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An Asymptotic Formula for the Green's Function of an Elliptic Operator

MARTINO BARDI (*)

1. Introduction

Let $G^\varepsilon(x,y)$ be the Green's function with pole at y of the Dirichlet problem for the uniformly elliptic operator L^ε, i.e., the weak solution of

$$\begin{cases} L^\varepsilon u := -\varepsilon a_{ij} u_{x_i x_j} + b_i u_{x_i} = \delta_y & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \varepsilon \leq 1$, δ_y is the Dirac measure at $y \in \Omega$, $\Omega \subseteq \mathbb{R}^N$ is bounded and open, and we adopt, here and in the following, the summation convention. In this paper we show that under certain conditions on the vector field b, as $\varepsilon \searrow 0$ $G^\varepsilon(\cdot, y)$ converges exponentially to 0, uniformly on compact subsets of $\Omega \backslash \{y\}$, with rate of decay $\dfrac{-I(x,y)+o(1)}{\varepsilon}$, $I(x,y) \geq 0$, and we give a representation formula for $I(x,y)$.

The exponential decay as $\varepsilon \searrow 0$ of the Green's functions of the parabolic operators $\dfrac{\partial}{\partial t} + L^\varepsilon$ was studied by Varadhan [17, 18] in the case $b \equiv 0$, and by Friedman [7, 8] in the general case. Friedman employed the Ventcel-Freidlin estimates from the theory of large deviations of stochastically perturbed dynamical systems and some rather delicate parabolic estimates due to Aronson. His result is the following. For any $x(\cdot) \in W^{1,2}_{\text{loc}}([0,\infty), \overline{\Omega})$ define

$$(1.1) \qquad \|\dot{x}(s) + b(x(s))\|^2 := a^{ij}(x(s)(\dot{x}(s) + b(x(s)))_i \; (\dot{x}(s) + b(x(s)))_j,$$

where $((a^{ij})) = a^{-1}$ is the inverse matrix of a. If $\partial\Omega$, a_{ij} and b_i are smooth,

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then the Green's function with pole in y, $q^\varepsilon(t,x,y)$, of $\frac{\partial}{\partial t}+L^\varepsilon$, satisfies

$$(1.2)\quad \lim_{\varepsilon\searrow 0} -\varepsilon \log q^\varepsilon(t,x,y) = \inf\{\frac{1}{4}\int_0^t \|\dot{x}(s)+b(x(s))\|^2 ds | x(0)=x,$$
$$x(t)=y,\ x(s)\in\Omega\ \forall\ 0\le s\le t\}.$$

The corresponding formula we propose for the elliptic case is the following:

$$(1.3)\quad \lim_{\varepsilon\searrow 0} -\varepsilon \log G^\varepsilon(x,y) = I(x,y) := \inf\{\frac{1}{4}\int_0^t \|\dot{x}(s)+b(x(s))\|^2 ds;$$
$$x(\cdot)\in W^{1,2}([0,t],\Omega),\ x(0)=x,\ x(t)=y,\ \text{for some } t\in[0,\infty)\}.$$

It is clear, however, that unlike the parabolic case, such a formula can be true only under some strong assumptions on the vector field b, since in the simple case $b\equiv 0$, $G^\varepsilon \to +\infty$ as $\varepsilon\searrow 0$ uniformly on compact subsets of $\Omega\backslash\{y\}$. The main result of this paper is the proof of formula (1.3) in the case that b satisfies the following condition:

$$(\text{B1})\quad \begin{cases} \text{if } x(\cdot)\in W^{1,2}_{\text{loc}}([0,\infty),\overline{\Omega}), & \text{then} \\ \int_0^\infty |\dot{x}(s)+b(x(s))|^2 ds = \infty. \end{cases}$$

Condition (B1) was first used by Fleming [5] in the study of a singular perturbation problem arising in stochastic control theory. Its physical meaning is that it takes an infinite amount of energy to resist the flow determined by $-b$ and stay forever in $\overline{\Omega}$. In particular, b is "regular", i.e., it has no zeroes in $\overline{\Omega}$. We remark that the definition of $I(x,y)$ coincides with that of "quasipotential" of the vector field b with respect to the point y in Freidlin-Wentzell [6, p. 108].

Our proof of (1.3) is completely independent of formula (1.2) and also of the probabilistic methods used by Friedman. Instead we follow the PDE approach to WKB-type results initiated in the recent paper by L.C. Evans and H. Ishii [4], where new, totally analytic and simpler proofs are given of three results due respectively to Varadhan, Fleming, and Ventcel-Freidlin. The idea of Evans-Ishii is basically the following: 1) apply a logarithmic transformation to the unknown function, in our case

$$v^\varepsilon(x,y) := -\varepsilon \log G^\varepsilon(x,y),$$

and find a PDE that v^ε solves; 2) prove estimates, independent of ε, on v^ε and its gradient; 3) show that a subsequence of v^ε converges as $\varepsilon\searrow 0$, to the *viscosity solution* of a Hamilton-Jacobi equation (see Crandall-Lions [2], Crandall-Evans-Lions [1] and P.L. Lions [14]); 4) by deterministic control theory methods find a representation formula for such a solution. .

The main difficulties in the implementation of this plan in our case are in the treatment of the two boundary layers that our problem exhibits, one at the boundary $\partial\Omega$, where v^ε goes to $+\infty$, and the other around the singularity y, where v^ε goes to $-\infty$: notice that the limit $I(x,y)$ is positive and bounded. To deal with these problems we shall establish in §2 suitable estimates of v^ε around y, and we shall introduce in §3 various approximating problems.

