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<http://www.numdam.org/item?id=ASNSP_1987_4_14_2_199_0>
Holomorphic Generators of Some Ideals in $C^\infty(\overline{D})$

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dedicated to B.V. Shabat

0. Introduction, notations and statement of the main results

Let $D \subset \mathbb{C}^{n+1}$ be a bounded domain with $C^\infty$-smooth boundary, $V$ a complex submanifold of a neighbourhood of $\overline{D}$ such that $\overline{D} \cap V = \overline{D} \cap V \neq \emptyset$, $\mathcal{F}_V$ the sheaf of ideals of $V$ and set:

$$\mathcal{S}^\infty(V) = \{ f \in C^\infty(\overline{D}) \mid f|_V = 0 \},$$

$$I^\infty(V) = \{ f \in A^\infty(D) = \mathcal{O}(D) \cap C^\infty(\overline{D}) \mid f|_V = 0 \}.$$

It is well known (see e.g. [7]) that if $g_1, \ldots, g_k \in \mathcal{O}(\overline{D})$ $g_j|_D \in I^\infty(V)$, $1 \leq j \leq k$, represent a complete defining system for $V$ (i.e. for every $x \in \overline{D}$, $g_{1,x}, \ldots, g_{k,x}$ generates $\mathcal{F}_{V,x}$ over $\mathcal{O}_x$), then $g_1, \ldots, g_k, \overline{g}_1, \ldots, \overline{g}_k$ generate $\mathcal{S}^\infty(V)$ over $C^\infty(\overline{D})$ if and only if $\overline{D}$ and $V$ are regularly separated in the sense of -Lojasiewicz, i.e. there exist $h \in \mathbb{Z}^+$ and $C > 0$ such that for every $x \in \overline{D}$ we

![Fig. 1](image-url)
have:

$$\text{dist}^h(x, V \cap \overline{D}) \leq C \text{dist}(x, V)$$

It is a natural question to ask under which assumptions, more in general, $I^\infty(V) \cup \overline{I^\infty(V)}$ generates $\mathcal{G}^\infty(V)$ over $C^\infty(\overline{D})$.

It is clear that is not always the case:

take e.g.: $V = L = \{ z_{n+1} = 0 \}$, $\Omega$ any bounded domain with $C^\infty$-smooth boundary such that $\Omega \cap L = \emptyset$ and $\overline{\Omega}$ and $L$ are not regularly separated somewhere; let $B$ a ball containing $\overline{\Omega}$ and let finally $D = B \setminus \overline{\Omega}$. Obviously we have $A^\infty(D) = A^\infty(B)$, so $I^\infty(V)$ is generated by $z_{n+1}$ (cf. [1] [4]), while $(z_{n+1}, \overline{z}_{n+1})C^\infty(\overline{D}) \subset \mathcal{G}^\infty(V)$.

Of course, pseudoconcavity of $D$ plays an essential role in this example.

The main result of this paper is the following:

**THEOREM.** Let $D \subset C^{n+1}$ be a bounded strictly pseudoconvex domain with $C^\infty$-smooth boundary, let $V$ be a complex submanifold of a neighbourhood of $\overline{D}$ such that $D \cap V = \overline{D} \cap V \neq \emptyset$, and let $g_1, \ldots, g_k$ be a complete defining system for $V$.

Then there exists $m \in \mathbb{Z}^+$ such that for every $f \in \mathcal{G}^\infty(V)$ one can find $\lambda_1, \ldots, \lambda_m \in I^\infty(V)$, $a_1, \ldots, a_k$, $b_1, \ldots, b_k$, $c_1, \ldots, c_m$, $d_1, \ldots, d_m \in C^\infty(\overline{D})$ in such a way that:

$$f = \sum_{j=1}^k (a_j g_j + b_j \overline{g}_j) + \sum_{h=1}^m (c_h \lambda_h + \overline{d}_h \overline{\lambda}_h).$$

Note that no requirement other than $\overline{D} \cap V = \overline{D} \cap V \neq \emptyset$ is made about the mutual position of $D$ and $V$.

The general ideas of the proof are the following:

1. Investigating the geometry of $D \cap V$ (Lemmas 1.1 and 1.2) we prove that, in the strictly pseudoconvex case, the area of bad contact (i.e. non regular separation) between $D$ and $V$, can be locally included in a totally real submanifold $\Sigma$ of $bD$.

