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On a Maximum Principle for Weak Solutions of the Stationary Stokes System

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1. - Introduction. Statement of the Result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We consider the homogeneous stationary Stokes system with unit viscosity:

$$-\Delta u + \nabla p = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\text{div } u = 0 \quad \text{in } \Omega; \tag{1.2}$$

here $u = \{u_1, u_2, u_3\}$ and p represent the velocity field of the flow, and the undetermined pressure, respectively ($\nabla p = \{p_{x_1}, p_{x_2}, p_{x_3}\}$ [1)]).

By $\partial\Omega$ we denote the boundary of Ω. Without any further reference, throughout the whole paper we suppose that $\partial\Omega \in C^2$ (cf. e.g. [8] for the definition). System (1.1), (1.2) will be completed by the boundary condition

$$u = f \text{ on } \partial\Omega \tag{1.3}$$

where f is a given vector field on $\partial\Omega$.

We introduce some notations used in what follows. Let $D \subset \mathbb{R}^3$ be any bounded domain with Lipschitz boundary ∂D (cf. e.g. [8]). Then $H^k(D) \equiv W_2^k(D)$ $(k = 1, 2, \cdots)$ denotes the usual Sobolev space of all functions in $L^2(D)$ having their generalized derivatives up to order k (including) in $L^2(D)$. Further, let

$$H_0^1(D) = \{v \in H^1(D) : v = 0 \text{ a.e. on } \partial D\},$$

$$H^1(D; \mathbb{R}^3) = [H^1(D)]^3$$

$$H_0^1(D; \mathbb{R}) = [H_0^1(D)]^3$$

and

$$V(D) = \{v \in H_0^1(D; \mathbb{R}^3) : \text{ div } v = 0 \text{ a.e. in } D\}.$$

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1) $\varphi_{x_i} = \frac{\partial \varphi}{\partial x_i}$ (with respect to a Cartesian frame; $i = 1, 2, 3$).

In order to define the notion of weak solution of (1.1)-(1.3) let $f \in H^{1/2}(\partial\Omega; \mathbb{R}^3) \equiv [W_2^{1/2}(\partial\Omega)]^3$ [2)] be given such that

$$\int_{\partial\Omega} f_i n_i dS = 0$$

($n = \{n_1, n_2, n_3\}$ = outward unit normal along $\partial\Omega$).

The function $u \in H^1(\Omega; \mathbb{R}^3)$ is called a *weak solution of* (1.1)-(1.3) if

$$\int_\Omega \nabla u_i \cdot \nabla \varphi_i dx = 0 \quad \forall \varphi \in V(\Omega), \tag{1.4}$$

$$\text{div } u = 0 \quad \text{a.e. in } \Omega, \tag{1.5}$$

$$u = f \quad \text{a.e. on } \partial\Omega. \tag{1.6}$$

It is well-known that the above conditions on f guarantee the existence and uniqueness of a weak solution of (1.1)-(1.3) (cf. e.g. [6]). Furthermore, (1.4) implies the existence of an element $\hat{p} \in L^2(\Omega)/\mathbb{R}$ such that

$$\int_\Omega \nabla u_i \cdot \nabla \chi_i dx = \int_\Omega p \text{ div } \chi dx \quad \forall \chi \in H_0^1(\Omega; \mathbb{R}^3), \ \forall p \in \hat{p} \tag{1.4'}$$

(cf. [5], [7], [11]). In addition, there holds $u \in [C^\infty(\Omega)]^3$ and $p \in C^\infty(\Omega)$ (for all $p \in \hat{p}$) (cf. [6], [7]).

The aim of the present paper is to prove a global L^∞-bound on the Euclidean norm of the weak solution of (1.1)-(1.3) in terms of f. We follow an idea of Cannarsa [4] and make essential use of results by Giaquinta, Modica [5] and Solonnikov, Ščadilov [11]. Moreover, our approach gives an additional information on p near the boundary $\partial\Omega$ (p according to (1.4′); cf. (3.2) below).

For any $\xi \in \mathbb{R}^3$, let

$$B_r(\xi) = \{x \in \mathbb{R}^3 : |x - \xi| < r\}.$$

The main result of our paper is the following

THEOREM. *Let* $f \in H^1(\Omega; \mathbb{R}^3)$. *Let* div $f = 0$ *a.e. in* Ω, *and let there exist an* $0 < R_0 \leq$ diam Ω *such that*

$$\Lambda_1 := \operatorname*{ess\,sup}_{\{x \in \Omega : \text{dist}(x, \partial\Omega) < R_0\}} |f|^2 < +\infty, \tag{1.7}$$

2) Cf. e.g. [8] for a discussion of the spaces $W_p^r(\partial\Omega)$ $(1 \leq p < +\infty,\ 0 < r < +\infty)$. In what follows, we do not make, however, any explicit use of these spaces. Throughout repeated Latin subscripts imply summation over 1,2,3.

$$\Lambda_2 := \sup\left\{ \frac{1}{r} \int\limits_{B_r(\xi)\cap\Omega} |\nabla f|^2 dx \,\middle|\, 0 < r \le R_0,\ \xi \in \partial\Omega \right\} < +\infty. \tag{1.8}$$

Let $u \in H^1(\Omega;\mathbb{R}^3)$ *be the weak solution of* (1.1)-(1.3). *Then*

$$\operatorname{ess\,sup}_{\Omega} |u|^2 \le c \left(\Lambda_1 + \Lambda_2 + \int\limits_{\Omega} (|f|^2 + |\nabla f|^2) dx \right) \tag{1.9}$$

where the constant c *depends on geometric properties of* $\partial\Omega$ *only.*

The paper is organized as follows. Section 2 is devoted to the proof of an inequality on the weak solution of the Stokes system in a semi-ball. This inequality is of an independent interest; it relies essentially on the square integrability of the second order derivatives of the solution near the base of the semi-ball, which we are going to prove in the appendix. The proof of our main theorem is then given in the third and fourth section.

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2. - The Stokes System in a Semi-Ball

Let

$$B_r^+ = B_r^+(0) = \{x \in \mathbb{R}^3 : |x| < r,\ x_3 > 0\}.$$

Suppose we are given a function $w \in H^1(B_r^+;\mathbb{R}^3)$ satisfying

$$\operatorname{div} w = 0 \text{ a.e. in } B_r^+,\quad w = 0 \text{ a.e. on } \partial B_r^+ \cap \{x_3 = 0\}. \tag{2.1}$$

By the Lax-Milgram lemma, there exists a uniquely determined function $U \in H^1(B_r^+;\mathbb{R}^3)$ such that

$$\int\limits_{B_r^+} \nabla U_i \cdot \nabla \varphi_i dx = 0 \quad \forall \varphi \in V(B_r^+), \tag{2.2}$$

$$\operatorname{div} U = 0 \quad \text{a.e. in } B_r^+, \tag{2.3}$$

$$U = w \quad \text{a.e. on } \partial B_r^+. \tag{2.4}$$

As above, (2.2) implies the existence of an element $\hat{q} \in L^2(B_r^+)/\mathbb{R}$ such that, for any $q \in \hat{q}$,

$$\int_{B_r^+} \nabla U_i \cdot \nabla \chi_i dx = \int_{B_r^+} \left(q - q_{B_r^+}\right) \text{ div } \chi dx \quad \forall \chi \in H_0^1(B_r^+;\mathbb{R}^3). \tag{2.2$'$}$$