The pioneering work about singular perturbation of elliptic operators is due to Levinson [13]. We refer to Schuss [15] for an introduction to the physical motivations and an extensive bibliography. The theory of viscosity solutions has been utilized for problems of this type also by P.L. Lions [14, Ch. 6] and Kamin [12]. For results in the nonregular case, i.e., b having one or more zeroes in Ω, we refer to Freidlin-Wentzell [6], Friedman [8, Ch. 14], Kamin [11], Day [3], and the papers quoted therein. Kamin [19] has also treated recently a nonregular problem where the relevant Hamilton-Jacobi equation has more than one viscosity solution.

The paper is organized as follows: in §2 we list the hypotheses, recall a few definitions and basic facts about the Green's function, prove the estimates for v^ε and deduce from them a convergence result; in §3 we prove the representation formula for the limit.

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2. Estimates and Convergence

Throughout the paper we assume the summation convention and $0 < \varepsilon \leq 1$. We will write for brevity $H := W^{1,2}_{\text{loc}}([0,\infty),\mathbb{R}^N)$.

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, satisfy

(A1) Ω is open, bounded and connected, with smooth boundary $\partial\Omega$.

Let $S^{N\times N}$ be the space on $N\times N$ symmetric matrices and let $a:\Omega\to S^{N\times N}$ satisfy

$$\text{(A2)}\qquad \begin{cases} a\in C^{1,\alpha}(\Omega) & \text{for some } \alpha>0 \quad \text{and} \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 & \text{for every } \xi\in\mathbb{R}^N,\ x\in\Omega. \end{cases}$$

Let $b:\overline{\Omega}\to\mathbb{R}^N$ satisfy

$$\text{(B0)}\qquad b\in C^{1,\alpha}(\Omega) \quad \text{for some} \quad \alpha>0.$$

and define

$$d_i^\varepsilon := b_i+\varepsilon a_{ij x_j},\quad D:=\sup_{\substack{0<\varepsilon\leq 1\\ i=1,\dots,N}}\|d_i^\varepsilon\|_{L^\infty(\Omega)}.$$

It is well known that in the above hypotheses, for every $f\in C^0(\overline{\Omega})$, the unique weak solution of

$$\text{(2.1)}\qquad \begin{cases} -(\varepsilon a_{ij}u_{x_i})_{x_j}+d_i^\varepsilon u_{x_i}=f & \text{in } \Omega, \\ u\in W_0^{1,2}(\Omega), \end{cases}$$

belongs to $W_{\text{loc}}^{2,N}(\Omega)\cap C^0(\overline{\Omega})$ and it solves

$$\text{(2.2)}\qquad \begin{cases} -\varepsilon a_{ij}u_{x_i x_j}+b_i u_{x_i}=f \text{ a.e. in } \Omega, \\ u=0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

see for instance [9, Ch. 9]. The problem adjoint to (2.1) is

$$\text{(2.3)}\qquad \begin{cases} -(\varepsilon a_{ij}v_{x_i}+d_j^\varepsilon v)_{x_j}=\psi & \text{in } \Omega, \\ v\in W_0^{1,2}(\Omega), \end{cases}$$

which is also uniquely solvable in the weak sense.

DEFINITION [16]. A function $G^\varepsilon(x,y)$ defined for $x,y\in\Omega$, $x\neq y$ is a Green's function for the problem (2.1) if $G^\varepsilon(\cdot,y)\in L^1(\Omega)$, $\forall y\in\Omega$, and

$$\int_\Omega G^\varepsilon(x,y)\psi(x)\mathrm{d}x=v(y) \quad \text{for all } y\in\Omega,$$

for every $\psi\in C^0(\overline{\Omega})$ and v the corresponding weak solution of (2.3).

From the theory of Stampacchia [16] it follows that there exists a unique Green's function for the problem (2.1), and it satisfies the following properties:

$$\text{(2.4)}\qquad \begin{cases} G^\varepsilon(x,y)=\tilde{G}^\varepsilon(y,x) \\ \text{where } \tilde{G}^\varepsilon \text{ is the Green's function for problem (2.3), i.e.} \\ \displaystyle\int_\Omega \tilde{G}^\varepsilon(x,y)f(x)\mathrm{d}x=u(y) \quad \forall y\in\Omega, \\ \text{for all } f\in C^0(\overline{\Omega}) \text{ and } u \text{ the corresponding solution of (2.1);} \end{cases}$$

$$\tilde{G}^\varepsilon(x,y) \geq 0; \tag{2.5}$$

$$\begin{cases} \text{for each } y \in \Omega \ G^\varepsilon(\cdot,y), \ \tilde{G}^\varepsilon(\cdot,y) \in W_0^{1,q}(\Omega) \\ \text{for every } q < \dfrac{N}{N-1}; \end{cases} \tag{2.6}$$

$$\begin{cases} \tilde{G}^\varepsilon(\cdot,y) \in W^{1,2}_{\text{loc}}(\Omega\backslash\{y\}) \\ \text{and it is a weak solution in } \Omega\backslash\{y\} \text{ of} \\ -(\varepsilon a_{ij} v_{x_i} + d_j^\varepsilon v)_{x_j} = 0. \end{cases} \tag{2.7}$$

PROPOSITION 2.8. *Under assumptions* (A1), (A2), (B0) *we have* $G^\varepsilon(\cdot,y) \in C^{3,\alpha}(\Omega\backslash\{y\})$ *and it satisfies*

$$-\varepsilon a_{ij} G^\varepsilon_{x_i x_j} + b_i G^\varepsilon_{x_i} = 0 \text{ for all } x \in \Omega\backslash\{y\}.$$

