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JENN-FANG HWANG

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Comparison Principles and Liouville Theorems for Prescribed Mean Curvature Equations in Unbounded Domains

JENN-FANG HWANG

1. - Introduction

The comparison principles of mean curvature equations in an unbounded domain has been opened for a long time, the uniqueness of solutions of the equations in such a domain is known only for some special cases.

Nitsche [8, p. 256] states "Let Ω_α be a sector in $\mathbb{R}^2$, with angle $0 < \alpha < \pi$, let $\Omega \subset \Omega_\alpha$ and let u be a solution of the minimal surface equation in Ω. Then if $u \leq$ constant on $\partial\Omega$ it follows that $u \leq$ constant in Ω." Thus the uniqueness for minimal surface equation in such a domain is known if the boundary data is constant.

If Ω is an exterior domain, Osserman [10] proved that the solution of minimal surfaces equation is unique in the class of bounded functions.

For capillary free surface equations, the uniqueness theorem in bounded domains is guaranteed by a comparison principle of Concus and Finn [2, 3]. However, when the domain is unbounded, the uniqueness is still open even in an infinite strip domain.

It was recently proved by Tam [14] that the solution of the capillary free surface equation in the absence of gravity must be a cylinder, in case of non-zero gravity, is the one-dimensional solution the unique solution? It is still open.

The purpose of this paper is trying to answer these problems; indeed, we have the following theorems.

1. Let Ω be a domain (bounded or unbounded) in $\mathbb{R}^2$, then the solutions of the generalized Dirichlet problem and the capillary free surface problem for a prescribed mean curvature equation in Ω is unique in the class of bounded functions.

2. Let $\Omega \subset \Omega_\alpha$ where Ω_α be as above, Nitsche raised a question "If u, v are two solutions of the minimal surface equation in Ω and $u - v < M$ on $\partial\Omega$, is $u - v < M$ in Ω also?"

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We answer Nitsche's problem by assuming that $|u|$, $|v|$ are bounded on $\partial\Omega$ (Theorem 2.8).

3. We find a sufficient condition, depended on the geometric properties of an unbounded domain Ω, such that the solutions of the capillary free surface equation in a gravitational field is unique (Theorem 2.9).

The comparison principles discussed in §2 will be generalized to nonlinear nonuniformly elliptic equations in §3 and §4, some Liouville theorems in $\mathbb{R}^2$ for nonlinear nonuniformly elliptic equations are obtained in §5, which extends part of the results of Meier [6] from uniformly elliptic to nonuniformly elliptic equations, and gives a new proof (analytically) of Liouville theorem for mean curvature equation in $\mathbb{R}^2$, which was proved (geometrically) by Cheng and Yau [1].

2. - Comparison principles for mean curvature equations in unbounded domains

Throughout the whole article, Ω will be a connected region (bounded or unbounded) in $\mathbb{R}^n$, and for any function $u \in C^1(\Omega)$, Tu will denote the vector $\dfrac{Du}{\sqrt{1+|Du|^2}}$, where Du is the gradient vector of u.

The extension of comparison principles of bounded domain to unbounded domain are based on the following inequality.

LEMMA 2.1. *For any functions u and v in $C^1(\Omega)$, we have*

$$
\begin{aligned}
(Tu - Tv)\cdot(Du - Dv) &\geq \frac{\sqrt{1+|Du|^2}+\sqrt{1+|Dv|^2}}{2}|Tu - Tv|^2\\
&\geq |Tu - Tv|^2
\end{aligned}
$$

and

$$(Tu - Tv)\cdot(Du - Dv) = 0 \quad \text{if and only if} \quad Du = Dv.$$

PROOF. Let

$$\alpha = (1, u_{x_1}, \cdots, u_{x_n}), \ \beta = (1, v_{x_1}, \cdots, v_{x_n}), \ \cos\,\theta = \frac{\alpha\cdot\beta}{|\alpha|\ |\beta|},$$

then we have

$$
\begin{aligned}
(Tu - Tv)\cdot(Du - Dv) &= \left(\frac{\alpha}{|\alpha|} - \frac{\beta}{|\beta|}\right)\cdot(\alpha - \beta)\\
&= |\alpha| + |\beta| - |\beta|\cos\,\theta - |\alpha|\cos\,\theta\\
&= (|\alpha| + |\beta|)\cdot(1 - \cos\,\theta)
\end{aligned}
$$

and

$$\begin{aligned}\frac{\sqrt{1+|Du|^2}+\sqrt{1+|Dv|^2}}{2}|Tu-Tv|^2 &\leq \frac{|\alpha|+|\beta|}{2}\left|\frac{\alpha}{|\alpha|}-\frac{\beta}{|\beta|}\right|^2 \\ &\leq \frac{|\alpha|+|\beta|}{2}(2-2\cos\ \theta) \\ &= (Tu-Tv)\cdot(Du-Dv).\end{aligned}$$

The other assertions follow immediately, and the Lemma is proved.