In addition, there holds

$$\int_{B_r^+} \left(q - q_{B_r^+}\right)^2 dx \leq c_0 \int_{B_r^+} |\nabla U|^2 dx \tag{2.5}$$

where

$$q_{B_r^+} = \frac{3}{2\pi r^3} \int_{B_r^+} q dy$$

and c_0 is an absolute constant. Indeed, B_r^+ being star-shaped with respect to any interior point of it, there exists a $\varsigma \in H_0^1(B_r^+;\mathbb{R}^3)$ such that

$$\text{div } \varsigma = q - q_{B_r^+} \text{ a.e. in } B_r^+, \quad \int_{B_r^+} |\nabla \varsigma|^2 dx \leq c_0 \int_{B_r^+} q^2 dx$$

(cf. [1]). By a homothetical argument, the constant c_0 can be easily seen to be independent of r. Now, letting $\chi = \varsigma$ in (2.2$'$) gives (2.5).

The proof of our main result is based on the estimate (2.6) below.

PROPOSITION (Campanato type estimate). *Let $U \in H^1(B_r^+;\mathbb{R}^3)$ satisfy (2.2)-(2.4). Then*

$$\int_{B_\rho^+} |\nabla U|^2 dx \leq c \left(\frac{\rho}{r}\right)^2 \int_{B_r^+} |\nabla U|^2 dx \quad \forall\ 0 < \rho \leq r \tag{2.6}$$

with $c =$ const independent of both ρ and r.

REMARK. Estimates of the type (2.6) [with $\left(\frac{\rho}{r}\right)^3$ in place of $\left(\frac{\rho}{r}\right)^2$; more general, with $\left(\frac{\rho}{r}\right)^n$ when $\mathbb{R}^n$ is the underlying space] have been firstly proved in [2] for weak solutions of homogeneous linear elliptic equations with constant coefficients (cf. [3] for a detailed discussion of estimates of this type).

We note that estimate (2.6) with $\left(\frac{\rho}{r}\right)^3$ can be proved when the third order derivatives of U are in L^2 near the boundary $\partial B_r^+ \cap \{x_3 = 0\}$ and appropriate estimates on these derivatives are available (cf. (2.8) below). However, (2.6) is sufficient for our later purposes.

PROOF OF THE PROPOSITION. We begin by observing that

$$U_{ix_kx_l},\ q_{x_k} \in L^2(B_{r/4}^+), \tag{2.7}$$

$$\int_{B^+_{r/4}} \left[(U_{ix_kx_l})^2 + (q_{x_k})^2\right] dx \leq \frac{c}{r^2} \int_{B^+_r} |\nabla U|^2 dx \tag{2.8}$$

($i, k, l = 1, 2, 3$; $c =$ const independent of r). The proof of (2.7) and (2.8) will be given in the appendix.

Estimate (2.6) is now easily deduced from (2.8). Indeed, let $0 < \rho \leq \frac{r}{4}$. By Hölder's inequality and Sobolev's imbedding theorem,

$$\int_{B^+_\rho} |\nabla U|^2 dx \leq \left(\frac{2\pi}{3}\right)^{2/3} \rho^2 \left(\int_{B^+_\rho} |\nabla U|^6 dx\right)^{1/3}$$

$$\leq c\rho^2 \left(\frac{1}{r^2} \int_{B^+_{r/4}} |\nabla U|^2 dx + \sum_{i,k,l=1}^{3} \int_{B^+_{r/4}} (U_{ix_kx_l})^2 dx\right)$$

where the constant c is independent of both ρ and r [3]. This can be readily seen by a homothetical argument. Thus, by (2.8),

$$\int_{B^+_\rho} |\nabla U|^2 dx \leq c \left(\frac{\rho}{r}\right)^2 \int_{B^+_r} |\nabla U|^2 dx.$$

This inequality is trivial for $\frac{r}{4} < \rho \leq r$. Whence (2.6).

3. - Proof of the Theorem

We begin by proving the following statement which is of an independent interest:

Let $f \in H^1(\Omega; \mathbb{R}^3)$ with $\operatorname{div} f = 0$ a.e. in Ω, and let $u \in H^1(\Omega; \mathbb{R}^3)$ be the weak solution of (1.1)-(1.3). *Suppose there exist constants $0 < R_0 \leq$ diam Ω and $0 < \lambda < 2$ such that*

$$(*) \qquad \Lambda := \sup \left\{ \frac{1}{r^\lambda} \int_{B_r(\xi)\cap\Omega} |\nabla f|^2 dx \,\middle|\, 0 < r \leq R_0,\ \xi \in \partial\Omega \right\} < +\infty.$$

Then there exists an $0 < R_1 \leq R_0$ and a constant $c > 0$ which both depend

[3] By c we denote different positive constants possibly changing their numerical value from line to line.

on λ and on geometric properties of $\partial\Omega$ only, such that

$$(3.1)\quad \int_{B_r(\xi)\cap\Omega} |\nabla(u-f)|^2 dx \le c\left(\Lambda + \int_\Omega |\nabla f|^2 dx\right) r^\lambda \ \forall\ 0 < r \le R_1,\ \forall \xi \in \partial\Omega.$$

REMARK. Let $x \in \Omega$ and $r = \operatorname{dist}(x, \partial\Omega) \le \frac{R_1}{2}$. Let $p \in L^2(\Omega)$ satisfy (1.4′). Then

$$(3.2)\quad \int_{B_r(x)} (p - p_{B_r(x)})^2\, dy \le c\left(\Lambda_2 + \int_\Omega |\nabla f|^2 dx\right) r$$

where

$$p_{B_r(x)} = \frac{3}{4\pi r^3} \int_{B_r(x)} p\,dy$$

and $c =$ const independent of x and r.

Indeed, there exists an $\eta \in H_0^1(B_r(x); \mathbb{R}^3)$ such that

$$\operatorname{div} \eta = p - p_{B_r(x)} \text{ a.e. in } B_r(x),$$

$$\int_{B_r(x)} |\nabla\eta|^2 dy \le c_0 \int_{B_r(x)} (p - p_{B_r(x)})^2\, dy$$

with an absolute constant c_0 (cf. [1], [10]). Let $\chi = \eta$ a.e. in $B_r(x)$ and $\chi = 0$ a.e. in $\Omega\backslash B_r(x)$. Then $\chi \in H_0^1(\Omega; \mathbb{R}^3)$, and (1.4′) implies

$$\int_{B_r(x)} (p - p_{B_r(x)})^2\, dy \le c \int_{B_r(x)} |\nabla u|^2 dy.$$

Let $\xi \in \partial\Omega$ satisfy $|\xi - x| = r$. Clearly, $B_r(x) \subset B_{2r}(\xi) \cap \Omega$, and (3.2) follows by combining (3.1) (with $\Lambda = \Lambda_2$ (from (1.8)) and $\lambda = 1$) and the latter estimate.

