular trajectories of minimal period 1 and speed of constant modulus, which lie on the sphere of radius R centred at the origin. The number $\vartheta_1(\psi, T)$ is determined by choosing the more convenient radius R. On the other hand, as we have seen in (2.6), $\vartheta_0(\psi, T)$ is the greatest lower bound of the Lagrangian integral on the trajectories which meet the origin.

Roughly speaking, our main purpose is to show that, if $V \in C^1(\mathbb{R}^n \setminus \{0\})$ is subjected to the double estimate

$$\psi_0\left(\frac{1}{|x|}\right) \leq -V(x) \leq \psi_1\left(\frac{1}{|x|}\right)$$

and

$$\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T),$$

for some $T > 0$, then there exists a T-periodic solution q of (2.1) with values in $\mathbb{R}^n \setminus \{0\}$.

More precisely, the double estimate on V is needed only in $B(0, \rho(T)) \setminus \{0\}$, where $\rho(T)$ can be determinated in the following way:

for every $s > 0$ let $r(s) > 0$ be such that

$$2(r(s))^2 = s^2 \psi_0(r(s)^{-1}).$$

Then $\rho :]0, +\infty[\to]0, +\infty[$ is any non-decreasing function, with $\lim_{s \to 0} \rho(s) = 0$, such that, for every $s > 0$

$$2(\rho(s))^2 \geq \vartheta_1(\psi_1, s) \text{ and } \psi_1(\rho(s)^{-1}) < \psi_0(r(s)^{-1}). \tag{2.9}$$

This last specification allows to consider also potentials V which are defined only near the origin.

Since the assumption $\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T)$ seems to be rather implicit, we list some properties of ϑ_0 and ϑ_1 which are helpful in verifying such a hypothesis.

For instance, it turns out that $\vartheta_1(\psi, T) < \vartheta_0(\psi, T)$ for all T, if $\psi(s) = s^\alpha$, with $\alpha > 1$, and strict inequality is stable under small perturbations of ψ.

(2.10) PROPOSITION. *Let $T, T_h > 0$ and $\psi, \psi_h :]0, +\infty[\to [0, +\infty[$ $(h \in \mathbb{N})$ be non-decreasing, non-constant and convex.*

Then the following facts hold:

i) $\vartheta_1(\psi, T) \leq \vartheta_0(\psi, T)$;

ii) *if $\vartheta_1(\psi, T) = \vartheta_0(\psi, T)$, we have that*

$$b := \lim_{s \to +\infty} \frac{\psi(s)}{s} < +\infty$$

and $\psi(s) - bs$ is constant on $\left[\left(\frac{\pi^2}{2bT^2}\right)^{1/3}, +\infty\right[$;

iii) *if $\lim_h T_h = T$ and $\lim_h \psi_h(s) = \psi(s)$ pointwise in s, we have*

$$\vartheta_0(\psi, T) \leq \liminf_h \vartheta_0(\psi_h, T_h),$$

$$\vartheta_1(\psi, T) = \lim_h \vartheta_1(\psi_h, T_h);$$

iv) *if* $\lambda > 0$, $\mu \in]0,1[$, *we have*

$$\vartheta_0(\lambda\psi, T) = \vartheta_0(\psi, T\sqrt{\lambda}), \ \ \vartheta_1(\lambda\psi, T) = \vartheta_1(\psi, T\sqrt{\lambda}),$$
$$\vartheta_0((1-\mu)\psi_0 + \mu\psi_1, T) \geq (1-\mu)\vartheta_0(\psi_0, T) + \mu\vartheta_0(\psi_1, T),$$
$$\vartheta_1((1-\mu)\psi_0 + \mu\psi_1, T) \geq (1-\mu)\vartheta_1(\psi_0, T) + \mu\vartheta_1(\psi_1, T);$$

v) *if* $\liminf_{s\to+\infty} \frac{\psi(s)}{s^2} > 0$, *we have* $\vartheta_0(\psi, T) = +\infty$;

vi) *if* ψ *is positively homogeneous of some degree* $\alpha \geq 1$, *then* $\vartheta_0(\psi, \cdot)$ *and* $\vartheta_1(\psi, \cdot)$ *are positively homogeneous of degree* $\frac{4}{\alpha+2}$.

PROOF. Let us prove i) and ii). If $\vartheta_0(\psi, T) = +\infty$, the result is true. Otherwise let $\gamma \in H_0^1(0,1)$, $\gamma \geq 0$ minimizing

$$\frac{1}{2}\int_0^1 (\gamma')^2 \mathrm{d}t + T^2 \int_0^1 \psi\left(\frac{1}{\gamma}\right) \mathrm{d}t.$$

By Jensen's inequality and (2.4) we have

$$\vartheta_0(\psi, T) \geq \frac{1}{2}\int_0^1 (\gamma')^2 \mathrm{d}t + T^2 \ \psi\left(\int_0^1 \frac{\mathrm{d}t}{\gamma}\right) \geq \vartheta_1(\psi, T).$$

Moreover, if equality holds, $\psi(s) = bs + \mathrm{d}$ on $[\|\gamma\|_{L^\infty}^{-1}, +\infty[$ and γ minimizes also

$$\frac{1}{2}\int_0^1 (\gamma')^2 \mathrm{d}t + T^2 \ \psi\left(\int_0^1 \frac{\mathrm{d}t}{\gamma}\right).$$

Then ψ is smooth in a neighbourhood of $\int_0^1 \frac{\mathrm{d}t}{\gamma}$ and, as in the proof of (2.4), it turns out that $\gamma :]0,1[\to]0,+\infty[$ is of class C^2, $\gamma'' + \frac{bT^2}{\gamma^2} = 0$ and $\gamma'\left(\frac{1}{2}\right) = 0$. By (2.3) we get

$$\|\gamma\|_{L^\infty} = \left(\frac{2bT^2}{\pi^2}\right)^{1/3}.$$

Now let (T_h) converge to T and let (ψ_h) pointwise converge to ψ. By weak convergence and Fatou's lemma we get

$$\vartheta_0(\psi, T) \leq \liminf_h \ \vartheta_0(\psi_h, T_h).$$

On the other hand, since ψ and ψ_h are convex and non-decreasing, it turns out that pointwise convergence implies uniform convergence on bounded

subsets of $]0,+\infty[$. Moreover $\lim_{s\to+\infty} \psi_h(s) = +\infty$ uniformly with respect to h. Then it is easy to conclude that

$$\vartheta_1(\psi, T) = \lim_h \vartheta_1(\psi_h, T_h),$$

proving iii).

Since ϑ_0 and ϑ_1 are obtained minimizing linear functions of ψ, they are concave with respect to ψ. The remaining part of iv) is obvious.

Since every γ in $H_0^1(0,1)$ is Hölder continuous of exponent $\frac{1}{2}$, we get v).