For every $R > 0$, let $B_R = \{x \in \mathbb{R}^n \mid |x| < R\}$, $\tilde{\Omega}_R = \Omega \cap B_R$, $\tilde{\Gamma}_R = \partial\tilde{\Omega}_R \cap \partial B_R$ and $|\tilde{\Gamma}| =$ the $n-1$ dimensional Hausdorff measure of $\tilde{\Gamma}_R$. With these notations, the comparison principles for prescribed mean curvature equations can be stated as in the following theorems.

THEOREM 2.2. *Let $\partial\Omega = \Sigma^\alpha + \Sigma^\beta$ be a decomposition of $\partial\Omega$ such that Σ^β is of class C^1 and for every u, $v \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma^\beta) \cap C^0(\overline{\Omega})$ let $M(R) = \max_{\tilde{\Omega}_R} u - v$. Suppose that*

(i) div $Tu \geq$ div Tv *in* Ω

(ii) $u \leq v$ *on* Σ^α

(iii) $Tu \cdot \nu \leq Tv \cdot \nu$ *on* Σ^β

where ν be the unit outer normal of Σ^β.

(iv) *if $M(R_1) > 0$ for some $R_1 > 0$ then* $\int_{R_1}^{\infty} \frac{1}{M^2(R)|\tilde{\Gamma}_R|} dR = \infty$.

Then if $\partial\Omega = \Sigma^\beta$, we have $u(x) \equiv v(x)$+positive constant or $u(x) \leq v(x)$. Otherwise, $u(x) \leq v(x)$ in Ω.

As an immediate consequence of Theorem 2.2, we have the following uniqueness theorem in $\mathbb{R}^2$.

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^2$, Σ^α, Σ^β, u and v be as in Theorem 2.2, suppose that*

(i) div $Tu =$ div Tv *in* Ω

(ii) $u = v$ *on* Σ^α

(iii) $Tu \cdot \nu = Tv \cdot \nu$ *on* Σ^β

(iv) $\max_{\tilde{\Omega}_R} |u-v| = O(\sqrt{\log\ R})$ *as* $R \to \infty$.

*Then if $\partial\Omega = \Sigma^\beta$, we have $u(x) \equiv v(x)$+*constant, *otherwise $u(x) = v(x)$.*

REMARK. (1) Theorem 2.2 generalizes the comparison principles of Concus and Finn [2, 3]. The main point in our theorem is to characterize the relation between the growth of the difference of the functions and the growth of the "width" of the domain.

(2) The growth condition (iv) of Theorem 2.3 cannot be improved too much. For considering the domain $\Omega = \{(x,y)|x^2+y^2>1\}$, we see that the Dirichlet problem of minimal surface equation with zero boundary data has two solutions $u \equiv 0$ and the catenoid $v = \cos\ h^{-1}\sqrt{(x^2+y^2)}$. Noting that $|u-v| = \cos\ h^{-1}R = O(\log\ R)$ as $R \to \infty$.

PROOF OF THEOREM 2.2. If $\{x \in \Omega | u(x) - v(x) > 0\}$ is non-empty, there exists $\epsilon > 0$ so that $\Omega' = \{x \in \Omega | u(x) - v(x) > \epsilon\}$ is non-empty and $\partial\Omega' \cap \Omega$ is smooth (Sard's Theorem). For any $R > 0$, set

$$\Omega_R = \Omega' \cap B_R,\ \Gamma_R = \partial\Omega_R \cap \partial B_R,\ \Gamma'_R = \partial\Omega_R \cap \partial\Omega,\ \text{and}\ \Gamma''_R = \partial\Omega_R - (\Gamma_R \cup \partial\Omega).$$

By the definition of Σ^β, we see that $\partial\Omega' \cap \partial\Omega \subset \Sigma^\beta$ and $\Gamma'_R \subset \Sigma^\beta$. Using the divergence theorem, we have

$$\begin{aligned}\int_{\partial\Omega_R} &(u-v-\epsilon)(Tu-Tv)\cdot\nu d\sigma \\ &= \int_{\Omega_R} (Du-Dv)\cdot(Tu-Tv)dx + \int_{\Omega_R}(u-v-\epsilon)(\text{div}\ Tu - \text{div}\ Tv)dx.\end{aligned}$$