We divide the proof of (3.1) into four steps.

1° Let $\xi \in \partial\Omega$ be arbitrary. We introduce Cartesian coordinates $y = \{y_1, y_2, y_3\}$ by

$$y = A(x - \xi)$$

where the direction of the negative y_3-axis coincides with the direction of the outward normal (with respect to Ω) at ξ, and $A = \{a_{ij}\}$ is an orthogonal matrix (with a_{ij} depending on ξ).

Our assumption $\partial\Omega \in C^2$ guarantees the existence of a real $\sigma = \sigma^{(\xi)} > 0$ and a function $F = F^{(\xi)} \in C^2(\Delta_\sigma)$ $(\Delta_\sigma = [-\sigma, \sigma] \times [-\sigma, \sigma])$ such that

$$\{y \in \mathbb{R}^3 : \{y_1, y_2\} \in \Delta_\sigma,\ y_3 = F(y_1, y_2)\} \subset \partial\Omega,$$

$$\{y \in \mathbb{R}^3 : \{y_1, y_2\} \in \Delta_\sigma,\ F(y_1, y_2) < y_3 \leq F(y_1, y_2) + \sigma\} \subset \Omega,$$

$$(3.3)\quad \begin{cases} F(0,0) = 0,\ \nabla F(0,0) = 0, \\ |\nabla F(y_1, y_2)| + \sum\limits_{\alpha,\beta=1}^{2} |F_{y_\alpha y_\beta}(y_1, y_2)| \leq M = \text{const}\ \forall \{y_1, y_2\} \in \Delta_\sigma. \end{cases}$$

Now, for all $\xi \in \partial\Omega$, the reals $\sigma = \sigma^{(\xi)}$ are uniformly bounded from below by a fixed positive constant, while the constants M (possibly depending on ξ) are uniformly bounded from above by a fixed constant. This can be established by the aid of the compactness of $\partial\Omega$. Thus, in all that follows, both σ and M are assumed to be independent of $\xi \in \partial\Omega$.

Set $\overline{u} = u - f$ a.e. in Ω. Then from (1.4) we get

$$(3.4)\quad \int_\Omega |\nabla \overline{u}|^2 dx \leq \int_\Omega |\nabla f|^2 dx,$$

$$(3.5)\quad \int_{B_\sigma(\xi)\cap\Omega} \nabla \overline{u}_i \cdot \nabla \varphi_i dx = - \int_{B_\sigma(\xi)\cap\Omega} \nabla f_i \cdot \nabla \varphi_i dx \quad \forall \varphi \in V(B_\sigma(\xi) \cap \Omega).$$

Next, for any $0 < r \leq \sigma$ let

$$C_r(0) = \{y \in \mathbb{R}^3 : |y| < r,\ y_3 > F(y_1, y_2)\}.$$

The orthogonality of A implies $B_r(\xi) \cap \Omega = C_r(0)$.

We introduce functions v and g on $C_\sigma(0)$ by setting

$$v(y) = A\overline{u}(x), \quad g(y) = Af(x) \text{ for a.a. } y \in C_\sigma(0).$$

Then (3.5) takes the form

$$(3.6)\quad \int_{C_\sigma(0)} \nabla v_i \cdot \nabla \chi_i dy = - \int_{C_\sigma(0)} \nabla g_i \cdot \nabla \chi_i dy \quad \forall \chi \in V(C_\sigma(0)).$$

Further,

$$\operatorname{div} v = 0 \text{ a.e. in } C_\sigma(0),\ v = 0 \text{ a.e. on } \partial C_\sigma(0) \cap \{y_3 = F(y_1, y_2)\},$$

$$(3.7)\quad \int_{B_r(\xi)\cap\Omega} |\nabla \overline{u}|^2 dx = \int_{C_r(0)} |\nabla v|^2 dy,$$

$$(3.8)\quad \int_{B_r(\xi)\cap\Omega} |\nabla f|^2 dx = \int_{C_r(0)} |\nabla g|^2 dy$$

for all $0 < r \le \sigma$.

2° We introduce new variables $z = \{z_1, z_2, z_3\}$ by the transformation

$$z = T(y) = \{y_1, y_2, y_3 - F(y_1, y_2)\}, \quad y \in C_\sigma(0).$$

Clearly, T is a one-to-one mapping (with Jacobian $\equiv 1$) from $C_\sigma(0)$ onto $D_\sigma = T(C_\sigma(0))$.

Define

$$\delta := (1 + \max\{1, 2M^2\})^{1/2} (M \text{ according to (3.3)}), \ r_1 := \frac{\sigma}{\delta},$$
$$B_{r_1}^+ := \{z \in \mathbb{R}^3 : |z| < r_1, z_3 > 0\}.$$

Then $B_{r_1}^+ \subset D_\sigma$. Indeed, $z \in B_{r_1}^+$ implies $\{z_1, z_2\} \in \Delta_\sigma$. Letting denote $y_1 = z_1, \ y_2 = z_2, \ y_3 = z_3 + F(z_1, z_2)$ we have $y_3 > F(y_1, y_2)$ and

$$|y|^2 \le |z|^2 + z_3^2 + 2(F(z_1, z_2))^2 \le |z|^2 + \max\{1, 2M^2\}|z|^2,$$

i.e. $y \in C_\sigma(0)$ and therefore $z = T(y) \in D_\sigma$. Furthermore, a simple calculation shows

$$\partial(T^{-1}(B_{r_1}^+)) = T^{-1}(\partial B_{r_1}^+). \tag{3.9}$$

Now we introduce new functions w and h by

$$w_\alpha(z) = v_\alpha(y) \ (\alpha = 1, 2), \ w_3(z) = v_3(y) - F_{y_\beta}(y_1, y_2) v_\beta(y) \ ^{4)},$$
$$h(z) = g(y)$$

for a.a. $y \in C_\sigma(0)$ $(z = T(y))$ (cf. [11]). Then

$$\begin{aligned} w_{\alpha z_\beta} &= v_{\alpha y_\beta} + F_{y_\beta} v_{\alpha y_3}, \quad w_{\alpha z_3} = v_{\alpha y_3}, \\ w_{3 z_\alpha} &= v_{3 y_\alpha} + F_{y_\alpha} v_{3 y_3} - F_{y_\alpha y_\gamma} v_\gamma - F_{y_\gamma}(v_{\gamma y_\alpha} + F_{y_\alpha} v_{\gamma y_3}), \\ w_{3 z_3} &= v_{3 y_3} - F_{y_\gamma} v_{\gamma y_3} \quad (\alpha, \beta = 1, 2). \end{aligned}$$

Thus, $w \in H^1(D_\sigma; \mathbb{R}^3)$, div $w = 0$ a.e. in D_σ and $w = 0$ a.e. on $\partial D_\sigma \cap \{z_3 = 0\}$. Analogously, $h \in H^1(D_\sigma; \mathbb{R}^3)$.