Now suppose that ψ is positively homogeneous of degree $\alpha \geq 1$. An explicit calculation shows that $\vartheta_1(\psi, \cdot)$ is positively homogeneous of degree $\frac{4}{(\alpha+2)}$. By the change of variable $\gamma = T^{\frac{2}{\alpha+2}}\eta$, we get also that $\vartheta_0(\psi, \cdot)$ is positively homogeneous of degree $\frac{4}{(\alpha+2)}$, proving vi). ∎

At this point we need a topological lemma which can be interpretated as an adaptation to our case of the technique involved in Lusternik-Fet theorem (see, for instance, [14]). A similar adaptation can be found also in [13].

Let us set

$$A = \{q \in H : \exists\ \nu \in \mathbb{R}^n \text{ with } (q(t)|\nu) > 0\ \forall\ t\},$$
$$\forall\ k \geq 1,\ D^k = \{x \in \mathbb{R}^k : |x| \leq 1\}.$$

Of course $A \cap X_0 = \emptyset$.

(2.11) LEMMA. *Let $n \geq 3$, R, $\rho > 0$ and let $\Theta : D^{n-2} \to H\backslash X_0$ be the continuous map defined by*

$$\Theta(x)(t) = \left(R(1-|x|^2)^{1/2} \cos 2\pi t,\ R(1-|x|^2)^{1/2} \sin 2\pi t,\ \rho x\right).$$

Then there do not exist any neighbourhood U of ∂D^{n-2} in D^{n-2} and any continuous map $\mathcal{H} : D^{n-2} \times [0,1] \to H\backslash X_0$ such that

$$\textit{for all } x \in D^{n-2},\quad \mathcal{H}(x,0) = \Theta(x) \quad \textit{and} \quad \mathcal{H}(x,1) \in A,$$
$$\textit{for all } (x,s) \in U \times [0,1],\quad \mathcal{H}(x,s) = \Theta(x).$$

PROOF. By contradiction, we could define a continuous map

$$\Phi : (D^2 \times D^{n-2}) \times [0,1] \to \mathbb{R}^n\backslash\{0\}$$

by

$$\Phi((x_1,x_2),s) = \begin{cases} (1-|x_1|)\displaystyle\int_0^1 \mathcal{H}(x_2,s)(t)\mathrm{d}t + |x_1|\mathcal{H}(x_2,s)(\chi(x_1)) & x_1 \neq 0 \\ \displaystyle\int_0^1 \mathcal{H}(x_2,s)(t)\mathrm{d}t & x_1 = 0, \end{cases}$$

where $\forall\ x \in D^2\backslash\{0\},\ 0 \leq \chi(x) < 1$ and

$$x = |x|\ (\cos\ 2\pi\chi(x),\ \sin\ 2\pi\chi(x)).$$

Let us remark that

$$\text{for all } ((x_1,x_2),s) \in (\partial(D^2 \times D^{n-2})) \times [0,1],\ \Phi((x_1,x_2),s) \neq 0.$$

Therefore (see, for instance, [17]) we have

$$\deg(0,\ \Phi(\cdot,0),\ D^2 \times D^{n-2}) = \deg(0,\ \Phi(\cdot,1),\ D^2 \times D^{n-2}).$$

Moreover we have

$$\forall\ (x_1,x_2) \in D^2 \times D^{n-2},\ \Phi((x_1,x_2),0) = \left(R(1-|x_2|^2)^{1/2}\ x_1,\ \rho x_2\right).$$

Therefore

$$\deg(0,\ \Phi(\cdot,0),\ D^2 \times D^{n-2}) = 1.$$

On the other hand, by the definition of A, we have

$$0 \notin \Phi(\cdot,1)(D^2 \times D^{n-2}),$$

hence

$$\deg(0,\ \Phi(\cdot,1),\ D^2 \times D^{n-2}) = 0$$

which is absurd. ∎

In the following lemma we state a simple extension of well known results about deformation techniques (see, for instance, [5, 15, 17]).

(2.12) DEFINITION. Let Y be a real Hilbert space, Λ an open subset of Y and $g : \Lambda \to \mathbb{R}$ a function of class C^1. If $c \in \mathbb{R}$, the function g is said to verify *the Palais-Smale condition at level* c, if for every sequence (u_h) in Λ with $\lim_h \operatorname{grad}\ g(u_h) = 0$, $\lim_h\ g(u_h) = c$, there exists a subsequence (u_{h_j}) converging to an element of Λ.

If $c \in \mathbb{R} \cup \{+\infty\}$, we set also $g^c = \{u \in \Lambda : g(u) \leq c\}$.

(2.13) LEMMA. *Let Y be a real Hilbert space, Λ an open subset of Y and $g : \Lambda \to \mathbb{R}$ a function of class C^1. Let $-\infty < \delta < m \leq +\infty$ be such that*

i) $\forall\ u \in \Lambda,\ \delta \leq g(u) \leq m \Rightarrow \text{ grad } g(u) \neq 0$;

ii) $\forall\ c \in [\delta, m[$, *the function g verifies the Palais-Smale condition at level c and g^c is closed in Y.*
Then g^δ is a strong deformation retract of g^m.

Now we can give the main results in this section. We recall that ϑ_0, ϑ_1 and ρ are defined in (2.7), (2.8) and (2.9), respectively.

(2.14) THEOREM. *Let $n \geq 3$, $V \in C^1(\mathbb{R}^n \backslash \{0\})$ with*

$$\lim_{|x| \to \infty} V(x) = 0, \quad \lim_{|x| \to \infty} \text{grad } V(x) = 0 \tag{2.15}$$

and let $\psi_0, \psi_1 :]0, +\infty[\to [0, +\infty[$ be non-constant and convex, with $\lim_{s \to 0} \psi_i(s) = 0$.
Then for every $T > 0$ satisfying

$$0 < |x| \leq \rho(T) \Rightarrow \psi_0\left(\frac{1}{|x|}\right) \leq -V(x) \leq \psi_1\left(\frac{1}{|x|}\right), \tag{2.16}$$

$$\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T), \tag{2.17}$$

there exists a T-periodic solution q_T of (2.1) *with $q_T(t) \neq 0$ for all t and the following further properties*:

i) *if (T_h) is a sequence satisfying* (2.16), (2.17) *and $\lim_h T_h = 0$, then*

$$\lim_h \|q_{T_h}\|_{L^\infty} = 0;$$

ii) *if $\forall\ x \in \mathbb{R}^n \backslash \{0\}$, $V(x) < 0$ and if (T_h) is a sequence satisfying* (2.16), (2.17) *and $\lim_h T_h = +\infty$, then*

$$\lim_h \|q_{T_h}\|_{L^\infty} = +\infty.$$

PROOF. Set $V(0) = -\infty$. Since $-V$ is non-negative and lower semicontinuous, by Rellich's theorem and Fatou's lemma we have that the Lagrangian integral $f : H \to \mathbb{R} \cup \{+\infty\}$ is sequentially weakly lower semicontinuous. Moreover it is of class C^1 on the open set $H \backslash X_0$.

For every $s > 0$ let $r(s) > 0$ be such that

$$2(r(s))^2 = s^2 \psi_0(r(s)^{-1}).$$

Now let $T > 0$ satisfying (2.16) and (2.17) and let $m = \vartheta_1(\psi_1, T)$.