Noting that $u-v-\epsilon \geq 0$ and $(Tu-Tv)\cdot\nu \leq 0$ on Γ'_R; $u-v-\epsilon = 0$ on Γ''_R, $u-v-\epsilon \geq 0$ and div Tu − div $Tv \geq 0$ in Ω', we obtain

$$\int_{\Gamma_R}(u-v-\epsilon)(Tu-Tv)\cdot\nu d\sigma \geq \int_{\Omega_R}(Du-Dv)\cdot(Tu-Tv)dx.$$

Let

$$g(R) = \int_{\Gamma_R}(u-v-\epsilon)(Tu-Tv)\cdot\nu d\sigma$$

$$h(R) = \int_{\Gamma_R}(Du-Dv)\cdot(Tu-Tv)d\sigma$$

and

$$k(R) = \int_0^R h(r)dr.$$

Since $(Du - Dv) \cdot (Tu - Tv) \geq 0$, by Fubini Theorem, we have

$$\begin{aligned} k(R) &= \int_0^R h(r)dr \\ &= \int_0^R \int_{\Gamma_R} (Du - Dv) \cdot (Tu - Tv) d\sigma dr \\ &= \int_{\Omega_R} (Du - Dv) \cdot (Tu - Tv) dx \\ &\leq g(R) \end{aligned}$$

for every $R \geq 0$, and

$$\begin{aligned} \left(\int_{\Gamma_R} |Tu - Tv| d\sigma \right)^2 &\leq \left(\int_{\Gamma_R} 1^2 d\sigma \right) \cdot \left(\int_{\Gamma_R} |Tu - Tv|^2 d\sigma \right) \\ &\leq |\Gamma_R| \int_{\Gamma_R} (Du - Dv) \cdot (Tu - Tv) d\sigma \\ &= |\Gamma_R| h(R). \end{aligned}$$

Thus

$$\begin{aligned} g^2(R) &\leq \left(\int_{\Gamma_R} M(R) |Tu - Tv| d\sigma \right)^2 \\ &\leq M^2(R) |\Gamma_R| h(R). \end{aligned}$$

Since $k(R) = \int_0^R h(r)dr$ and $h(r) \geq 0$, we have $k(0) = 0$, $k(R)$ increases as R increases and $k'(R) = h(R)$ for almost all R.

If there exists some $R_2 > 0$ such that $k(R_2) > 0$ then for $R \geq R_2$,

$-(\frac{1}{k})'$ exists almost everywhere and $-(\frac{1}{k})' = \frac{h}{k^2}$ almost everywhere. Thus

$$\begin{aligned} -\frac{1}{k}\Big|_{R_2}^{\infty} &\geq \int_{R_2}^{\infty} -(\frac{1}{k})' = \int_{R_2}^{\infty} \frac{h}{k^2} \\ &\geq \int_{R_2}^{\infty} \frac{g^2}{M^2(R)|\Gamma_R|k^2} \geq \int_{R_2}^{\infty} \frac{1}{M^2(R)|\Gamma_R|} \\ &\geq \int_{R_2}^{\infty} \frac{1}{M^2(R)|\tilde{\Gamma}_R|} = \infty. \end{aligned}$$

This is impossible since $-\frac{1}{k}$ is bounded in $[R_2, \infty)$ and therefore we must have $k(R) \equiv 0$ for every $R \geq 0$. Since $(Du - Dv) \cdot (Tu - Tv) \geq 0$, we have $(Du - Dv) \cdot (Tu - Tv) \equiv 0$ in Ω' and by Lemma 2.1, we conclude that $Du \equiv Dv$ in Ω'.

There are two cases:

(a) $\partial\Omega' \cap \Omega$ is empty, by the connectedness of Ω we have $\Omega' = \Omega$ and hence $u \equiv v$+constant in Ω.

(b) $\partial\Omega' \cap \Omega$ is non-empty, then $u \equiv v + \epsilon$ in Ω' and, by definition, Ω' must be empty. This is impossible and we conclude that $u(x) \leq v(x)$ for all x in Ω.

If $\Sigma^\beta = \partial\Omega$, supposing $\{x \in \Omega | u(x) - v(x) > 0\}$ is non-empty, the above arguments yield $u(x) - v(x) \equiv$ positive constant, otherwise $u(x) \leq v(x)$.

If $\Sigma^\beta \neq \partial\Omega$, since $u(x) \leq v(x)$ in $\partial\Omega - \Sigma^\beta$, either in (a) or (b), we have $u(x) \leq v(x)$ in Ω.

This completes the proof of the theorem.

THEOREM 2.4. *If the hypothesis* (i) *in Theorem* 2.2 *is replaced by*

(i′) div $Tu >$ div Tv *for all x in Ω such that*

$$u(x) - v(x) > 0$$

then $u(x) \leq v(x)$ in Ω.