PROOF. Fix ε and y and define $u(x) = G^\varepsilon(x,y)$. Let f_n be an approximation of the identity and let u^n be the weak solution of

$$\begin{cases} -(\varepsilon a_{ij} u^n_{x_i})_{x_j} + d_i^\varepsilon u^n_{x_i} = f_n & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega. \end{cases}$$

By [16, Thm. 9.1] we have

$$\|u^n\|_{W_0^{1,q}(\Omega)} \leq K \quad \text{for} \quad q < \frac{N}{N-1}.$$

Then a subsequence of u^n converges in $L^q(\Omega)$ and weakly in $W_0^{1,q}(\Omega)$ to u. Now fix $\Omega' \subset\subset \Omega\backslash\{y\}$ with smooth boundary. For n big enough u^n solves

$$-\varepsilon(a_{ij} u^n_{x_i})_{x_j} + d_i^\varepsilon u^n_{x_i} = 0 \quad \text{in } \Omega'.$$

Thus, by standard methods we have

$$\int_{\Omega'} |Du^n|^2 \, dx \leq C,$$

so that a subsequence of u_n converges in $L^2(\Omega')$ and weakly in $W^{1,2}(\Omega')$, necessarily to u. Thus u is a weak solution in Ω' of

$$-(\varepsilon a_{ij} u_{x_i})_{x_j} + d_i^\varepsilon u_{x_i} = 0.$$

Thus u is continuous and the proposition follows from the Schauder theory. □

By the above proposition and the strong maximum principle we have $G^\varepsilon(x,y) > 0$ for all $x \in \Omega$, $x \neq y$, so that we can define

$$v^\varepsilon(x,y) := -\varepsilon \log G^\varepsilon(x,y).$$

It is easy to check that $v^\varepsilon(\cdot,y)$ satisfies

$$\text{(2.9)} \qquad \begin{cases} -\varepsilon a_{ij} v^\varepsilon_{x_i x_j} + a_{ij} v^\varepsilon_{x_i} v^\varepsilon_{x_j} + b_i v^\varepsilon_{x_i} = 0 & \text{in} \quad \Omega\backslash\{y\}, \\ v^\varepsilon(x,y) \to -\infty & \text{as } x \to y, \\ v^\varepsilon(x,y) \to +\infty & \text{as } x \to \partial\Omega. \end{cases}$$

For the solution of the above PDE it is possible to obtain interior estimates for the gradient independent of ε, as shown by Evans-Ishii [4]:

LEMMA 2.10. *For each $\Omega' \subset\subset \Omega\backslash\{y\}$ there exists a constant $C(\Omega')$, independent of ε, such that every C^3 solution of the* PDE *in* (2.9) *satisfies*

$$\sup_{\Omega'} |D_x v^\varepsilon| \leq C(\Omega').$$

PROOF. See [4, Lemma 2.2]. □

We are now going to prove interior estimates, independent of ε, for $|v^\varepsilon|$. To do this we will estimate the Green's function of the adjoint problem $\tilde{G}^\varepsilon$ and exhibit the dependence of the constants on ε. The crucial exponential dependence on ε^{-1} of the bounds for $\tilde{G}^\varepsilon$ comes from the constant in the Harnack inequality:

LEMMA 2.11. *Let $\Omega' \subseteq \Omega$ be open and $u \in W^{1,2}(\Omega')$, $u \geq 0$, be a solution of*

$$-(\varepsilon a_{ij} u_{x_i} + d^\varepsilon_j u)_{x_j} = 0 \quad \text{in} \quad \Omega'.$$

Then, for any ball $B(z,4r) \subset \Omega'$, $r > \frac{4\varepsilon}{3}$, we have

$$\text{(2.12)} \qquad \sup_{B(z,r)} u \leq C^{r/\varepsilon} \inf_{B(z,r)} u,$$

where $C = C(N, D, \|a_{ij}\|_{L^\infty(\Omega)})$.

PROOF. Fix $x_0 \in B(z,r)$ and define

$$\tilde{a}_{ij}(x) := a_{ij}(x_0 + \varepsilon x),\ \tilde{d}_j(x) := d^\varepsilon_j(x_0 + \varepsilon x),\ \tilde{u}(x) := u(x_0 + \varepsilon x).$$

Then $\tilde{u}$ solves

$$-(\tilde{a}_{ij}\tilde{u}_{x_i} + \tilde{d}_j \tilde{u})_{x_j} = 0 \quad \text{in} \quad \tilde{\Omega} := \{x |\ x_0 + \varepsilon x \in \Omega'\},$$

$\tilde{a}$ satisfies (A2) and $\|\tilde{d}_j\|_{L^\infty(\tilde{\Omega})} \leq D$. Since $B(0,4) \subseteq \tilde{\Omega}$, by the Harnack inequality there exists C such that

$$\sup_{B(x_0,\varepsilon)} u = \sup_{B(0,1)} \tilde{u} \leq C \inf_{B(0,1)} \tilde{u} = C \inf_{B(x_0,\varepsilon)} u .$$

Since any two points in $B(z,r)$ can be connected by a chain of $\left[\frac{2r}{\varepsilon}\right]$ appropriately overlapping balls of radius ε, we obtain (2.12).

□

REMARK 2.13. The dependence on ε of the constant in the Harnack inequality displayed in (2.12) is sharp, as the following simple example shows:

$$-\varepsilon\Delta u + u_{x_i} = 0 \quad \text{in} \quad \mathbb{R}^N$$

has the positive solution $u(x) = e^{x_i/\varepsilon}$ that assumes the values $e^{r/\varepsilon}$ and $e^{-r/\varepsilon}$ on the boundary of $B(0,r)$.

