

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Geometric regularity versus functional regularity

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 14,
n° 2 (1987), p. 277-283

http://www.numdam.org/item?id=ASNSP_1987_4_14_2_277_0

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Geometric Regularity Versus Functional Regularity

E. AMAR

Introduction

Let Y be a real C^∞ submanifold of \mathbb{C}^n and Ω a pseudo-convex bounded domain in \mathbb{C}^n with smooth C^∞ boundary.

Let $X = \Omega \cap Y$ and suppose that X is a holomorphic subspace of Ω ; can X be defined by the annulation of holomorphic functions in Ω and C^∞ up to the boundary?

This very natural question (all hypothesis are C^∞) was asked to me by professor Forstneric.

In this work we study only the case where $\dim_{\mathbb{R}} Y = 2n - 2$ i.e. the condimension one for X in \mathbb{C}^n .

We show:

THEOREM 1: *Let Y be a real submanifold of \mathbb{C}^n , C^∞ and of real dimension $2n - 2$; let Ω a pseudo-convex bounded domain in \mathbb{C}^n , with smooth boundary, let $X = \Omega \cap Y$. If X is a holomorphic subspace of Ω and if Y and $\bar{\Omega}$ are regularly situated then:*

- A) $\forall z_0 \in \bar{X}$, $\exists U_{z_0}$ neighbourhood of z_0 in \mathbb{C}^n , $\exists u_{z_0} \in A^\infty(\Omega \cap U_{z_0})$ s.t.
 $X \cap U_z = \{u_{z_0} = 0\}$
- B) if moreover, $H^2(\Omega, \mathbb{Z}) = 0$ then $\exists u \in A^\infty(\Omega)$ s.t.: $X = \{u = 0\}$.

Let me recall the notion regular situation (R.S.) introduced by Lojaciwicz [6]:

- if A and B are closed sets in \mathbb{R}^n , then A and B are R. S.
- if $A \cap B = \emptyset$ or $\forall K \subset\subset \mathbb{R}^n$, $\exists C > 0$, $\exists \alpha > 0$ s.t:

$$\forall z \in K, d(z, A) + d(z, B) \geq Cd(z, A \cap B)^\alpha.$$

We can drop this R.S. condition, at least in \mathbb{C}^2 , if we add a stronger convexity condition:

THEOREM 2: *Let Y be a real submanifold of \mathbb{C}^2 , C^∞ and with real dimension 2; let Ω be a pseudo-convex domain in \mathbb{C}^2 , bounded with C^∞ smooth boundary and $X = Y \cap \Omega$.*

If X is a holomorphic subspace of Ω and if all points of $Y \cap \partial\Omega$ are points of strict pseudo-convexity of $\partial\Omega$ then A) and B) of theorem 1 hold.

Pervenuto alla Redazione il 16 Luglio 1986.

1. - Local results

§1 - Let Ω be an open pseudo-convex, bounded set in \mathbb{C}^n , with C^∞ smooth boundary

Let u in $C^\infty(\mathbb{C}^n)$ and let:

$$(1.1) \quad Y = \{z \in \mathbb{C}^n \text{ s.t. } u(z) = 0\}; X = \overline{\Omega} \cap Y.$$

Let us suppose that u verifies:

$$(1.2) \quad \begin{cases} 1 - \bar{\partial}u \text{ is flat on } X \\ 2 - u \text{ is flat in no point of } X. \end{cases}$$

Then we have:

PROPOSITION 1.1: *If Y and $\overline{\Omega}$ are regularity situated at $0 \in \overline{X}$, there is a neighbourhood U of 0 in \mathbb{C}^n , and a function v in $A^\infty(U \cap \Omega)$ such that:*

$$X \cap U = \{v = 0\}.$$

PROOF. Because of the R.S. (regular situation) we get:

$$(1.3) \quad \exists C > 0, \exists \alpha > 0, \text{ s.t. } \forall z \in \overline{\Omega} \cap U, d(z, Y) \geq Cd(z, X)^\alpha.$$

But u is flat in no point of X so we get;

$$(1.4) \quad \exists C > 0, \exists \beta > 0 \text{ s.t. } \forall z \in U, |u(z)| \geq Cd(z, Y)^\beta.$$

Now, using the fact that $\bar{\partial}u$ is flat on X :

$$(1.5) \quad \forall k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \exists C_{k,\mu} \text{ s.t. } \forall z \in U \quad |D^\mu(\bar{\partial}u)(z)| \leq C_{k,\mu}d(z, X)^k.$$

where, as usual, we used;

$$D^\mu f = \frac{\partial^{|\mu|} f}{\partial z_1^{\mu_1} \dots \partial z_n^{\mu_n} \partial \bar{z}_1^{\mu_{n+1}} \dots \partial \bar{z}_n^{\mu_{2n}}}.$$

Using Leibnitz's formula we get:

$$(1.6) \quad \exists E_{k,\mu} \text{ s.t. } : \left| D^\mu \left(\frac{\bar{\partial}u}{u} \right) \right| \leq E_{k,\mu} \frac{d(z, X)^k}{d(z, Y)^{\beta|\mu|}}; \forall k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \forall z \in U.$$