We claim that for all $q \in X_0$, $f(q) \geq m$. Indeed, if $\|q\|_{L^\infty} \leq \rho(T)$, by (2.6), (2.7), (2.16) and (2.17) we deduce $f(q) \geq \vartheta_0(\psi_0, T) \geq m$. Otherwise

$$2\rho(T) \leq \int_0^1 |q'| \mathrm{d}t \leq (2f(q))^{1/2},$$

so that $f(q) \geq 2(\rho(T))^2 \geq m$.

In particular, by lower semicontinuity of f, we deduce that, for every $c \in]0, m[$, $\{q \in H \backslash X_0 : f(q) \leq c\}$ is closed in H.

Now we claim that for every $c \in]0, m[$, the functional $f : H \backslash X_0 \to \mathbb{R}$ verifies the Palais-Smale condition at level c. Let (q_h) be a sequence in $H \backslash X_0$ with $\lim_h \operatorname{grad} f(q_h) = 0$, $\lim_h f(q_h) = c$. Of course (q_h') is bounded in L^2. We claim that (q_h) is bounded in L^∞, otherwise by (2.15)

$$\lim_h V(q_h) = 0, \ \lim_h |\operatorname{grad} V(q_h)| = 0 \text{ in } L^\infty. \tag{2.18}$$

Then we should find a sequence $(\varepsilon_h) \to 0$ such that

$$\forall\ h \in \mathbb{N},\ \forall\ \eta \in H, \ \left| \int_0^1 (q_h' | \eta') \mathrm{d}t \right| \leq \varepsilon_h \left(\int_0^1 (|\eta|^2 + |\eta'|^2) \mathrm{d}t \right)^{1/2}.$$

If we choose $\eta(t) = q_h(t) - q_h(0)$ and we remark that $\eta \in H_0^1$, we obtain

$$\left(\int_0^1 |q_h'|^2 \mathrm{d}t \right)^{1/2} \leq (1 + \pi^2)^{1/2}\ \varepsilon_h.$$

Combining this fact with (2.18), we get that $\lim_h f(q_h) = 0$, which is absurd. Therefore (q_h) is bounded in L^∞.

Up to a subsequence, (q_h) converges to some $q \in H$ in the weak topology of H and in the strong topology of L^∞. Since $f(q) \leq c < m$, we have $q \in H \backslash X_0$. Now it is standard to show that (q_h) converges to q also in the strong topology of H, proving the Palais-Smale condition at level c.

Now let $\delta(T) = T^2 \psi_0(r(T)^{-1}) = 2(r(T))^2$. First of all we remark that, since $\psi_0(r(T)^{-1}) > \psi_1(\rho(T)^{-1}) \geq \psi_0(\rho(T)^{-1})$, we have $r(T) < \rho(T)$.

We claim that $f^{\delta(T)} \subset A$. Indeed, if $q \in f^{\delta(T)}$ and q is not constant, it must be $\|q\|_{L^\infty} > r(T)$, otherwise by (2.16)

$$f(q) > -T^2 \int_0^1 V(q) \mathrm{d}t \geq T^2 \psi_0(r(T)^{-1}) = \delta(T).$$

On the other hand

$$\int_0^1 |q'|\mathrm{d}t \leq (2f(q))^{1/2} \leq (2\delta(T))^{1/2} = 2r(T)$$

so that $q \in A$.

Now we claim that there exists $q \in H\backslash X_0$ such that grad $f(q) = 0$, $\delta(T) \leq f(q) \leq \vartheta_1(\psi_1, T)$.

Let $R > 0$ be such that

$$m = \vartheta_1(\psi_1, T) = 2\pi^2 R^2 + T^2 \psi_1\left(\frac{1}{R}\right).$$

Of course $R < \rho(T)$. We can define a continuous map $\Theta : D^{n-2} \to H\backslash X_0$ by

$$\Theta(x)(t) = \left(R(1-|x|^2)^{1/2} \cos 2\pi t,\ R(1-|x|^2)^{1/2} \sin 2\pi t,\ \rho(T)x\right).$$

Then, for every x in D^{n-2}, we have $\|\Theta(x)\|_{L^\infty} \leq \rho(T)$, hence

$$f(\Theta(x)) \leq 2\pi^2 R^2 + T^2 \psi_1\left(\frac{1}{R}\right) = m.$$

Since $\Theta(0) \notin A$, it must be $\delta(T) < m$. Moreover, for every $x \in \partial D^{n-2}$,

$$f(\Theta(x)) \leq T^2 \psi_1(\rho(T)^{-1}) < T^2 \psi_0(r(T)^{-1}) = \delta(T).$$

Therefore, by (2.11), $f^{\delta(T)}$ cannot be a strong deformation retract of $f^m\backslash X_0$, otherwise, if $\mathcal{K} : (f^m\backslash X_0) \times [0,1] \to f^m\backslash X_0$ were the corresponding map, we could define $\mathcal{H} : D^{n-2} \times [0,1] \to H\backslash X_0$ by $\mathcal{H}(x,s) = \mathcal{K}(\Theta(x), s)$, against the thesis of (2.11).

By (2.13) we conclude that there exists $q \in H\backslash X_0$ with grad $f(q) = 0$, $\delta(T) \leq f(q) \leq \vartheta_1(\psi_1, T)$.

Now let $(T_h) \to 0$ with T_h satisfying (2.16), (2.17) and let q_h be given by the previous argument. We have to prove that $\lim_h q_h = 0$ in L^∞.

First of all, since $\frac{1}{2}\int_0^1 |q_h'|^2 \mathrm{d}t \leq \vartheta_1(\psi_1, T_h)$, we deduce

$$\lim_h \frac{1}{2} \int_0^1 |q_h'|^2 \mathrm{d}t = 0.$$

Therefore it is sufficient to show that

$$\lim_h \left(\inf_t |q_h(t)|\right) = 0.$$

Since $\int_0^1 q_h' \mathrm{d}t = 0$, we have

$$\int_0^1 |q_h'|^2 \mathrm{d}t \leq \int_0^1 |q_h''|^2 \mathrm{d}t = T_h^4 \int_0^1 |\mathrm{grad}\ V(q_h)|^2 \mathrm{d}t.$$

Then

$$T_h^2\ \psi_0(r(T_h)^{-1}) = \delta(T_h)$$
$$\leq T_h^2 \left(\frac{T_h^2}{2} \int_0^1 |\mathrm{grad}\ V(q_h)|^2 \mathrm{d}t - \int_0^1 V(q_h) \mathrm{d}t \right)$$

which gives the result by (2.15), as $\lim_h r(T_h) = 0$.

Finally, suppose that $V(x) < 0$, for every $x \in \mathbb{R}^n \backslash \{0\}$. Let $(T_h) \to +\infty$ and assume that each T_h satisfies (2.16) and (2.17). Let q_h be as in the previous argument. Then

$$0 \leq -T_h^2 \int_0^1 V(q_h) \mathrm{d}t \leq \vartheta_1(\psi_1, T_h) \leq 2\pi^2 T_h + T_h^2 \psi_1(T_h^{-1/2}),$$

which implies

$$\lim_h \int_0^1 V(q_h) \mathrm{d}t = 0.$$

Hence

$$\lim_h \|q_h\|_{L^\infty} = +\infty. \blacksquare$$

In the two-dimensional case, we can prove the same result, showing also the minimality of the period, under weaker assumptions. The argument is similar to that used in [10] in the case of strong forces.