□

PROPOSITION 2.14. *Assume* (A1), (A2), (B0). *The function* $\tilde{G}^\varepsilon(x,y)$ *defined in* (2.4) *satisfies the inequality*

$$\tilde{G}^\varepsilon(x,y) \geq \frac{C_1\, e^{-C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^{N-2}}, \quad \textit{for } \frac{4\varepsilon}{3} \leq |x-y| \leq 1 \wedge \text{dist }(y,\partial\Omega)/2$$

where C_1 *and* C_2 *are constants independent of* ε.

PROOF. Fix $x \neq y$ and define $r = |x-y|$, $u(z) := \tilde{G}^\varepsilon(z,y)$. Define $S_1 := \{z \in \Omega|\ \frac{r}{2} \leq |z-y| \leq r\}$, $S_2 := \{z \in \Omega|\ \frac{r}{4} \leq |z-y| \leq \frac{3r}{2}\}$ and let $\varsigma \in C_0^\infty(\Omega)$ be such that $0 \leq \varsigma \leq 1$, $\varsigma \equiv 1$ in S_1, $\varsigma \equiv 0$ in $\Omega\backslash S_2$, $|D\varsigma| \leq \frac{C}{r}$. By (2.7), using $\phi = u\varsigma^2$ as a test function, we get

$$\varepsilon \int_\Omega |Du|^2\varsigma^2 dz \leq (\frac{\varepsilon C}{r} + D) \int_\Omega \sum_i |u_{x_i}| u\varsigma dz + \frac{C}{r} \int_\Omega u^2\varsigma dz$$

where we indicate by C any constant depending only on N, D and $\|a_{ij}\|_{L^\infty(\Omega)}$. Thus

$$\text{(2.15)} \qquad \int_{S_1} |Du|^2 dz \leq \frac{C}{\varepsilon}(\frac{C\varepsilon}{r^2} + \frac{C}{r} + \frac{C}{\varepsilon}) \int_{S_2} u^2 dz \leq (\frac{C}{r^2} + \frac{C}{\varepsilon^2}) r^N \sup_{S_2} u^2.$$

Now let $\phi \in C_0^\infty(\Omega)$ be such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B(y, \frac{r}{2})$, $\phi \equiv 0$ in $\Omega \backslash B(y,r)$, $|D\phi| \le \frac{C}{r}$. As a consequence of (2.4) and the regularity of the coefficients we have

$$\int_\Omega (\varepsilon a_{ij} u_{x_i} \phi_{x_j} + d_i^\varepsilon u \phi_{x_i}) \mathrm{d}x = \phi(y).$$

Then

$$\begin{aligned} 1 &\le \varepsilon \, \frac{C}{r} \int_{S_1} |Du| \mathrm{d}z + \frac{C}{r} \int_{S_1} u \mathrm{d}z \\ &\le \frac{\varepsilon C}{r} \, r^{N/2} (r^N (\frac{C}{r^2} + \frac{C}{\varepsilon^2}) \sup_{S_2} \, u^2)^{1/2} + C r^{N-1} \sup_{S_2} \, u \\ &\le C r^{N-2} \sup_{S_2} \, u \end{aligned}$$

for $r \le 1$, where we have got the second inequality from Schwarz inequality and (2.15). Now, in order to apply the Harnack inequality (Lemma 2.11), we observe that any ball $B(z,R)$ with $z \in S_2$ and $R = \frac{r}{20}$ is such that $B(z,4R) \subseteq \Omega \backslash \{y\}$, and that any two points in S_2 can be connected by a chain of appropriately overlapping such balls whose number depends only on N. Then we obtain

$$1 \le C r^{N-2} \; C^{r/\varepsilon} \; u(x),$$

which yields the conclusion.

□

REMARK 2.16. For this proof we borrowed some ideas from Grüter-Widman [10].

□

In order to get the estimate from above of G^ε around the pole y we shall use hypothesis (B1). Define

$$\Omega_\gamma := \{x \in \mathbb{R}^N : \; \mathrm{dist}(x, \Omega) < \gamma\}.$$

The main consequence of (B1) is the following Lemma, which is a slight extension of Lemma 4.2 in [4]:

LEMMA 2.17. *Assume* (B0) *and* (B1) *and let* $\tilde{b}$ *be a Lipschitz extension of* b *to a neighbourhood of* Ω. *Then there exist* $\alpha > 0$, $T > 0$, $\gamma > 0$ *such that*

$$\int_0^S |\dot{x}(s) - \tilde{b}(x(s))|^2 \mathrm{d}s \ge \alpha S$$

for all $S \ge T$ *and for all* $x(\cdot) \in W^{1,2}([0,S], \overline{\Omega}_\gamma)$.

PROOF. First observe that (B1) is equivalent to

$$\int_0^\infty |\dot{x}(s) - b(x(s))|^2 ds = \infty$$

for all $x(\cdot) \in W^{1,2}_{\rm loc}([0,\infty),\overline{\Omega})$. Then the proof (by contradiction) is essentially the same as that of Lemma 4.2 in [4].

□

LEMMA 2.18. *Under the assumptions of Lemma* 2.17 *there exist* $\gamma > 0$ *and* $w \in W^{1,\infty}(\Omega_\gamma)$ *such that*

$$\tilde{b}_i w_{x_i} \geq 3, \qquad \text{a.e. in } \Omega_\gamma. \tag{2.19}$$

PROOF. For a given $x(\cdot) \in H$, $x(0) = x$, let t_x be the first exit time of $x(\cdot)$ from Ω_γ, i.e.

$$t_x := \inf \{t > 0 : x(t) \notin \Omega_\gamma\}.$$

Let α, T, γ be the constants provided by Lemma 2.17 and define for $x \in \overline{\Omega}_\gamma$

$$w(x) := \inf \{\int_0^{t_x} (\frac{3}{\alpha}|\dot{x}(s) - \tilde{b}(x(s))|^2 - 3)\, ds : x(\cdot) \in H,\ x(0) = x\}.$$

By standard arguments w is Lipschitz continuous in $\overline{\Omega}_\gamma$.