2. Since $\Sigma$ is totally real, functions in $I^\infty(V)$ are (relatively) flabby on $\Sigma$ and so, in some sense, they can be deformed on $\Sigma$ (Proposition 2.1) in order to reproduce locally any (possibly bad) behaviour of functions in $\mathcal{G}^\infty(V)$.

3. Using some arguments from [4], we pass from the local result to the Theorem (Lemma 3.1 and proposition 3.2).

As a corollary of the main Theorem, we obtain (Corollary 3.3) that regular separation is necessary and sufficient condition for $I^\infty(V)$ to be generated over $A^\infty(D)$ by $g_1, \ldots, g_k$.

The result of Corollary 3.3 can be found in the paper by E. Amar [2], which represented one of the starting points of the present investigation.

Some of the results presented in this paper where announced in [3].
1. - The geometrical situation.

The first step of the proof of the Theorem is to investigate the local geometry of \( D \cap V \), especially at those points where \( V \) and \( bD \) meet non-transversally.

In order to perform this investigation, let \( D \subset C^{n+1} \) be a strictly pseudoconvex domain with \( C^\infty \)-smooth boundary and let \( L \) be a complex hyperplane such that \( L \cap D = L \cap \overline{D} \neq \emptyset \) and \( L \) and \( bD \) are not transversal at \( x \in L \cap bD \); then it is possible to choose local complex coordinates \((z, z_{n+1})\), \( z = (z_1, \ldots, z_n) \) in a neighbourhood \( N \) of \( x \) in such a way that

\[
\begin{align*}
&i) \quad T^\mathbb{C}_{x} bD = \{z_{n+1} = 0\} = L, \quad T^\mathbb{R}_{x} bD = \{\text{Re } z_{n+1} = 0\} \\
&ii) \quad D \cap N = \{\text{Re } z_{n+1} > r(z, \text{Im } z_{n+1})\}
\end{align*}
\]

where:

\[
r(z, \text{Im } z_{n+1}) = p(z) + \varphi(z) + \psi(z, \text{Im } z_{n+1}),
\]

with

a) \( p(z) = \overline{z} A' z + \text{Re } z B' z \) with \( A, B \in M_{n,n}(\mathbb{C}), \quad A = A^* > 0, \quad B = B^* \)

b) \( \varphi(z) = o(|z|^2) \) for \( z \to 0 \)

c) \( \psi(z, \text{Im } z_{n+1}) = O(\text{Im } z_{n+1}^2) \) for \( \text{Im } z_{n+1} \to 0 \).

Let \( h(z) = p(z) + \varphi(z) \).

**Lemma 1.1.** Up to complex linear changes of coordinates, we can assume there exist \( k, r \in \mathbb{Z}^+ \), \( 0 \leq k \leq n \), \( 0 \leq r \leq n - k \), such that setting \( z_j = x_j + i y_j \) and \( T = (x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n) \) we have

\[
p(z) = p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 2 \sum_{j=1}^{k} y_j^2 + TP^* T,
\]

where \( P \) is a non-singular symmetric element of \( M_{2(n-k),2(n-k)}(\mathbb{R}) \) such that:

- \( P \) is positive definite on

\[
V^+ = \{x_j = 0, \quad k + 1 \leq j \leq k + r\}
\]

and negative definite on

\[
V^- = \{z_j = 0, \quad y_i = 0, \quad k + r + 1 \leq j \leq n, \quad k + 1 \leq i \leq k + r\}.
\]

**Proof.**

1. Up to an obvious complex linear change of coordinates (c.l.c.c.) we can assume \( p(z) = \overline{z}' z + \text{Re } z B' z \).

2. The space of degeneracy of \( p \) is given by \( W = \{dp = 0\} = \{t \overline{z} + B' z = 0\} \) and thus it is totally real: up to another c.l.c.c. we can assume there exists \( k \in \mathbb{Z}^+, \quad 0 \leq k \leq n \) such that

\[
W = \{z_{k+1} = \ldots = z_n = 0, \quad y_1 = \ldots = y_k = 0\}.
\]
This is equivalent to say

\[ B = \begin{pmatrix} -I_k & 0 \\ 0 & A \end{pmatrix}, \quad A = R + iS \]

and so we obtain the description of \( p \) we are looking for, setting:

\[ P = \begin{pmatrix} I + R & -S \\ -S & I - R \end{pmatrix}. \]