(2.19) THEOREM. *Let $V \in C^1(\mathbb{R}^2 \backslash \{0\})$ and let $\psi_0, \psi_1 :]0, +\infty[\to [0, +\infty[$ be non-constant and convex, with $\lim_{s \to 0} \psi_i(s) = 0$.*

Then, for every $T > 0$ satisfying

$$0 < |x| \leq \rho(T) \Rightarrow \psi_0\left(\frac{1}{|x|}\right) \leq -V(x) \leq \psi_1\left(\frac{1}{|x|}\right), \tag{2.20}$$

$$\vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T), \tag{2.21}$$

there exists a T-periodic solution q_T of (2.1) with $q_T(t) \neq 0$ for all t, minimal period T and the following further properties:

i) *if* (T_h) *is a sequence satisfying* (2.20), (2.21) *and* $\lim_h T_h = 0$, *then*

$$\lim_h \|q_{T_h}\|_{L^\infty} = 0;$$

ii) *if for all* $x \in \mathbb{R}^2 \backslash \{0\}$, $V(x) < 0$ *and if* (T_h) *is a sequence satisfying* (2.20), (2.21) *and* $\lim_h T_h = +\infty$, *then*

$$\lim_h \|q_{T_h}\|_{L^\infty} = +\infty.$$

PROOF. Set $V(0) = -\infty$. Let f, r be as in the proof of (2.14).

Let $T > 0$ satisfying (2.20) and (2.21), let $m = \vartheta_1(\psi_1, T)$ and let $\delta(T)$ be as in the proof of (2.14). Also in this case it turns out that $f^{\delta(T)} \subset A$ and $f(q) \geq m$ for every $q \in X_0$.

Let $X_1 = \{q \in H \backslash X_0 : q$ is not contractible in $\mathbb{R}^2 \backslash \{0\}\}$. Then X_1 is open and closed in $H \backslash X_0$ endowed with L^∞-topology. In particular, X_1 is open in H.

We claim that

$$\delta(T) < \inf_{X_1} f \leq \vartheta_1(\psi_1, T) \tag{2.22}$$

and the infimum is achieved.

Indeed, let $R > 0$ be as in the proof of (2.14) and let $\overline{q}(t) = (R \cos 2\pi t,\ R \sin 2\pi t)$. Of course $\overline{q} \in X_1$. Since $R < \rho(T)$, by (2.20) we deduce that $f(\overline{q}) \leq m$, proving the second inequality in (2.22). Since $f^{\delta(T)} \subset A$ and $A \cap X_1 = \emptyset$, if we show that the infimum is achieved, we get also the first inequality in (2.22).

Now, if $\inf_{X_1} f = m$, $\overline{q}$ is just a minimum point. Otherwise, let $\inf_{X_1} f < m$ and let (q_h) be a minimizing sequence.

Since $X_1 \subset H \backslash A$, we have

$$\text{for every } q \in X_1,\ \|q\|^2_{L^\infty} \leq \left(2 \int_0^1 |q'| \mathrm{d}t\right)^2 \leq 8 f(q). \tag{2.23}$$

Therefore, up to a subsequence, (q_h) converges to some $q \in H$ in the weak topology of H and in the strong topology of L^∞. Since $f(q) \leq \liminf_h f(q_h) < m$, we have $q \in H \backslash X_0$. Then $q \in X_1$ and the infimum is achieved.

We also claim that there does not exist $k \in \mathbb{N}$ with $k \geq 2$ and $q(t) = q\left(t - \frac{1}{k}\right)$, for every $t \in \left[\frac{1}{k}, 1\right]$. This means that q is the restriction to $[0,1]$ of a solution to (2.2) of *minimal* period 1.

Indeed, by contradiction, we could define $q_1 \in H \backslash X_0$ by $q_1(t) = q\left(\frac{t}{k}\right)$.

Then $q_1 \in X_1$, in particular q_1 is not constant, and

$$f(q) = \frac{k^2}{2}\int_0^1 |q_1'|^2 \mathrm{d}t - T^2 \int_0^1 V(q_1)\mathrm{d}t,$$

so that $k = 1$.

Now let (T_h) be a sequence of strictly positive numbers satisfying (2.20) and (2.21).

If $(T_h) \to 0$, we have $\lim_h \vartheta_1(\psi_1, T_h) = 0$. On the other hand, by (2.22) and (2.23) we deduce

$$\|q_{T_h}\|^2_{L^\infty} \leq 8\vartheta_1(\psi_1, T_h),$$

so that $(q_{T_h}) \to 0$ in L^∞.

Finally, if $V(x) < 0$, for every $x \in \mathbb{R}^2\backslash\{0\}$, and $(T_h) \to +\infty$, as in the proof of (2.14), it turns out that

$$\lim_h \|q_{T_h}\|_{L^\infty} = +\infty. \blacksquare$$

(2.24) REMARK. Theorems (2.14) and (2.19) can be applied to the particular case $V(x) = -\frac{b}{|x|^\alpha}$, in $\mathbb{R}^n\backslash\{0\}$, $(n \geq 2,\ b > 0,\ \alpha \geq 1)$, for any $T > 0$. It is sufficient to choose $\psi_0(s) = \psi_1(s) = bs^\alpha$ and to remark that, by $(2.10)_i$, $\vartheta_1(\psi, T) \leq \vartheta_0(\psi, T)$ for every $T > 0$.

Of course this case can also be treated in an elementary way, looking for circular solutions centred at the origin.

(2.25) COROLLARY. *Let $\psi :]0, +\infty[\to [0, +\infty[$ be a non-constant, convex function with $\lim_{s\to 0} \psi(s) = 0$.*

Let us assume that ψ is not linear on $]0, +\infty[$.

Then there exists $T_0 > 0$ such that, for every $T_1 \geq T_0$, there exists $\varepsilon > 0$ such that, for every $T \in [T_1 - \varepsilon, T_1 + \varepsilon]$ and for every $V \in C^1(\mathbb{R}^n\backslash\{0\})$, $n \geq 2$, with

$$\lim_{|x|\to+\infty} \operatorname{grad} V(x) = 0,$$

$$\forall\ x \in \mathbb{R}^n\backslash\{0\},\ (1-\varepsilon)\psi\left(\frac{1}{|x|}\right) \leq -V(x) \leq (1+\varepsilon)\psi\left(\frac{1}{|x|}\right),$$

there exists a T-periodic solution q of (2.1) with $q(t) \neq 0$ for every t.

Moreover, if $n = 2$, such a solution has minimal period T.

PROOF. By $(2.10)_{ii}$ there exists $T_0 > 0$ such that $\vartheta_1(\psi, T_1) < \vartheta_0(\psi, T_1)$ for every $T_1 \geq T_0$.