Now, observe that w is the value function of the control problem of minimizing

$$\int_0^{t_x} (\frac{3}{\alpha}|\beta(s) - \tilde{b}(x(s))|^2 - 3)\, ds$$

where $x(\cdot)$ satisfies $\dot{x}(s) = \beta(s)$, $x(0) = x$, and the control $\beta(\cdot) \in L^2_{\rm loc}([0,\infty),\mathbb{R}^N)$. The Hamilton-Jacobi-Bellman equation associated to this problem is

$$\frac{\alpha}{12}|Dw|^2 - \tilde{b}_i w_{x_i} + 3 = 0.$$

Since w is continuous it is a viscosity solution of this equation (see [14, Thm. 1.10]), then by Rademacher's theorem it also satisfies the equation a.e., which implies (2.19).

□

PROPOSITION 2.20. *Assume* (A1), (A2), (B0), (B1). *Then the function* $\tilde{G}^\varepsilon(x,y)$ *defined in* (2.4) *satisfies*

$$\tilde{G}^\varepsilon(x,y) \leq \frac{C_1\, e^{C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^N}, \text{ for } 20\varepsilon/3 < |x-y| < \text{ dist}(y,\partial\Omega)/2 \text{ and } \varepsilon < C_3,$$

where the constants C_1, C_2, C_3 *are independent of* ε.

PROOF. We extend b to a Lipschitz vector field $\tilde{b}$ defined in a neighbourhood of Ω and consider the function w constructed in Lemma 2.18. We call

$$w^\eta(x) := (w * \rho_\eta)(x), \qquad \text{for } 0 < \eta < \gamma, \ x \in \Omega,$$

the convolution of w with a mollifier ρ_η. Then w^η is smooth and satisfies

$$\sup_\Omega |w^\eta| \le C, \text{ for } 0 < \eta < \gamma,$$
$$b_i w^\eta_{x_i} \ge 3 - C\eta \quad \text{in } \Omega,$$

and

$$|w^\eta_{x_i x_j}| \le C/\eta^2 \quad \text{in } \Omega.$$

Now we choose η_0 small enough so that $v := w^{\eta_0}$ satisfies

$$b_i v_{x_i} \ge 2,$$

and

$$-\varepsilon a_{ij} v_{x_i x_j} \ge -\varepsilon C/\eta_0^2.$$

Therefore v satisfies

$$-\varepsilon a_{ij} v_{x_i x_j} + b_i v_{x_i} \ge 1, \text{ for all } \varepsilon \le \eta_0^2/C =: C_3, \ x \in \Omega.$$

Now let u^ε be the solution of

$$-\varepsilon a_{ij} u^\varepsilon_{x_i x_j} + b_i u^\varepsilon_{x_i} = 1, \text{ a.e. in } \Omega,$$
$$u^\varepsilon = 0, \text{ on } \partial\Omega.$$

By the Alexandrov-Bakelman-Pucci maximum principle (see e.g. [9, Thm. 9.1]) we then have

$$\sup_\Omega u^\varepsilon \le \sup_\Omega v + \sup_{\partial\Omega} v^- \le C, \text{ for } \varepsilon \le C_3.$$

Using the definition of $\tilde{G}^\varepsilon$ we get

$$\int_\Omega \tilde{G}^\varepsilon(x,y)\mathrm{d}x = u^\varepsilon(y) \le C, \text{ for } \varepsilon \le C_3, \ y \in \Omega.$$

Now taking $\rho = |x-y|/5$, $4\varepsilon/3 < \rho < \operatorname{dist}(x,\partial\Omega)/4$, by the Harnack inequality (Lemma 2.11) we have

$$\tilde{G}^\varepsilon(x,y) \le C^{\rho/\varepsilon} \inf_{z\in B(x,\rho)} \tilde{G}^\varepsilon(z,y) \le \frac{C^{\rho/\varepsilon}}{\rho^N} \int_\Omega \tilde{G}^\varepsilon(x,y)\mathrm{d}x.$$

□

THEOREM 2.21. *Assume* (A1), (A2), (B0), (B1). *For each* $y \in \Omega$ *there exists a sequence* $\varepsilon_k \searrow 0$ *and a function* $v(\cdot, y) \in C^{0,1}(\overline{\Omega})$ *such that* $\lim_k v^{\varepsilon_k}(\cdot, y) = v(\cdot, y)$ *uniformly on compact subsets of* $\Omega\backslash\{y\}$ *and* $v(\cdot, y)$ *is a viscosity solution of the Hamilton-Jacobi equation*

$$a_{ij}v_{x_i}v_{x_j} + b_i v_{x_i} = 0 \quad \text{in } \Omega\backslash\{y\}. \tag{2.22}$$

Moreover there exists a positive constant C *such that*

$$|v(x,y)| \le C|x-y|, \text{ for } |x-y| \le 1 \wedge \operatorname{dist}(y, \partial\Omega)/2. \tag{2.23}$$

PROOF. Lemma 2.10 and Propositions 2.14 and 2.20 imply that $\{v^{\varepsilon}(\cdot, y),\ 0 < \varepsilon \le 1\}$ is bounded in $W^{1,\infty}_{\text{loc}}(\Omega)$. Therefore a subsequence converges uniformly to $v(\cdot, y) \in W^{1,\infty}_{\text{loc}}(\Omega)$, which is a viscosity solution of (2.22) because $v^{\varepsilon}(\cdot, y)$ solves (2.9), see Crandall-Lions [2, §IV.1]. By the results of Crandall-Lions [2, §I.4], (2.22) implies $|Dv(\cdot, y)| \le C$ in $\Omega\backslash\{y\}$ and then $v(\cdot, y)$ has a unique Lipschitz extension to $\overline{\Omega}$.