Let $\psi \in V(B_{r_1}^+)$ be arbitrary. Set

$$\chi_\alpha(y) = \psi_\alpha(z) \ (\alpha = 1, 2), \ \chi_3(y) = \psi_3(z) + F_{z_\beta}(z_1, z_2) \psi_\beta(z)$$

for a.a. $z \in B_{r_1}^+$ $(y = T^{-1}(z))$. As above, $\chi \in H^1(T^{-1}(B_{r_1}^+); \mathbb{R}^3)$ and div $\chi = 0$ a.e. in $T^{-1}(B_{r_1}^+)$. By (3.9), $\chi = 0$ a.e. on $\partial(T^{-1}(B_{r_1}^+))$. Hence $\chi \in V(T^{-1}(B_{r_1}^+))$.

4) Repeated Greek subscripts imply summation over 1 and 2.

We extend χ by zero onto $C_\sigma(0)\backslash T^{-1}(B_{r_1}^+)$ and obtain an admissible test function for (3.6). This gives

$$\int_{B_{r_1}^+} \nabla w_i \cdot \nabla \psi_i dz = \int_{B_{r_1}^+} A(w,\psi)dz + \int_{B_{r_1}^+} B(h,\psi)dz \tag{3.10}$$

where

$$A(w,\psi) = A_{kl}^{ij} w_{iz_j}\psi_{kz_l} + A_\alpha^{ij}(w_{iz_j}\psi_\alpha + w_\alpha \psi_{iz_j}) + F_{z_\alpha z_\gamma} F_{z_\beta z_\gamma} w_\alpha \psi_\beta,$$

$$B(h,\psi) = -\nabla h_i \cdot \nabla \psi_i + B_{kl}^{ij} h_{iz_j}\psi_{kz_l} + F_{z_\alpha z_\beta}(h_{3z_\alpha} - F_{z_\alpha} h_{3z_3})\psi_\beta.$$

Here $A_{kl}^{ij} = B_{kl}^{ij} \equiv 0$ if $i+j+k+l \leq 5$, while the coefficients A_{kl}^{ij} and B_{kl}^{ij} with $6 \leq i+j+k+l \leq 12$ (at least one index = 3) are of the form $\pm F_{z_\alpha}$, $\pm F_{z_\alpha}F_{z_\beta}$, $\pm F_{z_\alpha}F_{z_\beta}F_{z_\gamma}$ or $F_{z_\alpha}F_{z_\beta}|\nabla F|^2$, respectively (e.g. $A_{\alpha 3}^{\alpha\beta} = -F_{z_\beta}$, $A_{\beta 3}^{\alpha 3} = F_{z_\alpha}F_{z_\beta}|\nabla F|^2$ $(\alpha,\beta = 1,2)$); the coefficients A_α^{ij} are composed by the functions $F_{z_\alpha z_\beta}$, $\pm F_{z_\alpha}F_{z_\beta z_\gamma}$ or $F_{z_\alpha}F_{z_\beta}F_{z_\beta z_\gamma}$, respectively. Thus, A_{kl}^{ij}, B_{kl}^{ij} and A_α^{ij} are continuous functions on Δ_σ and the following estimates hold:

$$|A_{kl}^{ij}|,\ |B_{kl}^{ij}| \leq c_0(1+|\nabla F|+|\nabla F|^2+|\nabla F|^3)|\nabla F|, \tag{3.11}$$

$$|A_\alpha^{ij}| \leq c_0(1+|\nabla F|+|\nabla F|^2)\sum_{\beta,\gamma=1}^{2}|F_{z_\beta z_\gamma}| \tag{3.12}$$

for all $z_1^2+z_2^2 \leq r_1^2$ $(i,j,k,l = 1,2,3, \alpha = 1,2;\ c_0 = \text{const})$.

3° Let $0 < r \leq r_1$ be arbitrary (recall that $r_1 = \sigma(1+\max\{1,2M^2\})^{-1/2}$). Let $U \in H^1(B_r^+;\mathbb{R}^3)$ denote the uniquely determined solution of (2.2)-(2.4) $[w = v\circ T^{-1}$ in (2.4)]. Then

$$\int_{B_\rho^+} |\nabla w|^2 dz \leq 4c_0\left(\frac{\rho}{r}\right)^2 \int_{B_r^+} |\nabla w|^2 dz + 2(1+2c_0)\int_{B_r^+} |\nabla(w-U)|^2 dz \tag{3.13}$$

for all $0 < \rho \leq r$ where c_0 is the constant occuring in (2.6).

The function

$$\psi = \begin{cases} w - U & \text{a.e. in } B_r^+, \\ 0 & \text{a.e. in } B_{r_1}^+\backslash B_r^+ \end{cases}$$

is admissible in (3.10). Adding (3.10) and (2.2) with $\varphi = w - U$ we find

$$\begin{aligned}\int_{B_r^+} |\nabla(w-U)|^2 dz &= \int_{B_r^+} A(w,w-U)dz + \int_{B_r^+} B(h,w-U)dz \\ &= I_1 + I_2.\end{aligned} \tag{3.14}$$

In order to estimate I_1 we first note that $|\nabla F| \le c(M)(z_1^2+z_2^2)^{1/2}$ [5] for all $z_1^2+z_2^2 \le r_1^2$ (cf. (3.3)). Thus, by (3.11),

$$\left| \int_{B_r^+} A_{kl}^{ij} w_{iz_j} (w-U)_{kz_l} dz \right| \le \frac{1}{8} \int_{B_r^+} |\nabla(w-U)|^2 dz + c(M) r^2 \int_{B_r^+} |\nabla w|^2 dz$$

[for what follows it is decisive that the factor of $\int_{B_r^+} |\nabla w|^2 dz$ can be made arbitrarily small to obtain (3.16) below; this explains the introduction of the coordinate system $y = A(x-\xi)$ *at each* $\xi \in \partial\Omega$]. The estimation of the remaining two integrals forming I_1, is readily seen when taking into account (3.3), (3.12) and $w - U = 0$ a.e. on ∂B_r^+. Thus,

$$I_1 \le \frac{1}{4} \int_{B_r^+} |\nabla(w-U)|^2 dz + c(M) r^2 \int_{B_r^+} |\nabla w|^2 dz.$$

Next, using (3.3) and (3.11) one easily obtains

$$I_2 \le \frac{1}{4} \int_{B_r^+} |\nabla(w-U)|^2 dz + c(\sigma, M) \int_{B_r^+} |\nabla h|^2 dz$$

$(0 < r \le r_1)$. Inserting these estimates into (3.14) and combining this result with (3.13) we find

$$\text{(3.15)} \qquad \int_{B_\rho^+} |\nabla w|^2 dz \le c(M) \left[\left(\frac{\rho}{r}\right)^2 + r^2 \right] \int_{B_r^+} |\nabla w|^2 dz + c(\sigma, M) \int_{B_r^+} |\nabla h|^2 dz$$

for all $0 < \rho \le r \le r_1$.