PROOF. The proof is identical to that of Theorem 2.2. Even in case (a) $u = v$+constant, we have div $Tu =$ div Tv and conclude that $u \leq v$ by (i′).

REMARK. If div $Tu = \kappa u$, div $Tv = \kappa v$ for some constant $\kappa > 0$, then (i′) still holds. In virtue of $|Tu| \leq 1$, Concus and Finn [2, 3] point out that the boundary situation for the comparison principles of prescribed mean curvature equations can be weakened. Theorem 2.2 and Theorem 2.4 can be improved by the concept of Concus and Finn.

Let $\partial\Omega = \Sigma^0 + \Sigma^\alpha + \Sigma^\beta$ be a decomposition of $\partial\Omega$ such that Σ^β is of class C^1 and Σ^0 can be covered, from within Ω, by a sequence of smooth surfaces $\{\Lambda\}$, each of which meets $\partial\Omega$ in a set of zero $(n-1)$- dimensional Hausdorff measure, and such that $\Lambda \to \Sigma^0$ and the area of Λ tends to zero. Let $\tilde{\Omega}_R$, B_R, $\tilde{\Gamma}_R$ and $|\tilde{\Gamma}_R|$ be as above, and let $\tilde{\Omega}_{R'R} = \tilde{\Omega}_R - \overline{B}_{R'}$, for $R > R' > 0$, a revised comparison principle can be stated as follows.

THEOREM 2.5. *Let u, $v \in C^2(\Omega)$. Suppose that*

(i) div $Tu \geq$ div Tv *in* Ω

(ii) lim sup $[u - v] \leq 0$ *for any approach to* Σ^α *from within* Ω

(iii) $(Tu - Tv) \cdot \nu \leq 0$ *almost everywhere on* Σ^β *as a limit from points of* Ω

(iv) $M_{R_0}(R) = \max\limits_{\tilde{\Omega}_{R_0 R}} (u - v) < \infty$ *for some positive constant* R_0 *and every* $R > R_0$.

(v) *If* $M_{R_0}(R_1) > 0$ *for some* $R_1 > R_0$, *then*

$$\int_{R_1}^{\infty} \frac{1}{M_{R_0}^2(R)|\tilde{\Gamma}_R|} dR = \infty.$$

Then, if Σ^α can be chosen such that $\Sigma^\alpha \subset \Sigma^0$, we have $u(x) \equiv v(x)$+positive constant or $u(x) \leq v(x)$. Otherwise, $u(x) \leq v(x)$.

PROOF. The proof is essentially identical to that of Theorem 2.2 with a modification by [2, Theorem 6]. We give here only a sketch of the proof.

If $\{x \in \Omega | u(x) - v(x) > 0\}$ is bounded or empty, then the theorem is an immediate consequence of [2, Theorem 6], so it suffices to assume the set is unbounded, hence there exists some constant $R_2 > R_1$ such that $M_{R_0}(R_2) > 0$. We may choose constants ϵ and M_2 such that $0 < \epsilon < M_{R_0}(R_2) < M_2$ and the set $\Omega' = \{x \in \Omega | u(x) - v(x) > \epsilon\} - \left\{x \in \bar{\tilde{\Omega}}_{R_2} | u(x) - v(x) \geq M_2\right\}$ is non-empty; by Sard's theorem we may also assume that $\partial\Omega' \cap \Omega$ is smooth. For every $R > R_2$, let $\Omega_R = \Omega' \cap B_R$, $\Gamma_R = \partial\Omega_R \cap \partial B_R$, $\Gamma'_R = \partial\Omega_R \cap \partial\Omega$, $\Gamma^\epsilon_R = \{x \in \partial\Omega_R | u(x) - v(x) = \epsilon\} - (\Gamma_R \cup \Gamma'_R)$ and let $\Gamma^{M_2} = \{x \in \partial\Omega_R | u(x) - v(x) = M_2\} - (\Gamma_R \cup \Gamma'_R)$. Noting that $\Gamma^{M_2} \subset B_{R_2}$ and that it is independent of R, using the divergence theorem we have

$$\int_{\Gamma_R \cup \Gamma'_R \cup \Gamma^\epsilon_R \cup \Gamma^{M_2}} (u - v - \epsilon)(Tu - Tv) \cdot \nu d\sigma$$

$$= \int_{\Omega_R} (u - v - \epsilon)(\text{div } Tu - \text{div } Tv) dx + \int_{\Omega_R} (Du - Dv) \cdot (Tu - Tv) dx.$$

Since $|Tu| \leq 1$ and $|Tv| \leq 1$, by using the same argument as [2, p. 193-195], we obtain

$$\int_{\Gamma_R} (u - v - \epsilon)(Tu - Tv) \cdot \nu d\sigma \leq \int_{\Omega_R} (Du - Dv) \cdot (Tu - Tv) dx;$$

the remainder of the proof is now identical to that of Theorem 2.2.