□

3. The Representation Formula for the Limit

We recall the definition:

$$I(x,y) := \inf\{\frac{1}{4}\int_0^t \|\dot{x}(s) + b(x(s))\|^2 ds \mid 0 \le t < \infty,$$
$$x(\cdot) \in W^{1,2}([0,t], \Omega), x(0) = x,\ x(t) = y\}$$

where $\|\dot{x}(s) + b(x(s))\|^2$ is defined by (1.1). Our main result is the following:

THEOREM 3.1. *Assume* (A1), (A2), (B0), (B1). *Then*

$$\lim_{\varepsilon \searrow 0} v^{\varepsilon}(\cdot, y) = I(\cdot, y)$$

uniformly on compact subsets of $\Omega\backslash\{y\}$.

PROOF. Let $v(\cdot, y) = \lim_k v^{\varepsilon_k}(\cdot, y)$ uniformly on compact subsets of $\Omega\backslash\{y\}$ for some $\varepsilon_k \searrow 0$. Our goal is to prove that $v(x,y) = I(x,y)$ for all $x, y \in \Omega$. For $x \in \Omega$ let τ_x be the first exit time of $x(\cdot) \in H$, $x(0) = x$, from $\Omega\backslash\{y\}$. Since by Theorem 2.21 $v(\cdot, y)$ is a viscosity solution of (2.22) and we are assuming

(B1), the following representation formula of Evans-Ishii [4, Thm. 4.1] holds:

$$v(x,y) = \inf\{\frac{1}{4}\int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds + v(x(\tau_x), y) | x(\cdot) \in H, x(0) = x\},$$

for all $x \in \Omega \backslash \{y\}$.

Since either $x(\tau_x) = y$ or $x(\tau_x) \in \partial\Omega$, we have

$$\begin{aligned} v(x,y) &\le \inf\{\tfrac{1}{4}\int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds + v(x(\tau_x), y) | x(\cdot) \in H,\ x(0) = x, \\ &\qquad\qquad x(\tau_x) = y\} \\ &= \inf\{\tfrac{1}{4}\int_0^{\tau_x} \|\dot{x}(s) + b(x(s))\|^2 ds | x(\cdot) \in H,\ x(0) = x,\ x(\tau_x) = y\} \\ &= I(x,y) \end{aligned}$$

because (2.23) implies $v(y,y) = 0$.

We are now going to prove that $v(x,y) \ge I(x,y)$ for all $x, y \in \Omega$. We fix $y \in \Omega$ and in order to simplify the notation we drop the second variable y in v^ϵ, v, I and in all the functions defined in the following. Define $\Omega' = \Omega \cup \{x \notin \Omega |\ \mathrm{dist}(x, \partial\Omega) < \beta\}$, for $\beta > 0$ small, let τ'_x be the exit time of $x(\cdot) \in H$ $x(0) = x \in \Omega'$ from $\Omega' \backslash \{y\}$, and extend a and b to be Lipschitz and bounded in all $\mathbb{R}^N$. For $\lambda \ge 0$, $x \in \Omega'$, define

$$I'_\lambda(x) := \inf\{\frac{1}{4}\int_0^{\tau'_x} e^{-\lambda s}\|\dot{x}(s) + b(x(s))\|^2 ds | x(\cdot) \in H,\ x(0) = x,$$
$$x(\tau'_x) = y \text{ if } \tau'_x < \infty\}.$$

For all $\lambda \ge 0$ I'_λ is locally Lipschitz with $|DI'_\lambda| \le \frac{1}{4}(1 + \|b\|_{L^\infty(\Omega)})^2$ so that

(3.2) $$|I'_\lambda| \le C, \qquad C \text{ independent of } \lambda,$$

and it is not difficult to show, using the argument in [4, Lemma 2.4], that I'_λ is a viscosity solution of

$$\begin{cases} \lambda I'_\lambda + a_{ij} I'_{\lambda x_i} I'_{\lambda x_j} + b_i I'_{\lambda x_i} = 0 & \text{in } \Omega' \backslash \{y\} \\ I'_\lambda(y) = 0. \end{cases}$$

Now define for $0 < \gamma < \beta$ the mollification

$$I^\gamma_\lambda(x) := (I'_\lambda * \rho_\gamma)(x), \qquad x \in \overline{\Omega},$$

where ρ_γ is an approximation of the identity. It is easy to deduce from Jensen's inequality, (A2), (B0) and (3.2) that

$$\lambda I^\gamma_\lambda + a_{ij} I^\gamma_{\lambda x_i} I^\gamma_{\lambda x_j} + b_i I^\gamma_{\lambda x_i} \le C\gamma, \qquad x \in \Omega,$$

where C is independent of γ and λ. (3.2) implies also

$$-\varepsilon a_{ij} I^\gamma_{\lambda x_i x_j} \le \frac{C\varepsilon}{\gamma^2}, \qquad \text{for all } x \in \Omega,$$

so that I^γ_λ satisfies

$$L^\varepsilon_\lambda I^\gamma_\lambda \le C_0(\gamma + \frac{\varepsilon}{\gamma^2}) \qquad \text{in } \Omega \tag{3.3}$$

for a suitable constant C_0 independent of ε, γ and λ, where L^ε_λ is the quasilinear elliptic operator

$$L^\varepsilon_\lambda w := -\varepsilon a_{ij} w_{x_i x_j} + a_{ij} w_{x_i} w_{x_j} + b_i w_{x_i} + \lambda w.$$

Furthermore, since $I'_\lambda(y) = 0$, we have $I^\gamma_\lambda(y) \le C_1\ \gamma$ and thus

$$I^\gamma_\lambda(x) \le C_1\gamma + C_2 R \text{ for all } x \in \partial B(y, R), \tag{3.4}$$

where the constants are independent of λ, γ and R, $0 < R < \operatorname{dist}(y, \partial\Omega)$.