3. By means of the ordinary spectral theorem, we can find an Euclidean-orthonormal, \( P \)-orthogonal basis \( B = \{v_1, \ldots, v_{2(n-k)}\} \) of \( \mathbb{C}^{n-k} \); assume the index of negativity of \( P \) is \( r \) and \( v_j P v_j < 0 \), \( 1 \leq j \leq r \); thus \( P \) is positive definite on \( V^+ = [v_{r+1}, \ldots, v_{2(n-k)}] \), which is the Euclidean-orthogonal complement of \( V^- = [v_1, \ldots, v_r] \); since \( p \) is strictly subharmonic when restricted to any complex direction in \( \mathbb{C}^{n-k} \), then \( V^- \) is totally real and so with a final orthogonal c.l.c.c., we can assume

\[ V^- = \{ z_j = 0 \; y_t = 0 \; k + r + 1 \leq j \leq n, \; k + 1 \leq i \leq k + r \} \]

and consequently:

\[ V^+ = \{ z_j = 0 \; k + 1 \leq j \leq k + r \}. \]

**Lemma 1.2.** Assume complex coordinates are chosen in such a way that \( p \) appears in the normalized form given by Lemma 1.1; thus:

a) if \( k = 0 \), then there exist a neighbourhood \( U \) of \( 0 \) and \( K > 0 \) such that if \( x \in U \cap \overline{D} \) then

\[ \text{dist}^2(x, L \cap \overline{D}) \leq K \text{dist}(x, L) \]

and so, in particular \( L \) and \( \overline{D} \) are regularly separated at 0;

b) if \( k > 0 \), then there exists a totally real \( (k + r) \)-dimensional \( C^\infty \)-submanifold \( S \) of \( L \), passing through 0 for which there exist a neighbourhood \( U \) of \( 0 \) and \( K > 0 \) such that if \( \Sigma = (S \times \text{Re} \mathbb{C}_{z_{n+1}}) \cap bD \) and \( Z = L \cup \Sigma \) then for every \( x \in U \cap \overline{D} \) we have

\[ \text{dist}^2(x, Z \cap \overline{D}) \leq K \text{dist}(x, Z) \]

and so, in particular \( Z \) and \( \overline{D} \) are regularly separated at 0.

**Proof.** First of all note that if \( x = (z, z_{n+1}) \in \overline{D} \) then we have

\[ \text{Re} \; z_{n+1} \geq r(z, \text{Im} \; z_{n+1}) = h(z) + O(|\text{Im} \; z_{n+1}|^2) \]
and so

\[ h(z) \leq \text{Re} \, z_{n+1} + O(1) \frac{1}{| \text{Im} \, z_{n+1} |^2) \leq c' \{ | \text{Re} \, z_{n+1} | + | \text{Im} \, z_{n+1} | \} \leq c |z_{n+1}|. \]

a) Assume \( k = 0 \).

1. Since we are interested only in those points \( x = (z, z_{n+1}) \in \overline{D} \) where \( h(z) > 0 \), in order to get (\#a), it is enough to prove

\[
\text{dist}^2(x, \overline{D} \cap L) \leq c |h(z)| \quad \text{for } z \in L \text{ near } 0
\]

and this condition, of course has nothing to do with the complex structure.

2. Up to a real linear change of coordinates, we can assume

\[
p(z) = p(u, v) = |u|^2 - |v|^2
\]

where \( u = (u_1, \ldots, u_p), v = (v_1, \ldots, v_q), p + q = 2n \).

Recall that \( h(u, v) = p(u, v) + \varphi(u, v) \) and \( \varphi(u, v) = o(|u|^2 + |v|^2) \) and so, given \( \lambda > 0 \), let \( \rho > 0 \) such that, if \( |u|^2 + |v|^2 \leq \rho^2 \) then \( |\varphi(u, v)| < \frac{1}{2} (|u|^2 + |v|^2) \); setting

\[
p_\lambda = p + \lambda(|u|^2 + |v|^2) \quad H_\lambda = \{ p_\lambda < 0 \} \quad A_\lambda = \mathcal{C} H_{-\lambda},
\]

in the ball \( B(0, \rho) \) we have:

\[
p_{-\lambda} < h < p_\lambda
\]

and therefore

i) if \( x \in H_\lambda \), then \( x \in L \cap \overline{D} \) i.e. \( H_\lambda \subset \overline{D} \cap L \)

ii) if \( x = (u, v) \in A_\lambda \) then \( p(u, v) \geq \lambda(|u|^2 + |v|^2) \) and

\[
h(x) > p(u, v) - \frac{\lambda}{2} (|u|^2 + |v|^2) \geq \frac{\lambda}{2} (|u|^2 + |v|^2) \geq c \text{ dist}^2(x, L \cap \overline{D}),
\]

so we have to consider only

\[
x \in C_\lambda = \mathcal{C} (H_\lambda \cup A_\lambda) = \left\{ (u, v) \in \mathbb{R}^p \times \mathbb{R}^q; \frac{1 - \lambda}{1 + \lambda} |v|^2 \leq |u|^2 \leq \frac{1 + \lambda}{1 - \lambda} |v|^2 \right\}.
\]