A parallel theorem of Theorem 2.4 can be stated as follows.

THEOREM 2.6. *If the hypothesis* (i) *in Theorem* 2.5 *is replaced by*

(i′) div $Tu >$ div Tv *for all* $x \in \Omega$ *such that*

$$u(x) > v(x).$$

Then $u(x) \leq v(x)$ *for all* x *in* Ω.

REMARK 2.7. (1) The hypothesis (iv) in Theorem 2.2 and Theorem 2.4 can be replaced by

(iv′) $|\tilde{\Gamma}_R| M(R) \leq O(R\sqrt{\log\ R})$ as $R \to \infty$.

If $n = 2$, since $|\tilde{\Gamma}_R| \leq 2\pi R$, it can furtherly be replaced by

(iv″) $M(R) \leq O(\sqrt{\log\ R})$ as $R \to \infty$.

Note that (iv″) only makes assumption on the growth of the difference between the functions and it is independent of the growth of the "width" of the domain.

(2) The hypothesis (v) in Theorem 2.5 and Theorem 2.6 can be replaced by

(v′) $|\tilde{\Gamma}_R| M_{R_0}(R) \leq O(R\sqrt{\log\ R})$ as $R \to \infty$.

By the above theorems, we have the following uniqueness theorem for minimal surface equation in $\mathbb{R}^2$, which generalizes the uniqueness theorem of Nitsche [8] from constant boundary data to arbitrary bounded continuous data.

THEOREM 2.8. *Let* $\Omega \subset \Omega_\alpha$, *where* Ω_α *is a sector domain in* $\mathbb{R}^2$ *with angle* $0 < \alpha < \pi$, *let* $\partial\Omega = \Sigma^0 + \Sigma^\alpha$, *where* Σ^0 *and* Σ^α *be as in Theorem 2.5, and let* $u,\ v \in C^2(\Omega)$. *Suppose that*

(i) div $Tu =$ div $Tv = 0$ *in* Ω

(ii) lim sup $u - v = 0$ *for any approach to* Σ^α *from within* Ω.

(iii) lim sup $|u| \leq M$ *and* lim sup $|v| \leq M$ *for any approach to* Σ^α *from within* Ω, *where* M *is a constant.*

Then $u \equiv v$.

PROOF. By [8, p. 256] we have $|u| \leq M$ and $|v| \leq M$ in Ω, the uniqueness then follows from Theorem 2.5.

We also have the following uniqueness theorem for capillary free surfaces.

THEOREM 2.9. *Let Ω be an unbounded domain in $\mathbb{R}^n$ with piecewise smooth boundary, let u, $v \in C^2(\Omega)$ and let $\tilde{\Gamma}_R$, $\tilde{\Omega}_{R'R}$ be as in Theorem 2.5. Suppose that*

(i) div $Tu = \kappa u$ *and* div $Tv = \kappa v$ *in Ω, where κ is a positive constant.*

(ii) $(Tu - Tv) \cdot \nu = 0$ *almost everywhere on $\partial\Omega$ as a limit from points of Ω, where ν is the outer normal of $\partial\Omega$.*

(iii) *There exists some $R_0 > 0$ such that $\delta(R) = \max\{r|$ the image of a ball of radius r moving in the interior of Ω covers $\tilde{\Omega}_{R_0R}\} > 0$.*

(iv) $\frac{|\Gamma_R|}{\delta(R)} = O(R\sqrt{\log R})$ *as* $R \to \infty$.

Then $u \equiv v$.

PROOF. For $r > 0$, let V_r be the solution of the equation

$$\text{div } TV_r = \kappa V_r \text{ in } B_r$$
$$TV_r \cdot \nu = 1 \text{ on } \partial B_r.$$

It follows from [13, Corollary of Theorem 7] that $0 < V_r <$ constant $\cdot\frac{1}{r}$. For $R > R_0$, $\delta(R)$ is a decreasing function of R, we have

$$\max_{\tilde{\Omega}_{R_0R}} |u| \le \max\ V_\delta(R) \le \text{ constant } \frac{1}{\delta(R)}$$

thus

$$|\tilde{\Gamma}_R| \cdot \max_{\tilde{\Omega}_{R_0R}} |u - v| \le \text{ constant } \frac{|\tilde{\Gamma}_R|}{\delta(R)} = O(R\sqrt{\log R})$$

as $R \to \infty$, and the theorem follows from Theorem 2.6 and Remark 2.7.