Now fix a constant M such that

$$I^\gamma_\lambda \le M, \text{ on } \partial\Omega, \text{ for all } \lambda, \gamma, \tag{3.5}$$

and define $v^\varepsilon_{\lambda,R}$ to be the solution of

$$\begin{cases} L^\varepsilon_\lambda\ v^\varepsilon_{\lambda,R} = 0 & \text{in } \Omega \backslash B(y, R), \\ v^\varepsilon_{\lambda,R} = C_1\gamma + C_2 R & \text{on } \partial B(y, R), \\ v^\varepsilon_{\lambda,R} = M & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

(for the existence and regularity of $v^\varepsilon_{\lambda,R}$ see e.g. [9, Thm. 15.10]). By the comparison principle [9, Thm. 10.1] and (3.3-4-5) we have

$$I^\gamma_\lambda \le v^\varepsilon_{\lambda,R} + \frac{C_0}{\lambda}\ (\gamma + \frac{\varepsilon}{\gamma^2}) \quad \text{in } \Omega \backslash B(y, R), \tag{3.7}$$

and

$$v^\varepsilon_{\lambda,R} \ge 0 \quad \text{in } \Omega \backslash B(y, R). \tag{3.8}$$

Again by the comparison principle, (3.6) (3.8) and (2.9) (2.20) we get

$$v^\varepsilon_{\lambda,R} \le v^\varepsilon + C_1\gamma + C_2 R + CR + \varepsilon(C - N \log R) \quad \text{in } \Omega \backslash B(y, R).$$

Combining this last inequality with (3.7) and letting $\varepsilon \to 0$, $\gamma \to 0$ and $R \to 0$ in this order we get

$$I'_\lambda(x) \le v(x) \quad \text{for all } x \in \Omega. \tag{3.9}$$

We are now going to show that

$$I_\lambda\ (x) := \inf\{\frac{1}{4}\int_0^{\tau_x} e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds | x(\cdot)\in H, x(0)=x, x(\tau_x)=y\}$$

satisfies

$$I_\lambda\ (x) \le \liminf_{\beta \searrow 0} I'_\lambda(x) \le v(x) \text{ for all } x\in\Omega. \tag{3.10}$$

Fix $\varepsilon > 0$. Let $\beta_n \searrow 0$ as $n\to\infty$, $\Omega_n := \Omega \cup \{x \notin \Omega | \text{dist}(x,\partial\Omega) < \beta_n\}$, let τ_x^n be the exit time from $\Omega_n\backslash\{y\}$ of $x(\cdot)\in H, x(0)=x$ and let $I'_{\lambda,n}$ be I'_λ for $\beta=\beta_n$. Now take $x_n(\cdot)\in H$ such that $x_n(0)=x$, $x_n(\tau_x^n)=y$ if $\tau_x^n<\infty$ and

$$\frac{1}{4}\int_0^{\tau_x^n} e^{-\lambda s}\|\dot{x}_n(s)+b(x_n(s))\|^2 ds \le I'_{\lambda,n}(x)+\varepsilon.$$

Define

$$z_n(s) := \begin{cases} x_n\ (s) & \text{for } s\le \tau_x^n \\ \text{the solution of} \begin{cases} \dot{z}=-b(z) \\ z(\tau_x^n)=y \end{cases} & \text{for } s>\tau_x^n, \text{ if } \tau_x^n<\infty. \end{cases}$$

It is easy to see, using (3.9), that for all $T>0$

$$\frac{1}{4}\int_0^{T} e^{-\lambda s}|\dot{z}_n(s)|^2 ds \le Cv(x)+C\varepsilon+CT,$$

so that the sequence $\{z_n(\cdot)\}$ is bounded in $W^{1,2}([0,T],\ \Omega_0)$. Hence there exists $z(\cdot)\in H$ and a subsequence of $z_n(\cdot)$, still denoted by $z_n(\cdot)$, which converges to $z(\cdot)$ weakly in $W^{1,2}([0,T],\ \Omega_0)$ and uniformly on $[0,T]$.

Now for $x(\cdot)\in H$, $x(0)=x$ define

$$s_x := \begin{cases} \inf\{s : x(s)=y\} & \text{if } \{s : x(s)=y\}\neq\emptyset \\ +\infty & \text{if } \{s : x(s)=y\}=\emptyset. \end{cases}$$

We claim that $z(s)\in\overline{\Omega}$ for all $0\le s\le s_x$. To prove this assume $z(\tilde{s})\notin\overline{\Omega}$. Let $\alpha :=\ \text{dist}(z(\tilde{s}),\partial\Omega)$ and fix $\overline{n}$ such that $|z_n(s)-z(s)|<\frac{\alpha}{2}$ for $0\le s\le\tilde{s}$, $n>\overline{n}$. Let $\tilde{n}$ be such that $\beta_{\tilde{n}}<\frac{\alpha}{2}$ and define $\overline{\overline{n}} := \max\{\overline{n},\tilde{n}\}$. Then for all $n>\overline{\overline{n}}$ we have $z_n(\tilde{s})\notin\Omega_n$ and thus $x_n(\tau_x^n)=y$ with $\tau_n := \tau_x^n[x_n(\cdot)]<\tilde{s}$. Hence τ_n has a subsequence converging to $\overline{s}\le\tilde{s}$ and it is easy to see that $z(\overline{s})=y$, which implies $\tilde{s}>s_x$ and proves the claim.