Let \( C = \{ p = 0 \} \) and let \( \nu \) be the outward pointing normal unit vector field to \( C - \{ 0 \} \), extended to \( C_\lambda - \{ 0 \} \); for a fixed small \( \lambda \), \( \nu \) defines a projection \( \pi: C_\lambda - \{ 0 \} \to C - \{ 0 \} \) thus, for \( x = (u, v) \in C_\lambda \), we have

\[
\frac{\partial h}{\partial \nu}(x) = \frac{\partial p}{\partial \nu} + o(|x|) \geq c |\pi(x)|;
\]

so if \( \hat{x} \in C_\lambda \cap L \cap bD \) is a point on the line from \( x \) parallel to \( \nu(\pi(x)) \), we have
and since $|\pi(x)| \geq |x - \hat{x}|$, we obtain

$$|h(x)| \geq c|x - \hat{x}| \geq c \text{ dist}^2(x, L \cap \overline{D}).$$

b) Assume $k > 0$.
1. Let

$$S = \left\{ (x_1, \ldots, x_n, y_1, \ldots, y_n) \in L \left| \frac{\partial h}{\partial x_l} = 0, \frac{\partial h}{\partial y_m} = 0, k + r + 1 \leq l \leq n, 1 \leq m \leq n \right. \right\}$$

we have $0 \in S$ and so, in virtue of the implicit functions theorem, there exists a neighbourhood $U$ of $0$ such that in $L \cap U$:

$$S = \left\{ (x_1, \ldots, x_n, y_1, \ldots, y_n) \in L \left| x_l = \eta_l(x_1, \ldots, x_{k+r}), y_m = \alpha_m(x_1, \ldots, x_{k+r}), k + r + 1 \leq l \leq n, 1 \leq m \leq n \right. \right\}$$

for $C^\infty$-smooth functions $\eta_l$, $\alpha_m$; so $S$ is totally real (cf. e.g. [5]); set $\Sigma = (S \times \text{Re} C_{x^k}) \cap bD$ and $Z = L \cup \Sigma$.
2. Write $D \cap U = \hat{M}_K \cup \hat{N}_K$ where:

$$\hat{M}_K = \{ x \in D \cap \overline{U} \mid \text{dist}^2(x, \Sigma) \leq K \text{ dist} (x, L) \}$$

and $\hat{N}_K = \overline{D \cap U} - \hat{M}_K$
if \( x \in \mathcal{M}_K \) then

\[
\text{dist}^2(x, Z \cap \overline{D}) = \min\{\text{dist}^2(x, \Sigma), \text{dist}^2(x, L \cap \overline{D})\} \leq \text{dist}^2(x, \Sigma)
\]

\[
\leq \begin{cases} 
C \text{dist}(x, \Sigma) \\
K \text{dist}(x, L)
\end{cases}
\]

\[
\leq c' \min\{\text{dist}(x, \Sigma), \text{dist}(x, L)\} = c' \text{dist}(x, Z).
\]

3. We have the following

CLAIM 1. Let

\[Q = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in L \cap U : \quad |h(x_1, \ldots, x_{k+r}, \eta_{k+r+1}, \ldots, \eta_n, \alpha_1, \ldots, \alpha_n) | \geq 0\};\]

if \( \pi: \mathbb{C}^{n+1} \rightarrow L \) is the natural projection, then there exists \( K > 0 \) such that if \( x \in \overline{D} \cap U \) and \( \pi(x) \in Q \), then \( x \in \mathcal{M}_K \).