By $(2.10)_{iii}$ there exists $\varepsilon > 0$ such that:

$$\text{for all } T \in [T_1 - \varepsilon, T_1 + \varepsilon], \quad \vartheta_1((1+\varepsilon)\psi, T) < \vartheta_0((1-\varepsilon)\psi, T).$$

Now we can apply (2.14) (or (2.19), if $n = 2$). ∎

(2.26) COROLLARY. *Let* $\psi :]0, +\infty[\to [0, +\infty[$ *be a non-constant convex function with* $\lim_{s \to 0} \psi(s) = 0$.

Let us assume that ψ *is not eventually linear, as* $s \to +\infty$.

Then, for every $T_1 > 0$, *there exists* $\varepsilon > 0$ *such that for every* $T \in [T_1 - \varepsilon, T_1 + \varepsilon]$ *and for every* $V \in C^1(\mathbb{R}^n \backslash \{0\})$, $n \geq 2$, *with*

$$\lim_{|x| \to +\infty} \operatorname{grad} V(x) = 0 \text{ and}$$

$$\text{for all } x \in \mathbb{R}^n \backslash \{0\}, \ (1 - \varepsilon)\psi\left(\frac{1}{|x|}\right) \leq -V(x) \leq (1 + \varepsilon)\psi\left(\frac{1}{|x|}\right),$$

there exists a T*-periodic solution* q *of* (2.1) *with* $q(t) \neq 0$ *for every* t.

Moreover, if $n = 2$ *such a solution has minimal period* T.

PROOF. By $(2.10)_{ii}$ we have $\vartheta_1(\psi, T_1) < \vartheta_0(\psi, T_1)$ for every $T_1 > 0$. Now we go on as in the proof of (2.25). ∎

(2.27) REMARK. We point out that (2.25) and (2.26) allow to treat some non-radial potential energies V with

$$\lim_{x \to 0} |x| V(x) > -\infty.$$

By (2.14) and (2.19) we can also deduce some results in the case in which V is defined only in a neighbourhood of the origin.

(2.28) THEOREM. *Let* $n \geq 2$, $r > 0$, $V \in C^1(B(0, r) \backslash \{0\})$ *and let* $\psi_0, \psi_1 :]0, +\infty[\to [0, +\infty[$ *be non-decreasing, non-constant and convex.*

Let us assume that

$$0 < |x| < r \Rightarrow \psi_0\left(\frac{1}{|x|}\right) \leq -V(x) \leq \psi_1\left(\frac{1}{|x|}\right);$$

$$\text{there exists } T_0 > 0 : \forall\ T \in]0, T_0], \ \vartheta_1(\psi_1, T) \leq \vartheta_0(\psi_0, T).$$

Then there exists $T_1 > 0$ *such that for every* $T \in]0, T_1]$ *there exists a* T*-periodic solution* q_T *of* (2.1) *with* $0 < |q_T(t)| < r$ *for every* t *and*

$$\lim_{T \to 0} \|q_T\|_{L^\infty} = 0.$$

Moreover, if $n = 2$, q_T *has minimal period* T.

PROOF. Let us remark that, if $\mathrm{d} \in \mathbb{R}$ and $\tilde{\psi}_i(s) = \psi_i(s) - \mathrm{d}$ for every sufficiently large s, then $\vartheta_i(\tilde{\psi}_i, T) = \vartheta_i(\psi_i, T) - \mathrm{d}T^2$ for every sufficiently small T. Therefore we can assume without loss of generality that $V \in C^1(\mathbb{R}^n \backslash \{0\})$,

$$\lim_{s \to 0} \psi_i(s) = 0, \ \lim_{|x| \to \infty} V(x) = 0, \ \lim_{|x| \to \infty} \operatorname{grad} V(x) = 0.$$

Let ρ be the function defined in (2.9).

We can find $\overline{T} > 0$ such that $\overline{T} \leq T_0$ and $\rho(T) < r$ for all $T \in]0, \overline{T}]$. Then for every T in $]0, \overline{T}]$ we can apply (2.14) ((2.19) resp.), obtaining some curves q_T. By $(2.14)_i$ ($(2.19)_i$ resp.) we get that for a sufficiently small $T_1 > 0$ it is $0 < |q_T(t)| < r$ for every t, whenever $0 < T \leq T_1$. Moreover, if $n = 2$, q_T has minimal period T by (2.19). ∎

Now for every $\alpha \geq 1$ let us define

$$\varphi(\alpha) = \left(\frac{\vartheta_0(\psi, 1)}{\vartheta_1(\psi, 1)}\right)^{\frac{\alpha+2}{2}} \tag{2.29}$$

where $\psi(s) = s^\alpha$.

(2.30) REMARK. *The real extended function* $\varphi : [1, +\infty[\to [1, +\infty]$ *is lower semicontinuous. Moreover we have*

i) $\varphi(1) = 1$;

ii) *for all* $\alpha \in]1, 2[$, $1 < \varphi(\alpha) < +\infty$;

iii) *for all* $\alpha \geq 2$, $\varphi(\alpha) = +\infty$.

PROOF. It is a consequence of (2.4), $(2.10)_i$, $(2.10)_{ii}$, $(2.10)_{iii}$ and $(2.10)_v$. ∎

(2.31) COROLLARY. *Let* $n \geq 2$, $r > 0$ *and* $V \in C^1(B(0, r)\backslash\{0\})$. *Let us assume that there exist* $\alpha \geq 1$, $b > 0$ *such that*

$$\textit{for every } x \in B(0, r)\backslash\{0\}, \ \frac{b}{|x|^\alpha} \leq -V(x) \leq \varphi(\alpha) \ \frac{b}{|x|^\alpha}.$$

Then the thesis of (2.28) *holds.*

PROOF. Let $\psi_0(s) = bs^\alpha$. If $\alpha < 2$, let $\psi_1(s) = \varphi(\alpha)bs^\alpha$. If $\alpha \geq 2$, let $\psi_1 :]0, +\infty[\to [0, +\infty[$ be any non-decreasing, non-constant, convex function such that

$$\text{for every } x \in B(0, r)\backslash\{0\}, \ -V(x) \leq \psi_1\left(\frac{1}{|x|}\right).$$

Let $T > 0$. If $\alpha \geq 2$, we have $\vartheta_1(\psi_1, T) < +\infty = \vartheta_0(\psi_0, T)$. Otherwise by $(2.10)_{iv}$, $(2.10)_{vi}$ and (2.29) we have

$$\varphi(\alpha) = \left(\frac{\vartheta_0(\psi_0, T)}{\vartheta_1(\psi_0, T)}\right)^{\frac{\alpha+2}{2}}.$$

Then, by $(2.10)_{iv}$ and $(2.10)_{vi}$, we deduce

$$\vartheta_1(\psi_1, T) = \vartheta_1(\psi_0, \varphi(\alpha)^{1/2}T) = \varphi(\alpha)^{\frac{2}{\alpha+2}} \ \vartheta_1(\psi_0, T) = \vartheta_0(\psi_0, T).$$