It remains to estimate the second integral on the right of (3.15). To this end, we note that $T^{-1}(B_r^+) \subset C_{\delta r}(0)$ [for $z \in B_r^+$ implies $|T^{-1}(z)|^2 \le |z|^2 + z_3^2 + 2(F(z_1,z_2))^2 \le \delta^2 |z|^2$]. Therefore,

$$\int_{B_r^+} |\nabla h|^2 dz = \int_{T^{-1}(B_r^+)} |\nabla h(T(y))|^2 dy \le \int_{C_{\delta r}(0)} |\nabla h(T(y))|^2 dy.$$

On the other hand, from $h(z) = g(y)$ $(z = T(y))$ we infer that $|\nabla h(z)| \le c_0(1 + \max_{\Delta_\sigma} |\nabla F|) |\nabla g(y)|$ for a.a. $y \in C_\sigma(0)$ $(c_0 = \text{const})$. Thus, by $(*)$, (3.3)

[5] In what follows, we denote by $c(M)$ (resp. $c(\sigma,M)$) different positive constants which only depend on M (resp. σ and M).

and (3.8),

$$\int_{B_r^+} |\nabla h|^2 dz \le c(M) \int_{B_{\delta r}(\xi)\cap\Omega} |\nabla f|^2 dx \le c(M)\Lambda(\delta r)^\lambda$$

for all $0 < r \le \min\left\{\frac{R_0}{\delta}, r_1\right\}$ [6].

Now, (3.15) gives

$$\int_{B_\rho^+} |\nabla w|^2 dz \le c(M)\left[\left(\frac{\rho}{r}\right)^2 + r^2\right] \int_{B_r^+} |\nabla w|^2 dz + c(\sigma,\lambda,M)\Lambda r^\lambda$$

for all $0 < \rho \le r \le \min\left\{\frac{R_0}{\delta}, r_1\right\}$. Hence there exists an $0 < r_2 \le \min\left\{\frac{R_0}{\delta}, r_1\right\}$ such that

$$\int_{B_r^+} |\nabla w|^2 dz \le c(\sigma,\lambda,M)\left(\frac{1}{r_2}\int_{B_{r_2}^+} |\nabla w|^2 dz + \Lambda\right) r^\lambda \tag{3.16}$$

for all $0 < r \le r_2$ (cf. e.g. [5; Lemma 0.6]). Here r_2 only depends on M via $c(M)$.

4° In (3.16) we return from w to u. To begin with, we note that $T(C_r(0)) \subset B_{\delta r}^+$ for any $0 < r \le \frac{r_1}{\delta}$ $(= \sigma(1+\max\{1, 2M^2\})^{-1})$. Observing that $|\nabla v(y)| \le c(M)(|w(z)| + |\nabla w(z)|)$ for a.a. $z \in B_{r_1}^+$ $(y = T^{-1}(z))$ we get by virtue of (3.7)

$$\begin{aligned}\int_{B_r(\xi)\cap\Omega} |\nabla \bar{u}|^2 dx &= \int_{T(C_r(0))} |\nabla v(T^{-1}(z))|^2 dz \\ &\le c(M)\int_{B_{\delta r}^+} (|w|^2 + |\nabla w|^2) dz \\ &\le c(\sigma, M)\int_{B_{\delta r}^+} |\nabla w|^2 dz \end{aligned}\tag{3.17}$$

for all $0 < r \le \frac{r_1}{\delta}$ [note that $w = 0$ a.e. on $\partial B_{\delta r}^+ \cap \{z_3 = 0\}$, $\frac{r_1}{\delta} < \sigma$].

Next, we have $T^{-1}(B_r^+) \subset C_{\delta r}(0)$ for all $0 < r \le r_1$, $|\nabla w(z)| \le c(M)(|v(y)| + |\nabla v(y)|)$ for a.a. $y \in C_\sigma(0)$ $(z = T(y))$ and $v = 0$ a.e. on

[6] We emphasize that the components a_{ij} of the matrix A occuring in $y=A(x-\xi)$, do not explicitely enter into (3.7) and (3.8). Therefore, all estimates in step 4° are independent of the a_{ij}'s and thus on $\xi\in\partial\Omega$, too.

$C_\sigma(0) \cap \{y_3 = F(y_1, y_2)\}$. Thus, (3.7) and (3.4) imply

$$\begin{aligned} \int_{B^+_{r_2}} |\nabla w|^2 dz &= \int_{T^{-1}(B^+_{r_2})} |\nabla w(T(y))|^2 dy \\ &\le c(\sigma, M) \int_{C_{\delta r_2}(0)} |\nabla v|^2 dy \\ &\le c(\sigma, M) \int_{B_{\delta r_2}(\xi)\cap\Omega} |\nabla \overline{u}|^2 dx \\ &\le c(\sigma, M) \int_\Omega |\nabla f|^2 dx \end{aligned} \tag{3.18}$$

[for $r_2 \le \min\left\{\frac{R_\Omega}{\delta}, r_1\right\}$, i.e. $\delta r_2 \le \delta r_1 = \sigma$].

Combining (3.16) with (3.17) and (3.18) one finally obtains

$$\begin{aligned} \int_{B_r(\xi)\cap\Omega} |\nabla \overline{u}|^2 dx &\le c(\sigma, M) \left(\frac{1}{r_2} \int_{B^+_{r_2}} |\nabla w|^2 dz + \Lambda \right) (\delta r)^\lambda \\ &\le c(\sigma, \lambda, M) \left(\int_\Omega |\nabla f|^2 dx + \Lambda \right) r^\lambda \end{aligned}$$

for all $0 < r \le \frac{r_2}{\delta}$ [$r_2 = r_2(M)$ being fixed]. Thus, (3.1) is satisfied with $R_1 := \frac{r_2}{\delta}$.

4. - Proof of the Theorem completed

Let $x \in \Omega$ be arbitrary. Let $d = \text{dist}(x, \partial\Omega)$. Then there holds

$$\int_{B_\rho(x)} |u|^2 dy \le c_0 \left(\frac{\rho}{d}\right)^3 \int_{B_d(x)} |u|^2 dy \quad \forall\ 0 < \rho \le d \tag{4.1}$$

with c_0 an absolute constant (cf. [5; Prop. 1.9]).

We distinguish two cases.

(i) $d \ge \frac{R_1}{2}$ (R_1 according to (3.1) with $\Lambda = \Lambda_2$ (from (1.8)) and $\lambda = 1$).

Then (4.1) combined with (3.4) gives

$$\int\limits_{B_\rho(x)} |u|^2 dy \le 16 c_0 R_1^{-3} \rho^3 \int\limits_\Omega (|f|^2 + |u-f|^2) dy$$
$$\le c\rho^3 \int\limits_\Omega (|f|^2 + |\nabla f|^2) dy$$

where the constant c depends on Ω only.