As a simple application of Theorem 2.9, we have the following Corollary, which shows that the one-dimensional solution of capillary free surfaces in a gravitational field over an infinite strip is the only solution.

COROLLARY 2.10. *Let Ω be an infinite strip or a sector domain in R^2, then the solution of the capillary free surface in a gravitational field is unique.*

3. - Comparison principles for non-uniformly elliptic equations in unbounded domains

Let Ω be a domain in $\mathbb{R}^n$, we consider the equation in divergence form:

$$\text{div } A(x, \omega, D\omega) = f(x, \omega, D\omega)$$

where $A = (A_1, \cdots, A_n)$, $A_i : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $A_i \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \cap C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $i = 1, \cdots, n$. Suppose that A satisfies the following structural conditions:
There exists a positive constant λ such that

$$|A(x,\omega,p) - A(x,\omega',q)|^2 \le \lambda(p-q) \cdot (A(x,\omega,p) - A(x,\omega',q)) \tag{3.1}$$

for all $x \in \overline{\Omega}$ and all (ω, p), $(\omega', q) \in \mathbb{R} \times \mathbb{R}^n$.

$$(p-q) \cdot (A(x,\omega,p) - A(x,\omega',q)) = 0 \text{ if and only if } p = q \tag{3.2}$$

for all $x \in \overline{\Omega}$ and all ω, $\omega' \in \mathbb{R}$.

We will write $A\omega$ instead of $A(x,\omega,D\omega)$ if there is no ambiguity.

Now we have

THEOREM 3.1. *Let Ω, Σ^α, Σ^β, $\tilde{\Gamma}_R$, $|\tilde{\Gamma}_R|$, u, v, $M(R)$ be as in Theorem* 2.2. *Suppose that*

(i) *A satisfies conditions* (3.1) *and* (3.2).

(ii) div $Au \ge$ div Av *in* Ω

(iii) $u \le v$ *on* Σ^α

(iv) $Au \cdot \nu \le Av \cdot \nu$ *on* Σ^β

(v) *if for some* $R_1 > 0$, *we have* $M(R_1) > 0$, *then*

$$\int_{R_1}^{\infty} \frac{1}{M^2(R)|\tilde{\Gamma}_R|} dR = \infty.$$

Then, if $\partial\Omega = \Sigma^\beta$, *we have* $u(x) \equiv v(x)+$*positive constant or* $u(x) \le v(x)$. *Otherwise,* $u(x) \le v(x)$.

PROOF. The proof is identical to that of Theorem 2.2.

THEOREM 3.2. *If the hypothesis* (ii) *of Theorem* 3.1 *is replaced by*

(ii′) div $Au >$ div Av *for all* $x \in \Omega$ *such that* $(u-v)(x) > 0$. *Then* $u(x) \le v(x)$ *in* Ω.

If $|A(x,\omega,p)| \le$ *constant, then the boundary conditions of Theorem* 3.1 *and* 3.2 *can be weakened also.*

THEOREM 3.3. *Let Ω, Σ^0, Σ^α, Σ^β, $\tilde{\Gamma}_R$, $|\tilde{\Gamma}_R|$, $\tilde{\Omega}_{R'R}$, u and v be as in Theorem* 2.5. *Suppose that*

(i) *A satisfies conditions* (3.1) *and* (3.2)

(ii) $|A(x,\omega,p)| \le$ *constant for every* $(x,\omega,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$

(iii) div $Au \ge$ div Av *in* Ω

(iv) lim sup $(u-v) \le 0$ *for any approach to* Σ^α *from within* Ω.

(v) $(Au - Av) \cdot \nu \leq 0$ *a.e. on* Σ^β *as a limit from points of* Ω.

(vi) *there exists some* $R_0 > 0$ *such that* $M_{R_0}(R) = \max_{\tilde{\Omega}_{R_0 R}} u - v < \infty$ *for every* $R > R_0$.

(vii) *if for some* $R_1 > R_0$, *we have* $M_{R_0}(R_1) > 0$, *then*

$$\int_{R_1}^{\infty} \frac{1}{M_{R_0}^2(R)|\tilde{\Gamma}_R|} dR = \infty.$$

Then, if Σ^α *can be chosen so that* $\Sigma^\alpha \subset \Sigma^0$, *we have* $u(x) \equiv v(x) +$*positive constant or* $u(x) \leq v(x)$. *Otherwise,* $u(x) \leq v(x)$.

PROOF. The proof is identical to that of Theorem 2.5.

THEOREM 3.4. *If the hypothesis* (iii) *of Theorem* 3.3 *is replaced by*

(iii′) div $Au >$ div Av *for all* $x \in \Omega$ *such that* $(u - v)(x) > 0$. *Then* $u(x) \leq v(x)$ *in* Ω.