With (1.3) it becomes:

$$(1.7) \quad \left| D^\mu \left(\frac{\bar{\partial}u}{u} \right) \right| \leq \frac{E_{k,\mu}}{C^\beta} \frac{d(z, X)^k}{D(z, X)^{\alpha\beta|\mu|}}, \forall k \in \mathbb{N}, \forall \mu \in \mathbb{N}^{2n}, \forall z \in U \cap \overline{\Omega}$$

Taking $k \geq \alpha\beta|\mu|$ we have:

$$\left| D^\mu \left(\frac{\bar{\partial}u}{u} \right) (z) \right| \text{ is bounded in } U \cap \bar{\Omega}.$$

Now, using Sobolev's theorem we get:

$$(1.9) \quad \omega = \frac{\bar{\partial}u}{u} \text{ is } C^\infty \text{ in } U \cap \bar{\Omega}.$$

Let U' be an admissible neighbourhood of 0 in $\bar{\Omega}$ [1], $U' \subset U \cap \bar{\Omega}$ i.e.:

$$(1.10) \quad U' \text{ is pseudo-convex with } C^\infty \text{ smooth boundary and } \partial U' \cap \partial \Omega \text{ contains a neighbourhood of 0 in } \partial \Omega$$

$$(1.11) \quad \bar{\partial}G = \omega, \text{ with } G \in C^\infty(\bar{U}').$$

Let now $h = e^{-G}$, we have:

$$(1.12) \quad h \in C^\infty(\bar{U}'), h \neq 0 \text{ in } \bar{U}',$$

and:

$$(1.13) \quad \bar{\partial}(hu) = h\bar{\partial}u + u\bar{\partial}h \equiv 0 \text{ in } \bar{U}.$$

So the function $v - hu$ is the solution asked for in proposition 1.1.

2. - Manifold

Let Y be a real submanifold of $\mathbb{C}^n (= \mathbb{R}^{2n})$ of real dimension $2n - 2$.
Let X be an open set in Y .

DEFINITION 2.1 : We call X a holomorphic submanifold of \mathbb{C}^n if:

$$(2.1) \quad \forall z \in X, T_z Y \text{ is a } \mathbb{C}\text{-linear hyperplane of } \mathbb{C}^n$$

where $T_z Y$ is the tangent space of Y at z .

The continuity of the complex structure of \mathbb{C}^n implies:

$$(2.2) \quad \text{if } X \text{ is holomorphic, then } \forall z \in \bar{X}, T_z Y \text{ is } \mathbb{C}\text{-linear.}$$

Let now z_0 be a point of \bar{X} ; using a \mathbb{C} -linear change of coordinates we may suppose:

$$(2.3) \quad z_0 = 0 \quad \text{and} \quad T_0 Y = \{z_n = 0\}$$

Then we have:

LEMMA 2.1: *There is a neighbourhood U of 0 in \mathbb{C}^n such that:*

$$Y \cap U = \{z_1, \dots, z_{n-1}\}$$

where $f \in C^\infty(\mathbb{C}^{n-1})$ and f is holomorphic in $\pi_n(X \cap U)$.

Here π_n is the canonical projection from \mathbb{C}^n onto \mathbb{C}^{n-1}

$$(z_1, \dots, z_n) \sim \rightarrow (z_1, \dots, z_{n-1}, 0).$$

PROOF: Using the implicit function theorem we get:

There is U_1 , neighbourhood of 0 in \mathbb{C}^{n-1} , and $f \in C^\infty(U_1)$ such that:

$$(2.4) \quad Y \cap U = \{z_n = f(z_1, \dots, z_{n-1})\},$$

where U is a neighbourhood of 0 in \mathbb{C}^n .

Let us write the tangent hyperplane in z to Y :

$$(2.5) \quad (Z_1 - z_n) - \sum_{i=1}^{n-1} (Z_i - z_i) \frac{\partial f}{\partial z_i}(z') - \sum_{i=1}^{n-1} (\bar{Z}_i - \bar{z}_i) \frac{\partial f}{\partial \bar{z}_i}(z') = 0$$

with $z' = (z_1, \dots, z_{n-1}) \in U_1$.

But, if $z' \in \pi_n(X \cap U)$, then this hyperplane must be \mathbb{C} -linear so:

$$(2.6) \quad \forall z' \in \pi_n(X \cap U), \frac{\partial f}{\partial \bar{z}_i}(z') = 0, i = 1, \dots, n-1$$

and f must be holomorphic in $\pi_n(x \cap U)$.

REMARK 1: π_n is a C^∞ diffeomorphism of Y on $\pi_n(y)$ near 0, because at 0 we have $d\pi_n = \text{identity}$; so $\pi_n(X \cap U)$ is an open set in \mathbb{C}^{n-1} .

REMARK 2: Using directly the finer results of G. Galusinsky [3] we can shown that \bar{X} is a regular holomorphic retract of an open set in \mathbb{C}^n .