(ii) $d < \frac{R_1}{2}$. There exists a $\xi \in \partial\Omega$ such that $|\xi - x| = d$. Following [4] we combine (1.7) and (3.1) to obtain

$$\int\limits_{B_d(x)} |u|^2 dy \le \frac{8}{3}\pi\ d^3 \operatorname*{ess\,sup}_{B_d(x)} |f|^2 + 2 \int\limits_{B_d(x)} |u-f|^2 dy$$
$$\le \frac{8}{3}\pi\ d^3 \Lambda_1 + c_0 d^2 \int\limits_{B_{2d}(\xi)\cap\Omega} |\nabla(u-f)|^2 dy$$
$$\le c(\sigma, M)\left(\Lambda_1 + \Lambda_2 + \int\limits_\Omega |\nabla f|^2 dy\right) d^3,$$

c_0 being an absolute constant.

Thus, in both cases,

$$\int\limits_{B_\rho(x)} |u|^2 dy \le c\left(\Lambda_1 + \Lambda_2 + \int\limits_\Omega (|f|^2 + |\nabla f|^2) dy\right)\rho^3$$

for all $0 < \rho \le d = \operatorname{dist}(x, \partial\Omega)$. Since almost all points $x \in \Omega$ are Lebesgue points of $|u|^2$, the latter inequality implies the assertion of the Theorem.

Appendix: Proof of (2.7) and (2.8)

We apply an idea from Solonnikov, Ščadilov [11] (cf. step 3 below). In that paper, the authors prove the square integrability of the second order derivatives of any generalized solution to the inhomogeneous Stokes system near the boundary of a bounded domain with C^3-boundary (i.e. after introducing the new variables $z = \{z_1, z_2, z_3\}$ (cf. above) the reasoning in [11] refers to an equation of type (3.10)). In contrast to that, we start immediately from (2.2). Therefore, our proof of (2.7) is technically simpler that the one in [11]. In addition, we establish the estimate (2.8) which is crucial for the proof of (2.6).

To begin with, we introduce the following notations. Let $\varsigma \in L^1(B_r^+)$. We extend ς by zero onto $\mathbb{R}^3_+ \setminus B_r^+$ [7] and denote this function on $\mathbb{R}^3_+$ again by ς.

[7] $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$.

Then define

$$\varsigma_\varepsilon(x) = \int_{\mathbb{R}^2} \omega_\varepsilon(x'-y')\varsigma(y',x_3)dy'$$

for any $\varepsilon > 0$ and almost all $x \in \mathbb{R}^3_+$, where $x' = \{x_1, x_2\}$, $y' = \{y_1, y_2\} \in \mathbb{R}^2$ and $\omega_\varepsilon(x') = \frac{1}{\varepsilon^2}\omega\left(\frac{x'}{\varepsilon}\right)$, ω being the standard mollifying kernel in $\mathbb{R}^2$. We have:

(i) Let $\varsigma \in L^2(B_r^+)$. Then

$$\int_{B_r^+} \varsigma_\varepsilon^2 dx \le \int_{B_r^+} \varsigma^2 dx \quad \forall \varepsilon > 0.$$

(ii) Let $\varsigma \in H^1(B_r^+)$. Then $\varsigma_{\varepsilon x_i} = (\varsigma_{x_i})_\varepsilon$ a.e. in $B^+_{3r/4}$ for $i = 1,2,3$ and $0 < \varepsilon < \frac{r}{4}$.

1. Proof of

$$\text{(A.1)} \qquad U_{ix_jx_\alpha} \in L^2(B^+_{r/2}), \quad \int_{B^+_{r/2}} (U_{ix_jx_\alpha})^2 dx \le \frac{c}{r^2}\int_{B_r^+} |\nabla U|^2 dx$$

($i, j = 1,2,3, \alpha = 1,2,$ $c = \text{const} > 0$ *independent of* r).

Let $\psi \in [C^\infty(B_r^+)]^3$, $\operatorname{supp}(\psi) \subset B^+_{3r/4}$. We extend ψ by zero onto $\mathbb{R}^3_+ \setminus B_r^+$, denote this function on $\mathbb{R}^3_+$ again by ψ and form

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^2} \omega_\varepsilon(x'-y')\psi(y',x_3)dy'$$

for a.a. $x \in B_r^+$ and all $0 < \varepsilon < \frac{r}{4}$. Then $\psi_{\varepsilon x_\alpha} = 0$ near ∂B_r^+ $(\alpha = 1,2)$. Using $\psi_{\varepsilon x_\alpha}$ as test function in (2.2′) (in place of χ), changing variables and observing (ii) gives

$$\int_{B^+_{3r/4}} \nabla U_{\varepsilon i} \cdot \nabla \psi_{ix_\alpha} dx = \int_{B^+_{3r/4}} \left(q - q_{B_r^+}\right)_\varepsilon \operatorname{div} \psi_{x_\alpha} dx.$$

Thus, by integration by parts,

$$\text{(A.2)} \qquad \int_{B^+_{3r/4}} \nabla U_{\varepsilon i x_\alpha} \cdot \nabla \psi_i dx = \int_{B^+_{3r/4}} \left(q - q_{B_r^+}\right)_{\varepsilon x_\alpha} \operatorname{div} \psi dx.$$

By an approximation argument, (A.2) is in fact true for any $\psi \in H^1_0(B^+_{3r/4}; \mathbb{R}^3)$ (cf. e.g. [8; Th. 4.10, p. 87]).

Let $\eta \in C^\infty(\mathbb{R}^3)$ be a cut-off function for $B_{3r/4} : \eta \equiv 1$ on $B_{r/2}$, $\eta \equiv 0$ in $\mathbb{R}^3 \backslash B_{3r/4}$ and $0 \le \eta \le 1$, $|\nabla \eta| \le \frac{c_0}{r}$ and $|\eta_{x_i x_j}| \le \frac{c_0}{r^2}$ in $\mathbb{R}^3$ $(i,j = 1,2,3$, $c_0 = \text{const} > 0$ independent of r). Then $\psi = U_{\varepsilon x_\alpha}\eta^2 \in \overset{\circ}{H}{}^1_0(B^+_{3r/4};\mathbb{R}^3)$ $(0 < \varepsilon < \frac{r}{4}$, $\alpha = 1,2)$ [note that $U_{\varepsilon x_\alpha} = 0$ a.e. on $\partial B^+_{3r/4} \cap \{x_3 = 0\}$ by virtue of (2.1) and (2.4)]. Observing that div $U_{\varepsilon x_\alpha} = (\text{div } U)_{\varepsilon x_\alpha} = 0$ a.e. in $B^+_{3r/4}$ (cf. (2.3) and (ii) above) we obtain from (A.2)

$$
\begin{aligned}
&\int_{B^+_{3r/4}} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx \\
&= -2 \int_{B^+_{3r/4}} U_{\varepsilon i x_j x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_j} dx + 2 \int_{B^+_{3r/4}} \left(q - q_{B^+_r}\right)_{\varepsilon x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_i} dx \\
\text{(A.3)} \quad &= -2 \int_{B^+_{3r/4}} U_{\varepsilon i x_j x_\alpha} U_{\varepsilon i x_\alpha} \eta \eta_{x_j} dx \\
&\quad - 2 \int_{B^+_{3r/4}} \left(q - q_{B^+_r}\right)_\varepsilon [U_{\varepsilon i x_\alpha x_\alpha} \eta \eta_{x_i} + U_{\varepsilon i x_\alpha} (\eta_{x_\alpha} \eta_{x_i} + \eta \eta_{x_\alpha x_i})] dx \\
&= I_1 + I_2
\end{aligned}
$$

[no summation over α].