REMARK 3.5. Let $D_R = \{(x_1, \cdots, x_n) | x_1 < R\}$ and Ω be a domain such that $\Omega \cap D_R$ is compact for every $R > 0$. Let $\tilde{\Omega}_R = \Omega \cap D_R$, $\tilde{\Gamma}_R = \partial\tilde{\Omega}_R \cap \partial D_R$, $|\tilde{\Gamma}_R| =$ the $(n-1)$-dimensional Hausdorff measure of $\tilde{\Gamma}_R$. Then we have the same results as Theorem 3.1 - 3.4.

4. - Examples for the structural conditions (3.1) and (3.2)

EXAMPLE 4.1. Let Ω be a domain in $\mathbb{R}^n$ and let $A_i : (x,p) \to \mathbb{R}$ for every $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$, $i = 1, \cdots, n$. Suppose that $A_i \in C^1(\Omega \times \mathbb{R}^n)$ and $\frac{\partial A_i}{\partial p_j}\varsigma_i\varsigma_j > 0$ for every $\varsigma \in \mathbb{R}^n - \{0\}$. We are going to find a sufficient condition which implies (3.1) and (3.2).

Let $F_i(p,x,A) = A_i - A_i(x,p)$, $i = 1, \cdots, n$. Since $\det[\frac{\partial F_i}{\partial p_j}] = (-1)^n \det[\frac{\partial A_i}{\partial p_j}] \neq 0$ and $F_i(p,x,A) = 0$, by implicit function theorem, we can solve $p = p(x,A)$ so that $A_i = A_i(x, p(x,A))$ for $i = 1, \cdots, n$ and $[\frac{\partial A_i}{\partial p_j}]^{-1} = [\frac{\partial p_i}{\partial A_j}]$.

Fix q, $q' \in R^n$ and let $E = A(x,q)$, $E' = A(x,q')$. Then $q = p(x,E)$, $q' = p(x,E')$.

$$
\begin{aligned}
(A(x,q)-A(x,q'))\cdot(q-q') &= (E-E')\cdot(p(x,E)-p(x,E')) \\
&= (E_i-E_i')(p_i(x,E_j)-p_i(x,E_j')) \\
&= (E_i-E_i')(p_i(x,E_j+s(E_j-E_j')))|_{s=0}^{1} \\
&= \int_0^1 \frac{d}{ds}(E_i-E_i')(p_i(x,E_j+s(E_j-E_j')))ds \\
&= \int_0^1 (E_i-E_i')\frac{\partial p_i}{\partial A_j}(E_j-E_j')ds.
\end{aligned}
$$

If $\frac{\partial p_i}{\partial A_j}\varsigma_i\varsigma_j \geq \lambda|\varsigma|^2$ for every $\varsigma \in \mathbb{R}^n$, then $(A(x,q)-A(x,q'))\cdot(q-q') \geq \lambda|E-E'|^2 = \lambda|A(x,q)-A(x,q')|^2$. But $\frac{\partial p_i}{\partial A_j}\varsigma_i\varsigma_j \geq \lambda|\varsigma|^2$ for every $\varsigma \in R^n$ if and only if $0 < \frac{\partial A_i}{\partial p_j}\varsigma_i\varsigma_j \leq \frac{1}{\lambda}|\varsigma|^2$ for every $\varsigma \in R^n - \{0\}$ and we find a sufficient condition which implies (3.1). It is easy to see that (3.2) is true under this condition also.

REMARK. In Example 4.1, $A = (A_i)$, $E = (E_i)$, $E' = (E_i')$, $p = (p_i)$, $q = (q_i)$, $q' = (q_i')$, $\varsigma = (\varsigma_i)$, $i = 1, \cdots, n$.

EXAMPLE 4.2. Consider the variation problem

$$J(u) = \int F(p_1, p_2, \cdots, p_n)dx$$

where $p_i = \frac{\partial u}{\partial x_i}$, $i = 1, \cdots, n$. Then the Euler equation is expressed as

$$\text{div } DF = \frac{\partial}{\partial x_i}F_{p_i} = F_{p_i p_j}\frac{\partial p_j}{\partial x_i}.$$

A sufficient condition which implies (3.1) and (3.2) is $0 < F_{p_i p_j}\varsigma_i\varsigma_j \leq \frac{1}{\lambda}|\varsigma|^2$ for every $\varsigma \in \mathbb{R}^n - \{0\}$.