Now we can apply (2.28). ∎

(2.32) COROLLARY. *Let* $n \geq 2$, $r > 0$ *and let* $V \in C^1(B(0,r)\backslash\{0\})$ *be of the form*

$$-V(x) = \frac{b}{|x|^\alpha} + W(x)$$

with $b > 0$, $\alpha > 1$ *and* $W \in C^1(B(0,r)\backslash\{0\})$ *such that*

$$\lim_{x \to 0} |x|^\alpha \, W(x) = 0.$$

Then the thesis of (2.28) *holds.*

PROOF. By (2.30) we have $\varphi(\alpha) > 1$. Let $\varepsilon > 0$ with $(1-\varepsilon)\varphi(\alpha) - 1 > 0$. By decreasing r, we can suppose that

$$\text{for every } x \in B(0,r)\backslash\{0\}, \quad -\varepsilon b \leq |x|^\alpha W(x) \leq ((1-\varepsilon)\varphi(\alpha) - 1)b,$$

namely

$$\text{for every } x \in B(0,r)\backslash\{0\}, \quad \frac{(1-\varepsilon)b}{|x|^\alpha} \leq -V(x) \leq \varphi(\alpha) \, \frac{(1-\varepsilon)b}{|x|^\alpha}.$$

Now we can apply (2.31). ∎

(2.33) REMARK. We point out that we do not know whether or not (2.32) holds true in the case $\alpha = 1$.

3. - The even case

Further results concerning Newtonian potentials can be obtained under an eveness assumption on the potential energy V. Under strong force hypothesis, results for even singular potentials have been obtained also in [6].

Throughout this section

$$V : \Omega \to \,]-\infty, 0]$$

will denote a function of class C^1 defined in an open subset Ω of $\mathbb{R}^n$ with $n \geq 2$. We recall that φ denotes the function defined in (2.29).

(3.1) THEOREM. *Let* $V \in C^1(\mathbb{R}^n\backslash\{0\})$ *be such that*

i) *for every* $x \in \mathbb{R}^n\backslash\{0\}$, $V(-x) = V(x)$;

ii) *there exist* $\alpha \geq 1$, $b > 0$ *such that,*

$$\textit{for every } x \in \mathbb{R}^n\backslash\{0\}, \quad \frac{b}{|x|^\alpha} \leq -V(x) \leq 2^\alpha \varphi(\alpha) \, \frac{b}{|x|^\alpha}.$$

Then for every $T > 0$ there exists a T-periodic solution q_T of (2.1) *which does not cross the origin and has minimal period T. Moreover q_T is symmetric with respect to the origin* $\left(\text{that is } q_T\left(t+\frac{T}{2}\right) = -q_T(t)\right)$ *and*

$$\lim_{T\to 0} \|q_T\|_{L^\infty} = 0.$$

Furthermore, if ii) *is substituted by the stronger assumption*

ii)′ *there exist $\alpha \geq 1$, $b > 0$, $\mu \in [1, 2^\alpha \varphi(\alpha)[$ such that,*

$$\text{for every } x \in \mathbb{R}^n \backslash \{0\}, \quad \frac{b}{|x|^\alpha} \leq -V(x) \leq \mu \frac{b}{|x|^\alpha},$$

then we have also

$$\lim_{T\to+\infty} (\min_t |q_T(t)|) = +\infty.$$

PROOF. Let $H_s = \left\{q \in H : q(t) = -q\left(t-\frac{1}{2}\right),\ \forall\, t \in \left[\frac{1}{2}, 1\right]\right\}$, set $V(0) = -\infty$ and let $f : H_s \to \mathbb{R} \cup \{+\infty\}$ be the Lagrangian integral associated with V. Of course f is of class C^1 on the open (in H_s) set $H_s \backslash X_0$.

We need the following results.

(3.2) LEMMA. *Since V is even, we have* grad $f(q) \in H_s$ *for every $q \in H_s \backslash X_0$.*

PROOF. Let $\eta \in H$ be defined by

$$\eta(t) = \begin{cases} -(\text{grad } f(q))\left(t+\frac{1}{2}\right), & 0 \leq t \leq \frac{1}{2} \\ -(\text{grad } f(q))\left(t-\frac{1}{2}\right), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since grad V is odd, a simple change of variable shows that $\eta = \text{grad} f(q)$, hence $\text{grad} f(q) \in H_s$. ∎

Because of the previous lemma, to get a solution of (2.1), it is sufficient to find a critical point of the functional $f_{|H_s \backslash X_0}$.

(3.3) LEMMA. *We have that inf $f_{|H_s}$ is achieved at a non-constant curve.*

PROOF. Because of symmetry, the length of a curve $q \in H_s$ is such that $\int_0^1 |q'| dt \geq 4|q(s)|$ for all s; therefore we have $\int_0^1 |q'|^2 dt \geq 16\|q\|^2_{L^\infty}$. So a minimizing sequence for f in H_s is bounded in L^∞, hence in H. By a standard argument we have that such a sequence has a subsequence weakly converging in H_s to a minimum point.

Let us observe that any minimum point for $f_{|H_s}$ is non-constant, because the only constant curve in H_s is the origin and $f(0) = +\infty$. ∎

(3.4) LEMMA. *Every minimum point for $f_{|H_s}$ is the restriction to $[0,1]$ of a curve having minimal period* 1.

PROOF. Let q be a minimum point. By contradiction, we should find $k \in \mathbb{N}$ such that $k \geq 2$ and $q(t) = q\left(t - \frac{1}{k}\right)$ for all $t \in \left[\frac{1}{k}, 1\right]$. Let, for every $t \in [0,1]$, $\overline{q}(t) = q\left(\frac{t}{k}\right)$. Of course $\overline{q} \in H$. Since q is not the constant 0, k must be odd; therefore $\overline{q} \in H_s$ and $\overline{q}$ is not constant.

But

$$f(q) = \frac{k^2}{2}\int_0^1 |\overline{q}'|^2 \mathrm{d}t - T^2 \int_0^1 V(\overline{q})\mathrm{d}t > f(\overline{q})$$

and this is absurd, because q is a minimum point. ∎

(3.5) LEMMA. *Let $\gamma_R(t) = (R\cos 2\pi t)e_1 + (R\sin 2\pi t)e_2$, where $R > 0$ and $(e_i|e_j) = \delta_{ij}$. Then, there exists $\overline{R} > 0$ such that*

$$f(\gamma_{\overline{R}}) \leq \inf_{H_s \cap X_0} f.$$

PROOF. By ii) we have

$$\inf_{H_s\cap X_0} f \geq \inf\left\{\frac{1}{2}\int_0^1 |q'|^2\mathrm{d}t + T^2\int_0^1 \frac{b}{|q|^\alpha}\,\mathrm{d}t : q \in H_s \cap X_0\right\}$$

$$\geq \inf\left\{\frac{1}{2}\int_0^1 (\gamma')^2\mathrm{d}t + T^2\int_0^1 \frac{b}{\gamma^\alpha}\,\mathrm{d}t :\right.$$

$$\left.\gamma \in H_0^1(0,1),\ \gamma \geq 0,\ \gamma(t) = \gamma\left(t - \frac{1}{2}\right),\ \forall\ t \in \left[\frac{1}{2}, 1\right]\right\}$$

$$= 2\inf\left\{\frac{1}{2}\int_0^{1/2} (\gamma')^2\mathrm{d}t + T^2\int_0^{1/2} \frac{b}{\gamma^\alpha}\,\mathrm{d}t : \gamma \in H_0^1\left(0, \frac{1}{2}\right),\ \gamma \geq 0\right\}.$$

By the change of variable $\gamma(t) = \overline{\gamma}(2t)$ and $(2.10)_{vi}$, we deduce

$$\inf_{H_s\cap X_0} f \geq 4\vartheta_0\left(\psi, \frac{T}{2}\right) = 4\left(\frac{T}{2}\right)^{\frac{4}{\alpha+2}} \vartheta_0(\psi, 1)$$

(3.6)

$$= \left(2^\alpha T^2\right)^{\frac{2}{\alpha+2}} \vartheta_0(\psi, 1),$$

where $\psi(s) = bs^\alpha$.