Now define

$$J_\lambda(x) := \inf\{\frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds | x \in H, x(0)=x, \ x(s)\in\overline{\Omega} \quad \text{for } 0\le s\le s_x\}.$$

We claim that

$$J_\lambda(x) \le \liminf_n \ I'_{\lambda,n}(x). \tag{3.11}$$

To prove this we recall that for each $T>0$ the functional

$$x(\cdot) \mapsto \int_0^T e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds$$

is sequentially weakly lower semicontinuous by a classical theorem of Tonelli. Then

$$\int_0^T e^{-\lambda s}\|\dot{z}(s) \ + \ b(z(s))\|^2 ds \le \liminf_n \int_0^T e^{-\lambda s}\|\dot{z}_n(s)+b(z_n(s))\|^2 ds$$
$$\le \liminf_n \int_0^{\tau_x^n} e^{-\lambda s}\|\dot{x}_n(s)+b(x_n(s))\|^2 ds$$
$$\le \liminf_n \ 4I'_{\lambda,n}(x)+4\varepsilon.$$

Thus

$$J_\lambda(x) \le \frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{z}(s)+b(z(s))\|^2 ds$$
$$\le \liminf_n \ I'_{\lambda,n}(x)+\varepsilon,$$

and the claim is proved by the arbitrariness of ε.

Next we claim that

$$I_\lambda(x) \le J_\lambda(x). \tag{3.12}$$

We first observe that by a Lemma of Evans-Ishii (see [4, Remark 4.3]) hypothesis (B1) implies that there exist T_0, λ_0 such that

$$\frac{1}{4}\int_0^T e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds \ge v(x)+1 \ge J_\lambda(x)+1,$$

for all $T \geq T_0$, $0 \leq \lambda \leq \lambda_0$, $x(\cdot) \in H$ satisfying $x(s) \in \overline{\Omega}$ for all $0 \leq s \leq T_0$. Then, if we assume $\lambda \leq \lambda_0$ and fix $0 < \varepsilon < 1$, we can find $x(\cdot) \in H$ such that

$$\text{(3.13)} \qquad \begin{cases} x(0) = x, \ x(s_x) = y, \ s_x \leq T_0, \ x(s) \in \overline{\Omega} \text{ for } 0 < s < s_x, \\ \dfrac{1}{4} \displaystyle\int_0^{s_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds \leq J_\lambda(x) + \varepsilon. \end{cases}$$

Since $\partial\Omega$ is smooth there exists a smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{cases} \Omega = \{x \in \mathbb{R}^N | \phi(x) > 0\}, \ \partial\Omega = \{x \in \mathbb{R}^N : \phi(x) = 0\}, \\ |D\phi| = 1 \text{ on } \partial\Omega. \end{cases}$$

Define

$$\text{(3.14)} \qquad \overline{x}(s) := \begin{cases} x + sD\phi(x) & \text{for } 0 \leq s \leq \varepsilon \\ x(s-\varepsilon) + \varepsilon D\phi(x(s-\varepsilon)) & \text{for } \varepsilon \leq s < s_x + \varepsilon \\ y + (2\varepsilon + s_x - s)D\phi(y) & \text{for } s_x + \varepsilon \leq s \leq s_x + 2\varepsilon. \end{cases}$$

Clearly

$$I_\lambda(x) \leq \frac{1}{4} \int_0^{s_x+2\varepsilon} e^{-\lambda s} \|\dot{\overline{x}}(s) + b(\overline{x}(s))\|^2 ds,$$

and using the definitions (1.1) (3.13) (3.14) and the smoothness of a, b, and ϕ, it is not hard to show that

$$\int_0^{s_x} |\dot{x}(s)|^2 ds \leq C,$$

and to deduce from it that

$$I_\lambda(x) \leq \frac{1}{4} \int_0^{s_x} e^{-\lambda s} \|\dot{x}(s) + b(x(s))\|^2 ds + 0(\varepsilon) \leq J_\lambda(x) + 0(\varepsilon).$$

This proves the claim (3.12), and then, by (3.11) and the arbitrariness of $\beta_n \searrow 0$, the proof of (3.10) is complete. It remains to show that

$$\text{(3.15)} \qquad I_\lambda(x) \to I(x) \text{ for all } x \in \Omega.$$

Using [4, Remark 4.3] as above, we find λ_0, T_0 such that for all $\lambda \leq \lambda_0$ and fixed $\varepsilon > 0$ there exists $x_\lambda(\cdot) \in H$ such that

$$\begin{cases} x_\lambda(0) = x, \ x_\lambda(\tau_x) = y, \ \tau_x < T_0, \\ \dfrac{1}{4} \displaystyle\int_0^{\tau_x} e^{-\lambda s} \|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 ds \leq I_\lambda(x) + \varepsilon. \end{cases}$$

Then

$$I_\lambda(x) + \varepsilon \geq \frac{1}{4}\, e^{-\lambda T_0} \int_0^{\tau_x} \|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 \mathrm{d}s$$
$$\geq e^{-\lambda T_0}\; I(x) \geq I(x) - \varepsilon$$

for λ small enough. This gives (3.15) and completes the proof.

□

REMARK 3.16. Several ideas in this proof are taken from Evans-Ishii [4, §2].

□

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