EXAMPLE 4.3. Consider the prescribed mean curvature equation

$$\text{div } D\sqrt{1+|p|^2} = \text{div } \frac{p}{\sqrt{1+|p|^2}} = \frac{(1+|p|^2)\delta_{ij} - p_i p_j}{(1+|p|^2)^{3/2}}\frac{\partial p_j}{\partial x_i}.$$

Since $0 < \frac{(1+|p|^2)\varsigma_i^2 - p_i p_j \varsigma_i \varsigma_j}{(1+|p|^2)^{3/2}} \leq |\varsigma|^2$, we have that (3.1) and (3.2) hold.

5. - Liouville Theorems over $\mathbb{R}^2$

Let $A = (A_1, A_2)$ where $A_i : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $A_i \in C^1(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2)$, $i = 1, 2$. In this chapter, we generalize Meier's structural conditions [6] as follows: There exists some positive constant λ such that

$$|A(x, \omega, p)|^2 \leq \lambda p \cdot A(x, \omega, p) \tag{5.1}$$

for every $(x, \omega, p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$.

$$p \cdot A(x, \omega, p) = 0 \text{ if and only if } p = 0 \tag{5.2}$$

for every $(x, \omega) \in \mathbb{R}^2 \times \mathbb{R}$.

REMARK. (1) If A satisfies (3.1), (3.2) and $A(x, 0, D0) = 0$, then A satisfies (5.1) and (5.2) also.

(2) If A satisfies (3.1) and (3.2), then $A(x, \omega, p) + DF(x)$ satisfies (3.1) and (3.2) also, but $A(x, 0, D0) + DF(x) \neq 0$ in general.

THEOREM 5.1. *Let $u \in C^2(\mathbb{R}^2)$. Suppose that*

(i) *A satisfies conditions* (5.1) *and* (5.2)

(ii) div $Au \geq 0$ *in* $\mathbb{R}^2$

(iii) $\max_{B_R} u \leq O(\sqrt{\log R})$ *as* $R \to \infty$.

Then u is a constant.

PROOF. The proof is identical to that of Theorem 2.2 for the case $\Sigma^\beta = \partial\Omega$.

The following theorem is essentially a generalization of [6, Theorem 2].

THEOREM 5.2. *Let $u \in C^2(\mathbb{R}^2)$ and let α be a positive constant. Suppose that*

(i) *A satisfies conditions* (5.1) *and* (5.2)

(ii) div $Au \geq -\alpha|Au|^2$

(iii) $\sup_{\mathbb{R}^2} u = M < \infty$

Then u is a constant.

PROOF. Suppose u is not a constant, there exists $0 < \epsilon < \frac{\lambda}{2\alpha}$ such that the set $\Omega' = \{x \in \Omega | u(x) > M - \epsilon\}$ is non-empty and $\partial\Omega'$ is smooth (Sard's Theorem).

Let $\Omega_R = \Omega' \cap B_R$, $\Gamma_R = \partial\Omega_R \cap \partial B_r$ and $\Gamma'_R = \partial\Omega_R - \Gamma_R$ for every

$R > 0$. By divergence theorem, we have

$$\int_{\partial\Omega_R} (u - M + \epsilon) \cdot Au \cdot \nu d\sigma = \int_{\Omega_R} Du \cdot Audx + \int_{\Omega_R} (u - M + \epsilon)\text{div } Audx$$
$$\geq \lambda \int_{\Omega_R} |Au|^2 dx - \int_{\Omega_R} \epsilon \ \alpha |Au|^2 dx$$
$$\geq \frac{\lambda}{2} \int_{\Omega} |Au|^2 dx.$$

The remainder of the proof is identical to that of Theorem 2.2.

Let $f \in C^2(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2)$. Suppose that

$$\omega f \geq -\mu |p|^2 \text{ for all } (x, \omega, p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2, \tag{5.3}$$

where μ is a constant less than λ.

The following theorem is essentially a generalization of [6, Theorem 4].

THEOREM 5.3. *Let* $u \in C^2(\mathbb{R}^2)$. *Suppose that*

(i) A *satisfies conditions* (5.1) *and* (5.2)

(ii) f *satisfies condition* (5.3)

(iii) div $Au = f$

(iv) $\max_{B_R} u \leq O(\sqrt{\log R})$ *as* $R \to \infty$.

Then u *is a constant.*

In particular, we have

COROLLARY 5.4. (Cheng and Yau [1, Corollary 2 of Theorem 2]) *Let* $u \in C^2(\mathbb{R}^2)$. *Suppose that*

(i) div $Tu \leq 0$ *in* $\mathbb{R}^2$

(ii) $u \geq$ *constant in* $\mathbb{R}^2$.

Then u *is a constant.*

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Department of Mathematics
National Taiwan University
Taipei, Taiwan, R.O.C.

Current address:
Institute of Mathematics
Academia Sinica
Nankang, Taipei
Taiwan 11529 R.O.C.