Let $\overline{R} > 0$ be such that

$$2\pi^2\overline{R}^2 + (2^\alpha\varphi(\alpha)T^2)\ \frac{b}{\overline{R}^\alpha} = \vartheta_1\left(\psi, (2^\alpha\varphi(\alpha))^{\frac{1}{2}}\ T\right).$$

By ii) and $(2.10)_{vi}$ we have

$$f(\gamma_{\overline{R}}) \leq \vartheta_1\left(\psi, (2^\alpha\varphi(\alpha))^{\frac{1}{2}}\ T\right)$$

(3.7)

$$= \left(2^\alpha T^2\right)^{\frac{2}{\alpha+2}}\ \left(\varphi(\alpha)\right)^{\frac{2}{\alpha+2}}\ \vartheta_1(\psi, 1).$$

Combining (2.29) with (3.6) and (3.7), we get the thesis. ∎

Let us return to the proof of (3.1). By (3.3) and (3.5) there exists a minimum point q_T for $f_{|H_s}$ which does not cross the origin. By (3.2) q_T is a critical point of f on $H\backslash X_0$. Moreover, by (3.4) q_T is the restriction to $[0,1]$ of a curve of minimal period 1.

Since

$$f(q_T) \leq \min\ \left\{2\pi^2R^2 - T^2\int_0^1 V(\gamma_R)\mathrm{d}t : R > 0\right\},$$

where $\gamma_R(t) = (R\ \cos\ 2\pi t)e_1 + (R\ \sin\ 2\pi t)e_2$, $(e_i|e_j) = \delta_{ij}$, we have

$$\lim_{T\to 0}\int_0^1 |q_T'|^2\mathrm{d}t = 0.$$

If we remember that $\int_0^1 |q_T'|^2\mathrm{d}t \geq 16\ \|q_T\|^2_{L^\infty}$, we get

$$\lim_{T\to 0}\ \|q_T\|_{L^\infty} = 0.$$

Now let us assume ii)′.

As in the proof of (3.5), (3.7), we have

$$\frac{1}{2}\int_0^1 |q_T'|^2\mathrm{d}t - T^2\int_0^1 V(q_T)\mathrm{d}t \leq (\mu\ T^2)^{\frac{2}{\alpha+2}}\ \vartheta_1(\psi, 1), \tag{3.8}$$

where $\psi(s) = bs^\alpha$.

Let us set for every $T > 0$ and for every $\rho > 0$,

$$\chi(T,\rho) = \inf \left\{ \int_0^{\frac{1}{2}} (\gamma')^2 \mathrm{d}t + 2T^2 \int_0^{\frac{1}{2}} \frac{b}{\gamma^\alpha}\, \mathrm{d}t : \right. \tag{3.9}$$

$$\left. \gamma \in H^1\left(0, \frac{1}{2}\right),\ \gamma \geq 0,\ \gamma\left(\frac{1}{2}\right) = \gamma(0),\ |\gamma(0)| \leq \rho \right\}.$$

Then for every $T > 0$,

$$\chi(T,0) \leq \liminf_{\rho \to 0} \chi(T,\rho). \tag{3.10}$$

Moreover, as in the proof of (3.5), we have for every $T > 0$,

$$\chi(T,0) = (2^\alpha\ \varphi(\alpha)\ T^2)^{\frac{2}{\alpha+2}}\ \vartheta_1(\psi,1). \tag{3.11}$$

By (3.10) and (3.11), we can find $\rho_1 > 0$ such that

$$\chi(1,\rho_1) > \mu^{\frac{2}{\alpha+2}}\ \vartheta_1(\psi,1). \tag{3.12}$$

Let us remark that, by the change of variable $\gamma = T^{\frac{2}{\alpha+2}}\eta$, we get

$$\chi(T, \rho_1\ T^{\frac{4}{\alpha+2}}) = T^{\frac{4}{\alpha+2}}\ \chi(1,\rho_1). \tag{3.13}$$

Combining (3.8) with (3.12) and (3.13), we obtain

$$\begin{aligned}
\chi(T, \rho_1 T^{\frac{4}{\alpha+2}}) &> \frac{1}{2} \int_0^1 |q_T'|^2 \mathrm{d}t - T^2 \int_0^1 V(q_T)\mathrm{d}t \\
&\geq \frac{1}{2} \int_0^1 (|q_T|')^2 \mathrm{d}t + T^2 \int_0^1 \frac{b}{|q_T|^\alpha}\, \mathrm{d}t \\
&= \int_0^{1/2} (|q_T|')^2 \mathrm{d}t + 2T^2 \int_0^{1/2} \frac{b}{|q_T|^\alpha}\, \mathrm{d}t.
\end{aligned}$$

Therefore $|q_T(t)| > \rho_1\ T^{\frac{4}{\alpha+2}}$ for every t, hence

$$\lim_{T \to +\infty} (\min_t\ |q_T(t)|) = +\infty. \ \blacksquare$$

(3.14) COROLLARY. *Let A be a symmetric matrix in $\mathbb{R}^n$ with eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and let $V(x) = -(Ax|x)^{-\alpha/2}$ with $\alpha \geq 1$. Let us assume that $\frac{\lambda_n}{\lambda_1} < 4(\varphi(\alpha))^{2/\alpha}$.*

Then for every $T > 0$ there exists a T-periodic solution q_T of (2.1) *which does not cross the origin and has minimal period T. Moreover*

$$\lim_{T\to 0} \|q_T\|_{L^\infty} = 0, \quad \lim_{T\to+\infty} (\min_t |q_T(t)|) = +\infty.$$

PROOF. Since

$$\lambda_1|x|^2 \leq (Ax|x) \leq \lambda_n|x|^2,$$

we have

$$\begin{aligned} \lambda_n^{-\alpha/2}\ |x|^{-\alpha} \leq -V(x) &\leq \lambda_1^{-\alpha/2}\ |x|^{-\alpha} \\ &= \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha/2}\ \lambda_n^{-\alpha/2}\ |x|^{-\alpha}. \end{aligned}$$

Since $\left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha/2} < 2^\alpha\ \varphi(\alpha)$, we can apply (3.1). ∎