3. - Proof of Theorem 1

Let Y be a submanifold (real) of \mathbb{C}^n , $\dim_{\mathbb{R}} Y = 2n - 2$, and Ω a pseudoconvex domain in \mathbb{C}^n , bounded with smooth boundary.

Let $X = \Omega \cap Y$ and let us suppose that X is a holomorphic submanifold of \mathbb{C}^n .

Using the results of §2, with $u = z_n - f(z')$ we get: $\forall z_0 \in \bar{X}$, $\exists U_{z_0}$ and $u \in C^\infty(U_{z_0})$ with:

$$(3.1) \quad Y \cap U_{z_0} = \{z \in U_{z_0} \text{ s.t. } u(z) = 0\}$$

and, more over:

$$(3.2) \quad \left. \begin{array}{l} \bar{\partial}u = 0 \\ \partial u \neq 0 \end{array} \right\} \text{ on } \bar{X} \cap U_z.$$

Shrinking U_{z_0} if necessary, if Y and $\bar{\Omega}$ are R.S. in z_0 , we can apply the results of §1 and get the A) of theorem 1.

II - Strictly pseudo-convex case

Now we suppose $n = 2$ i.e. Ω is a pseudo-convex domain in \mathbb{C}^2 , smoothly bounded.

Y is a (real) submanifold of \mathbb{C}^2 , $X = Y \cap \Omega$, and we suppose that: $0 \in \bar{X} \cap \partial\Omega$ is a point of strict pseudo-convexity of $\partial\Omega$.

Using again the notations of I.2, let us make the change of variables:

$$(4.1) \quad \begin{cases} Z_1 = z_1 \\ Z_2 = z_2 - f(z_1). \end{cases}$$

This is a C^∞ change of variables in a neighbourhood of 0, which is “holomorphic” in 0, i.e. $\bar{\partial}Z_2$ is flat in 0.

Let us denote Ω' the image the Ω by this change of variables, then $\partial\Omega'$ is still strictly pseudo-convex in 0.

Y' , image of Y , has for equation:

$$(4.2) \quad Y' = \{Z \in \mathbb{C}^2 \text{ s. t. } Z_2 = 0\}.$$

Now, using the results in [2], we know that the points making obstruction to the regular situation of Y' and $\bar{\Omega}'$ are localized on a curve $\Gamma' \subset \partial\Omega'$, C^∞ and smooth.

So, going back to Y and Ω , we get that the points making obstruction to the regular situation of Y and $\bar{\Omega}$ are localizable on a curve $\Gamma \subset \partial\Omega$, C^∞ and smooth near 0.

Using again the method introduced in [2], we used the projection of Γ onto $\{z_2 = 0\}$:

$$(4.3) \quad \Gamma'' = \text{proj } \{z_2 = 0\} \Gamma.$$

And, because a C^∞ smooth curve is always totally real, we can modify f on $\Gamma'' \cap \pi_2(\bar{\Omega})^c$ in such a way that ([2]):

$$(4.4) \quad \bar{\partial}f \text{ is flat on } \Gamma.$$

Going on reproducing the methods used in [2], we get, again because of the strict pseudo-convexity:

$$(4.5) \quad \frac{\bar{\partial} f}{z_2 - f} \text{ is in } C^\infty(\bar{\Omega}) \text{ near } 0.$$

Now, using §1 we get the theorem 2, locally.

III - Globalization

Using the results of I or II we have:

$$(5.1) \quad \begin{aligned} \forall z_0 \in \bar{X}, \quad \exists U_{z_0} \in A^\infty(U_{z_0} \cap \Omega) \text{ s.t.} \\ X \cap U_{z_0} = \{u_{z_0} = 0\} \text{ and } \partial u_{z_0} \neq 0 \end{aligned}$$

This gives us a divisor on Ω : let us cover $\bar{\Omega}$ by open sets U_i , with associated functions u_i such that:

$$(5.2) \text{ if } U_i \cap X \neq \emptyset, \text{ then } U_i \text{ is a } U_z \text{ and } u_i \text{ is a } u_z \text{ by (5.1)}$$

$$(5.3) \text{ if } U_i \cap \bar{X} = \emptyset \text{ then } u_i = 1.$$

We then get the transitions functions:

$$(5.4) \text{ on } U_i \cap U_j, \quad g_{ij} = u_i u_j^{-1} \in A^\infty(U_i \cap U_j \cap \Omega) \text{ and } g_{ij} \neq 0.$$

Shrinking U_i if necessary we can assume that:

$$(5.5) \quad U_i \cap \Omega \text{ and } U_i \cap U_j \cap \Omega \text{ are admissible [1] and simply connected.}$$

Using then Hörmander's method [4, chap V-5], we take logarithms to go to an additive Cousin problem, we solve it in $A^\infty(\Omega)$ using J. Kohn's estimates as soon as $H^2(\Omega, \mathbb{Z}) = 0$. This ends the proof of theorem 1 and of theorem 2, part B.

REMARK: A. Sebbar [7] proved that a fiber bundle in $A^\infty(\Omega)$ which is topologically trivial is also $A^\infty(\Omega)$ trivial under some hypothesis on Ω .

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