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Partial Regularity of Functions of Least Gradient in $\mathbb{R}^8$

HAROLD R. PARKS

0. INTRODUCTION. Let Ω be a bounded open subset of $\mathbb{R}^n$ and suppose $u \in BV(\Omega)$. We say that u is of *least gradient* (*with respect to* Ω) if, for every $v \in BV(\Omega)$ such that $v = u$ outside some compact subset of Ω,

$$\int_\Omega |Du| \leq \int_\Omega |Dv|.$$

In [PZ], the following result, [PZ; 4.5], was demonstrated.

Suppose $2 \leq n \leq 7$. *If* $\Omega \subset \mathbb{R}^n$ *is bounded and open,* $\mathrm{Bdry}(\Omega)$ *is a class* $n-1$ *submanifold of* $\mathbb{R}^n, \varphi : \mathrm{Bdry}(\Omega) \to \mathbb{R}$ *is of class* $n-1$, *and* $u :$ Clos $(\Omega) \to \mathbb{R}$ *is lipschitzian and of least gradient with* $u|\mathrm{Bdry}(\Omega) = \varphi$, *then there exists an open dense set* $W \subset \Omega$ *such that* $u|W$ *is of class* $n-3$.

In this paper, we use the recent work of L. Simon on Uniqueness of Tangent Cones and of R. Hardt and L. Simon on Isolated Singularities of Area Minimizing Hypersurfaces, to extend the result of [PZ] to $n = 8$. We would, of course, conjecture that the result of [PZ] holds for any integer $n \geq 2$, but progress towards a proof is blocked by the mysterious nature of singularities in area minimizing surfaces of dimension greater than 7.

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1. PRELIMINARIES.

(1) Let $\Omega \subset \mathbb{R}^8$ be non-empty, bounded, open with boundary Γ of class $C^q, q \geq 4$.

(2) Let $u :$ Clos $(\Omega) \to \mathbb{R}$ be a lipschitzian function of least gradient such that $\varphi = u|\Gamma$ is of class q. Set $M = \mathrm{Lip}(u)$.

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(3) For $\inf\ u < r < \sup\ u$, set

$$T_r = [\partial(\mathbb{E}^n L\Omega \cap \{x : u(x) \leq r\})] \lfloor \Omega.$$

Assume that 0 is a regular value of φ. It is known by [FH; 5.4.16] and [HS1; 11.1] that T_0 has only isolated singularities, $p_1, p_2, ..., p_k \in \Omega$ and the tangent cone to T_0 at each p_i is regular (i.e. has a singularity only at the origin); thus also, the Uniqueness of Tangent Cones result of L. Simon will be applicable (see [SL; Theorem 5.7]).

(4) We denote the tangent cone at p_i by $\mathcal{C}_i$. By [HS2; Theorem 2.1], each component of the complement of $\mathcal{C}_i$ contains a unique area minimizing current without boundary, which we will refer to as the Hardt-Simon current. Let $\mathcal{S}_i$ denote the Hardt-Simon current in the component of the complement of $\mathcal{C}_i$ for which $\{x : u(x) > 0\}$ has positive density.

(5) For $l \in \{1, 2, ..., k\}$ and $r > 0$, we set

$$\rho_l(r) = \text{ dist } [\text{spt } (T_r), p_l].$$

By the Lipschitz condition we know $\rho_l(r) > r/M$, but presumably $\rho_l(r)$ is much larger than that. We will need to know that $\rho_l(r)$ goes to 0 as r goes to zero, so we can use it for blowing-up.

2. LEMMA. *Let $T \in \mathbb{I}_m(\mathbb{R}^{m+1})$ be absolutely area minimizing.*

(1) *Suppose W is open with* $\text{spt}(\partial T) \subset W$, *and suppose $\varepsilon > 0$. Then there exists $\delta' > 0$ (which will depend on T, W, and ε) such that, if $S \in \mathbb{I}_m(\mathbb{R}^{m+1})$ is area minimizing,*

$$\text{spt}(\partial S) \subset \{x : \text{ dist } (x, \partial T) < \delta'\}, \textit{and}$$

$$\mathcal{F}(S - T) < \delta',$$

then the Hausdorff distance between $\text{spt}(T) \sim W$ *and* $\text{spt}(S) \sim W$ *is less than* ε.