The estimation of I_1 is standard:

$$
\begin{aligned}
I_1 &\le \frac{1}{4} \int_{B^+_{3r/4}} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B^+_{3r/4}} |\nabla U_\varepsilon|^2 dx \\
&\le \frac{1}{4} \int_{B^+_{3r/4}} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B^+_{3r/4}} |\nabla U|^2 dx
\end{aligned}
$$

(cf. (i) and (ii) above). Next, to estimates I_2 we make use of (2.5), (i) and (ii):

$$
\begin{aligned}
&-2 \int_{B^+_{3r/4}} \left(q - q_{B^+_r}\right)_\varepsilon U_{\varepsilon i x_\alpha x_\alpha} \eta \eta_{x_i} dx \\
&\le \frac{1}{4} \int_{B^+_{3r/4}} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B^+_{3r/4}} \left[\left(q - q_{B^+_r}\right)_\varepsilon\right]^2 dx \\
&\le \frac{1}{4} \int_{B^+_{3r/4}} |\nabla U_{\varepsilon x_\alpha}|^2 \eta^2 dx + \frac{c}{r^2} \int_{B^+_r} |\nabla U|^2 dx,
\end{aligned}
$$

$$-2\int\limits_{B^+_{3r/4}} (q-q_{B^+_r})_\varepsilon U_{\varepsilon i x_\alpha}(\eta_{x_i}\eta_{x_\alpha}+\eta\eta_{x_i x_\alpha})dx$$
$$\le \frac{c}{r^2}\int\limits_{B^+_{3r/4}} |\nabla U|^2 dx.$$

Inserting these estimates into (A.3) we get

$$\int\limits_{B^+_{r/2}} |\nabla U_{\varepsilon x_\alpha}|^2 dx \le \frac{c}{r^2}\int\limits_{B^+_r} |\nabla U|^2 dx \quad \forall\ 0<\varepsilon\le \frac{r}{4}. \tag{A.4}$$

Letting $\varepsilon \to 0$ implies (A.1).

2. Proof of

$$\begin{cases} U_{3x_3x_3},\ q_{x_3} \in L^2(B^+_{r/2}), \\ \int\limits_{B^+_{r/2}} \left[(U_{3x_3x_3})^2+(q_{x_3})^2\right] dx \le \frac{c}{r^2}\int\limits_{B^+_r} |\nabla U|^2 dx. \end{cases} \tag{A.5}$$

Firstly, div $U=0$ a.e. in B^+_r and (A.1) imply

$$U_{3x_3x_3} = -(U_{1x_1}+U_{2x_2})_{x_3} = -U_{1x_3x_1} - U_{2x_3x_2}$$

a.e. in $B^+_{r/2}$. Whence the statement on $U_{3x_3x_3}$ in (A.5).

Secondly, let $h \in H^1_0(B^+_{r/2})$. We extend h by zero onto $B^+_r\backslash B^+_{r/2}$ and denote this function on B^+_r again by h. Then $\chi=\{0,0,h\}$ is admissible in (2.2.′):

$$-\int\limits_{B^+_{r/2}} (\Delta U_3)h dx = \int\limits_{B^+_{r/2}} q h_{x_3} dx.$$

The statement on q_{x_3} in (A.5) is now readily seen.

3. Proof of

$$q_{x_\alpha} \in L^2(B^+_{r/4}),\ \int\limits_{B^+_{r/4}} (q_{x_\alpha})^2 dx \le \frac{c}{r^2}\int\limits_{B^+_r} |\nabla U|^2 dx \quad (\alpha=1,2). \tag{A.6}$$

In order to prove (A.6) we need the following result.

Let $f \in H^1(\mathbb{R}^3_+)$ have bounded support. Then there exists a function $\phi \in H^2(\mathbb{R}^3_+;\mathbb{R}^3)$ such that

$$\text{div } \phi = f \ \textit{a.e. in}\ \mathbb{R}^3_+, \tag{A.7}$$

(A.8) $$\phi = 0 \text{ a.e. on } \{x_3 = 0\},$$

(A.9) $$\int_{\mathbb{R}^3_+} |\nabla\phi|^2 dx \le c \int_{\mathbb{R}^3_+} f^2 dx,$$

(A.10) $$\int_{\mathbb{R}^3_+} |\nabla\phi_{x_\alpha}|^2 dx \le c \int_{\mathbb{R}^3_+} (f_{x_\alpha})^2 dx \quad (\alpha = 1, 2)$$

($c = \text{const} > 0$ *independent of* f).

This result is stated without proof in [11]. A proof of (A.7)-(A.10) can be given by using the explicit representation of the solution of

$$-\Delta v + \nabla p = 0,\ \text{div } v = f \text{ in } \mathbb{R}^3_+,\ v = 0 \text{ on } \{x_3 = 0\}$$

in terms of potentials the kernels of which involve only differences with respect to x_1 and x_2 ($x = \{x_1, x_2, x_3\} \in \mathbb{R}^3_+$; cf. [9; pp. 163-165]) [private communication by V.A. Solonnikov].

An entirely different and more elementary solution of (A.7)-(A.10) can be given as follows [private communication by V.A. Solonnikov]. Define

$$\phi_i(x) = \int_{\mathbb{R}^3} K\left(\frac{x-y}{|x-y|}\right) \frac{x_i - y_i}{|x-y|^3}\, \tilde{f}(y)dy,\ x \in \mathbb{R}^3_+\ (i = 1, 2, 3);$$

here K is any function in $C^2(\mathbb{R}^3)$ with $\text{supp}(K) \subset \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$ and $\int_{\partial B_1(0)} K\, dS = 1$, and

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for a.a. } x \in \mathbb{R}^3_+, \\ 0 & \text{for a.a. } x \in \mathbb{R}^3 \setminus \mathbb{R}^3_+. \end{cases}$$

Then (A.7) and (A.8) are easily verified. Further, the derivatives ϕ_{ix_k} as well as $\phi_{ix_\alpha x_k}$ ($i, k = 1, 2, 3; \alpha = 1, 2$) give rise to a singular integral to which the well-known Calderon-Zygmund theorem applies. Whence (A.9) and (A.10) (cf. also [10; Lemma 2.1, p. 252]).