(3.15) COROLLARY. *Let E be a measurable subset of $\mathbb{R}^3$, $b > 0$ and let $V \in C^1(\mathbb{R}^3\backslash\overline{E})$ be defined by*

$$V(x) = -b\int_E \frac{dy}{|x-y|}.$$

Let us suppose that

i) *for all $x \in E$, $-x \in E$;*

ii) *there exist r and R such that $R^3 < 2r^3$ and*

$$B(0,r) \subset E \subset B(0,R).$$

Then there exists $T_0 > 0$ such that, for every $T \geq T_0$, there exists a T-periodic solution q_T of (2.1), *having minimal period T, such that $q_T(t) \in \mathbb{R}^3\backslash\overline{E}$ for all t.*

Moreover

$$\lim_{T\to+\infty} \left(\min_t |q_T(t)|\right) = +\infty.$$

PROOF. By i) we have

$$\text{for every } x \in \mathbb{R}^3\backslash\overline{E}, \quad V(-x) = V(x).$$

Moreover, by ii), for every $x \in \mathbb{R}^3 \backslash \overline{B(0,R)}$,

$$\frac{4}{3}\pi r^3 b\ |x|^{-1} = \int_{B(0,r)} \frac{b\mathrm{d}y}{|x-y|} \leq -V(x)$$

$$\leq \int_{B(0,r)} \frac{b\mathrm{d}y}{|x-y|} = \frac{4}{3}\pi R^3 b\ |x|^{-1} = \left(\frac{R}{r}\right)^3 \frac{4}{3}\pi r^3 b\ |x|^{-1}.$$

Let $\tilde{V} \in C^1(\mathbb{R}^3 \backslash \{0\})$ be such that

$$\text{for all } x \in \mathbb{R}^3 \backslash \{0\},\ \tilde{V}(-x) = \tilde{V}(x);$$

$$\text{for all } x \in \mathbb{R}^3 \backslash \{0\},\quad \frac{4}{3}\pi r^3 b\ |x|^{-1} \leq -\tilde{V}(x) \leq \left(\frac{R}{r}\right)^3 \frac{4}{3}\pi r^3 b\ |x|^{-1};$$

$$\text{for all } x \in \mathbb{R}^3 \backslash B(0,2R),\ \tilde{V}(x) = V(x).$$

Since $\left(\frac{R}{r}\right)^3 < 2 = 2\ \varphi(1)$, we apply (3.1) and we remember that

$$\lim_{T \to +\infty} (\min_t\ |q_T(t)|) = +\infty. \blacksquare$$

(3.16) COROLLARY. *Let*

$$-V(x) = \frac{b}{|x|^\alpha} + W(x),$$

where $\alpha \geq 1$, $b > 0$, $W \in C^1\left(\mathbb{R}^n \backslash \overline{B(0,R)}\right)$, $R > 0$, $W(-x) = W(x)$ *and*

$$\lim_{|x| \to \infty} |x|^\alpha\ W(x) = 0.$$

Then, there exists $T_0 > 0$ *such that, for every* $T \geq T_0$ *there exists a* T*-periodic solution* q_T *of* (2.1), *having minimal period* T, *such that* $|q_T(t)| > R$ *for all* t.

Moreover

$$\lim_{T \to +\infty} \left(\min_t\ |q_T(t)|\right) = +\infty.$$

PROOF. Let $\mu \in]1, 2^\alpha \varphi(\alpha)[$. There exist $\varepsilon > 0$, $R' > R$ such that

$$\text{for every } x \in \mathbb{R}^n \backslash \overline{B(0,R')},\ \frac{(1-\varepsilon)b}{|x|^\alpha} \leq -V(x) \leq \mu\ \frac{(1-\varepsilon)b}{|x|^\alpha}.$$

Now we go on as in the proof of (3.15). $\blacksquare$

Also in the even case, we can give a result whenever V is defined only in a neighbourhood of the origin.

(3.17) THEOREM. *Let* $V \in C^1(B(0,r)\backslash\{0\})$, $r > 0$, *be such that*

i) *for every* $x \in B(0,r)\backslash\{0\}$, $V(-x) = V(x)$;

ii) *there exist* $\alpha \geq 1$ *and* $b > 0$ *such that*

$$\text{for every } x \in B(0,r)\backslash\{0\}, \quad \frac{b}{|x|^\alpha} \leq -V(x) \leq 2^\alpha \ \varphi(\alpha) \ \frac{b}{|x|^\alpha}.$$

Then there exists $T_0 > 0$ *such that, for every* $T \in]0, T_0]$, *there exists a* T-*periodic solution* q_T *of* (2.1), *having minimal period* T, *such that* $0 < |q_T(t)| < r$ *for every* t *and*

$$\lim_{T \to 0} \|q_T\|_{L^\infty} = 0.$$

PROOF. Let $0 < \rho < r$ and $\tilde{V} \in C^1(\mathbb{R}^n\backslash\{0\})$ such that for every $x \in B(0,\rho)\backslash\{0\}$, $\tilde{V}(x) = V(x)$, for every $x \in \mathbb{R}^n\backslash\{0\}$, $\tilde{V}(-x) = \tilde{V}(x)$ and

$$\frac{b}{|x|^\alpha} \leq -\tilde{V}(x) \leq 2^\alpha \varphi(\alpha) \ \frac{b}{|x|^\alpha}.$$

Now we apply (3.1) and we remember that

$$\lim_{T \to 0} \|q_T\|_{L^\infty} = 0. \ \blacksquare$$

(3.18) COROLLARY. *Let*

$$-V(x) = \frac{b}{|x|^\alpha} + W(x),$$

where $\alpha \geq 1$, $b > 0$, $W \in C^1(B(0,r)\backslash\{0\})$, $r > 0$, $W(-x) = W(x)$ *and*

$$\lim_{x \to 0} |x|^\alpha \ W(x) = 0.$$

Then there exists $T_0 > 0$ *such that, for every* $T \in]0, T_0]$, *there exists a* T-*periodic solution* q_T *of* (2.1), *having minimal period* T, *such that* $0 < |q_T(t)| < r$ *for every* t *and*

$$\lim_{T \to 0} \|q_T\|_{L^\infty} = 0.$$

PROOF. We have $2^\alpha \ \varphi(\alpha) > 1$. Let $\varepsilon > 0$ be such that $(1-\varepsilon) \ 2^\alpha \ \varphi(\alpha) - 1 > 0$. By decreasing r we can suppose that

$$\forall \ x \in B(0,r)\backslash\{0\}, \ -\varepsilon b \leq |x|^\alpha \ W(x) \leq ((1-\varepsilon) \ 2^\alpha \ \varphi(\alpha) - 1) \, b,$$

namely

$$\forall \ x \in B(0,r)\backslash\{0\}, \ \frac{(1-\varepsilon)b}{|x|^\alpha} \leq -V(x) \leq 2^\alpha \ \varphi(\alpha) \ \frac{(1-\varepsilon)b}{|x|^\alpha}.$$

Now we can apply (3.17). $\blacksquare$

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