(2) *Suppose W is open with* $\text{spt}(\partial T) \subset W$, *suppose all singular points of* $\text{spt}(T)$ *are contained in W, and suppose $\varepsilon > 0$. Then there exists $\delta > 0$ (which will depend on T, W, and ε) such that, if $S \in \mathbb{I}_m(\mathbb{R}^{m+1})$ is area minimizing,*

$$\text{spt}(\partial S) \subset \{x : \text{ dist } (x, \partial T) < \delta\}, \textit{ and}$$

$$\mathcal{F}(S - T) < \delta,$$

then S is regular in $\text{spt}(S) \sim W$, *and* $x \in \text{spt}(S) \sim W, y \in \text{spt}(T), |x - y| < \delta$, *imply the normal to* $\text{spt}(S)$ *at x and the normal to* $\text{spt}(T)$ *at y differ by less than ε.*

PROOF.

(1) We argue by contradiction. If not then there exist sequences S_i, δ_i such that

$$\operatorname{spt}(\partial S_i) \subset \{x : \operatorname{dist}(x, \partial T) < \delta_i\}, \text{and}$$
$$\mathcal{F}(S_i - T) < \delta_i,$$
$$\lim_i \delta_i = 0$$

and for each i the Hausdorff distance from $\operatorname{spt}(T) \sim W$ to $\operatorname{spt}(S_i) \sim W$ is at least ε. Passing to a subsequence if necessary, but without changing notation, we may suppose either

(i) for each i there is $p_i \in \operatorname{spt}(S_i) \sim W$ with $\mathbb{U}^n(p_i, \varepsilon) \cap \operatorname{spt}(T) \sim W = \emptyset$

or

(ii) for each i there is $p_i \in \operatorname{spt}(T) \sim W$ with $\mathbb{U}^n(p_i, \varepsilon) \cap \operatorname{spt}(S_i) \sim W = \emptyset$.

We may also assume that $p_i \to p$. In the first case, we have a contradiction between the lower bound on density for S_i in $\mathbb{U}^n(p, \varepsilon/2)$ and the weak convergence of S_i to T, while in the second case we have a contradiction between $p \in \operatorname{spt}(T)$ and the weak convergence of S_i to T.

(2) Conclusion (2) follows from (1), [AS; Theorem 1.2], and the fact that $\operatorname{spt}(T)$ is regular.

■

3. NOTATION. For convenience, in sections 3 and 4 we fix $l \in \{1, 2, ..., k\}$, suppose $p_l = 0$, and suppress the subscripts on ρ_l, $\mathcal{C}_l$, and $\mathcal{S}_l$.

LEMMA. *$\rho(r) \to 0$ as $r \to 0+$.*

PROOF. Clearly,

$$(*) \qquad \mathcal{F}(T_r, T_0) \leq \mathcal{L}^n(\Omega \cap \{x : 0 < u(x) < r\}) + \mathcal{H}^n(\Gamma \cap \{x : 0 < \varphi(x) < r\}),$$

so the lemma follows from 2(1).

■

4. NOTATION. For $r > 0$, we set

$$S_r = \mu_{1/\rho(r)\#}(T_r),$$
$$Q_r = \mu_{1/\rho(r)\#}(T_0).$$

PROPOSITION. *$Q_r \to \mathcal{C}_l$ in the flat topology as $r \to 0+$ and $S_r \to \mathcal{S}_l$ in the flat topology as $r \to 0+$.*

PROOF. This is immediate by Uniqueness of Tangent Cones, [SL; Theorem 5.7], and [HS2; Theorem 2.1].

■

5. THEOREM. *There exists $r_0 > 0$ such that* $\operatorname{spt}(T_r) \sim \Gamma$ *is regular for* $0 < r < r_0$.

PROOF. By 2(2) and 4, for all small enough r, T_r is regular in a neighborhood of the singular points. Away from the singularities we use 2(2) and (*).

■

COROLLARY. *For $0 < r < r_0$, if $u - r$ satisfies the conditions of* [PZ; $1.1(5i - v)$], *then there exists an open set $W \subset \Omega$ with*

$$\Omega \subset u^{-1}(r) \subset W$$

such that $u|W$ is C^{q-2}.

PROOF. We proceed as in [PZ] through the proof of [PZ; 4.4], but the interior curvature estimate used in Section 2 of [PZ] must be obtained using [AS; Theorem 1.2].

■

6. THEOREM. *If Γ is a C^7 submanifold of $\mathbb{R}^8$, and φ is C^7, then there exists an open dense set $U \subset \Omega$ such that $u|U$ is C^5.*

PROOF. We proceed as in the proof of [PZ; 4.5], but after choosing $t \notin \varphi(N_1)$, we apply 5 above to $u - t$.

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