Now, let $\eta \in C^\infty(\mathbb{R}^3)$ be a cut-off function for $B_{r/2} : \eta \equiv 1$ on $B_{r/4}$, $\eta \equiv 0$ in $\mathbb{R}^3 \setminus B_{r/2}$ and $0 \le \eta \le 1$, $|\nabla\eta| \le \frac{c_0}{r}$ and $|\eta_{x_i x_j}| \le \frac{c_0}{r^2}$ in $\mathbb{R}^3$ ($i, j = 1, 2, 3$; $c_0 = \text{const} > 0$ independent of r). We apply the result just stated with $f = \left(q - q_{B_r^+}\right)_\varepsilon \eta$ a.e. in B_r^+, $f = 0$ a.e. in $\mathbb{R}^3_+ \setminus B_r^+$ $\left(0 < \varepsilon < \frac{r}{4}\right)$. Thus, there exists a function $\phi^{(\varepsilon)} \in H^2(\mathbb{R}^3_+; \mathbb{R}^3)$ such that

(A.7$_\varepsilon$) $$\text{div } \phi^{(\varepsilon)} = \left(q - q_{B_r^+}\right)_\varepsilon \eta \quad \text{a.e. in } B_r^+,$$

(A.8$_\varepsilon$) $$\phi^{(\varepsilon)} = 0 \quad \text{a.e. on } B_r^+ \cap \{x_3 = 0\},$$

$$\text{(A.9}_\varepsilon\text{)} \qquad \int\limits_{\mathbb{R}^3_+} |\nabla \phi^{(\varepsilon)}|^2 dx \le c \int\limits_{B^+_{r/2}} \left(q - q_{B^+_r}\right)^2 dx,$$

$$\text{(A.10}_\varepsilon\text{)} \qquad \int\limits_{\mathbb{R}^3_+} |\nabla \phi^{(\varepsilon)}_{x_\alpha}|^2 dx \le c \int\limits_{B^+_{r/2}} \left[\left(q_{\varepsilon x_\alpha}\right)^2 \eta^2 + \frac{1}{r^2} \left(q - q_{B^+_r}\right)^2 \right] dx$$

($\alpha = 1, 2$; $c = \text{const} > 0$ independent of r; note that $\left(q - q_{B^+_r}\right)_\varepsilon = q_\varepsilon - q_{B^+_r}$).

Clearly, $\phi^{(\varepsilon)}_{x_\alpha} \eta \in H^1_0(B^+_{3r/4}; \mathbb{R}^3)$ ($\alpha = 1, 2$). Thus, $\chi = \phi^{(\varepsilon)}_{x_\alpha} \eta$ is admissible in (A.2). Taking into account (A.7_ε) and

$$\int\limits_{B^+_{3r/4}} \left(q - q_{B^+_r}\right)_{\varepsilon x_\alpha} \phi^{(\varepsilon)}_{i x_\alpha} \eta_{x_i} dx$$

$$= - \int\limits_{B^+_{3r/4}} \left(q - q_{B^+_r}\right)_\varepsilon \left(\phi^{(\varepsilon)}_{i x_\alpha x_\alpha} \eta_{x_i} + \phi^{(\varepsilon)}_{i x_\alpha} \eta_{x_i x_\alpha} \right) dx$$

we get

$$\int\limits_{B^+_{r/2}} (q_{\varepsilon x_\alpha})^2 \eta^2 dx$$

$$\text{(A.11)} \qquad = \int\limits_{B^+_{r/2}} \nabla U_{\varepsilon i x_\alpha} \cdot \nabla \eta \phi^{(\varepsilon)}_{i x_\alpha} dx + \int\limits_{B^+_{r/2}} \nabla U_{\varepsilon i x_\alpha} \cdot \nabla \phi^{(\varepsilon)}_{i x_\alpha} \eta dx$$

$$+ \int\limits_{B^+_{r/2}} \left(q - q_{B^+_r}\right)_\varepsilon \left(\phi^{(\varepsilon)}_{i x_\alpha x_\alpha} \eta_{x_i} + \phi^{(\varepsilon)}_{i x_\alpha} \eta_{x_i x_\alpha} \right) dx$$

$$- \int\limits_{B^+_{r/2}} q_{\varepsilon x_\alpha} \left(q - q_{B^+_r}\right)_\varepsilon \eta \eta_{x_\alpha} dx$$

$$= J_1 + J_2 + J_3 + J_4$$

[no summation over α]. To estimate J_1 and J_2 we combine (2.5) and (A.4), (A.9_ε), (A.10_ε):

$$J_1 \le \frac{1}{2} \int\limits_{B^+_{r/2}} |\nabla U_{\varepsilon x_\alpha}|^2 dx + \frac{c_0^2}{2r^2} \int\limits_{B^+_{r/2}} |\nabla \phi^{(\varepsilon)}|^2 dx$$

$$\le \frac{c}{r^2} \int\limits_{B^+_r} |\nabla U|^2 dx,$$

$$J_2 \le \left(\int\limits_{B^+_{r/2}} |\nabla U_{\varepsilon x_\alpha}|^2 dx \right)^{1/2} \left(\int\limits_{B^+_{r/2}} |\nabla \phi^{(\varepsilon)}_{x_\alpha}|^2 dx \right)^{1/2}$$
$$\le \frac{1}{4} \int\limits_{B^+_{r/2}} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int\limits_{B^+_r} |\nabla U|^2 dx.$$

Analogously, by (2.5) and ($A.9_\varepsilon$), ($A.10_\varepsilon$),

$$J_3 \le \frac{c}{r} \left\{ \int\limits_{B^+_r} |\nabla U|^2 dx \right\}^{1/2} \left\{ \int\limits_{B^+_{r/2}} \left(\frac{1}{r^2} |\nabla \phi^{(\varepsilon)}|^2 + |\nabla \phi^{(\varepsilon)}_{x_\alpha}|^2 \right) dx \right\}^{1/2}$$
$$\le \frac{1}{4} \int\limits_{B^+_{r/2}} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int\limits_{B^+_r} |\nabla U|^2 dx.$$

Finally,

$$J_4 \le \frac{1}{4} \int\limits_{B^+_{r/2}} (q_{\varepsilon x_\alpha})^2 \eta^2 dx + \frac{c}{r^2} \int\limits_{B^+_r} |\nabla U|^2 dx.$$

Inserting the estimates on $J_1, \cdots, J_4$ into (A.11) and letting $\varepsilon \to 0$ we get (A.6).

4. PROOF OF

$$U_{\alpha x_3 x_3} \in L^2(B^+_{r/4}), \quad \int\limits_{B^+_{r/4}} (U_{\alpha x_3 x_3})^2 dx \le \frac{c}{r^2} \int\limits_{B^+_r} |\nabla U|^2 dx \quad (\alpha = 1, 2).$$

Let $h \in H^1_0(B^+_{r/4})$. We extend h by zero onto $B^+_r \backslash B^+_{r/4}$ and denote this function on B^+_r again by h. Then we let $\chi = \{h, 0, 0\}$ in (2.2′) and find

$$\int\limits_{B^+_{r/4}} U_{1x_3} h_{x_3} dx = - \int\limits_{B^+_{r/4}} (U_{1x_1} h_{x_1} + U_{1x_2} h_{x_2}) dx + \int\limits_{B^+_{r/4}} q h_{x_1} dx.$$

Hence, the claim follows for $\alpha = 1$ when observing (A.4) and (A.6). To prove the claim for $\alpha = 2$ we let $\chi = \{0, h, 0\}$ in (2.2′) and argue analogously.

The proof of (2.7) and (2.8) is complete.