PROOF OF CLAIM 1. Let \( x \in \overline{D} \), \( x = (z, z_{n+1}) \) with

\[z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in Q\]

let \( x' = (z, 0) \), \( x'' = (\hat{z}, 0) \) where \( \hat{z} = (x_1, \ldots, x_{k+r}, \eta_{k+r+1}, \ldots, \eta_n, \alpha_1, \ldots, \alpha_n) \); of course \( \hat{z} \in Q \cap \overline{S} \); then

\[h(x) = h(\hat{z}) + \frac{1}{2} \text{Hess}(h)(\hat{z})(z - \hat{z}) + O(|z - \hat{z}|^3)\]

where \( \text{Hess}(h)(\hat{z}) \) is the Hessian quadratic form of \( h \) at \( \hat{z} \); we have \( \text{Hess}(h) = \text{Hess}(p) + \text{Hess}(\varphi) \) and, since \( p \) is positive definite on \( L^+ = \{z \in L | x_j = 0, \quad 1 \leq j \leq k + r\}, \ z - \hat{z} \in L \) and \( \varphi(z) = o(|z|^3) \), we obtain

\[h(z) \geq h(\hat{z}) + c|z - \hat{z}|^2 \geq h(\hat{z}) + c' \text{dist}(z, S);\]

so

\[\text{dist}(x, \Sigma) \leq \text{dist}(x, x') + \text{dist}(x', \Sigma) = |z_{n+1}| + \text{dist}(x', \Sigma) \leq |z_{n+1}| + \text{dist}(x', x'') + \text{dist}(x'', \Sigma).\]

Now we have:

i) \( \text{dist}(x', x'') \leq c_2 \text{dist}(x', S) \)

ii) since \( (\hat{z}, h(\hat{z})) \in \Sigma: \)

\[\text{dist}(x'', \Sigma) \leq \text{dist}(x'', (\hat{z}, h(\hat{z}))) = h(\hat{z}) < h(z);\]

so:

\[\text{dist}^2(x, \Sigma) \leq c_3(|z_{n+1}|^2 + \text{dist}^2(x, S) + h^2(z)) \leq c_4(|z_{n+1}|^2 + h(z)) \leq K|z_{n+1}| = K \text{dist}(x, L)\]
and the proof of claim 1 is complete.

4. Next step is the following:

**CLAIM 2.** If \( x \in \overline{D \cap U} \) and \( \pi(x) \not\in Q \), then there exists \( K > 0 \) such that

\[
\text{dist}^2(x, L \cap \overline{D}) \leq K \text{dist}(x, L).
\]

**PROOF OF CLAIM 2.** It is enough to show that if \( x = (z, z_{n+1}) \in \overline{D \cap U} \) and \( z \not\in Q \cup (L \cap D) \) then \( h(z) \geq c \text{dist}^2(z, L \cap D) \); now for such an \( x \) we have \( h(z) > 0 \) while \( h(\tilde{z}) = h(x_1, \ldots, x_{k+1}, \eta_{k+1}, \ldots, \eta_n, \alpha_1, \ldots, \alpha_n) < 0 \); in the segment \([\tilde{z}, z]\), consider the last point \( \tilde{z} \) such that \( h(\tilde{z}) = 0 \) and let \( f(t) = h((1-t)\tilde{z} + tz) \) since \( f''(t) = \text{Hess}(h)((1-t)\tilde{z} + tz)(z - \tilde{z}) \geq c|z - \tilde{z}|^2 \), then \( f(t) \) is a convex increasing function in \([0,1]\); moreover we have:

\[
h(z) = f(1) = f(0) + f'(0) + \frac{1}{2}f''(\hat{t}) \quad \text{for} \quad \hat{t} \in [0,1];
\]

since \( f(0) = h(\tilde{z}) = 0 \), \( f'(0) \geq 0 \), we obtain precisely

\[
h(z) \geq c \text{dist}^2(x, L \cap \overline{D}).
\]

5. **Summing up:**

Given \( x \in \overline{D \cap U} \), if \( \pi(x) \in Q \), then by claim 1, \( x \in M_K \) and so \( \text{dist}^2(x, Z \cap \overline{D}) \leq c_1 \text{dist}(x, Z) \); if \( \pi(x) \not\in Q \), then by claim 2, \( \text{dist}^2(x, L \cap \overline{D}) \leq c_2 \text{dist}(x, L) \) and so

\[
\text{dist}^2(x, Z \cap \overline{D}) = \min\{\text{dist}^2(x, \Sigma), \text{dist}^2(x, L \cap \overline{D})\}
\leq c_2 \min\{\text{dist}(x, \Sigma), \text{dist}(x, L)\}
= c_2 \text{dist}(x, Z)
\]

and the proof of Lemma 1.2 is complete.

**REMARK 1.3.** a) Lemma 1.2 asserts essentially that if \( D \) is strictly pseudoconvex, then \( \overline{D} \) and \( L \) are not regularly separated at most “along” a totally real submanifold \( \Sigma \) of \( bD \) (see [2] for some partial results in this direction);

b) it follows from Lemma 1.2 and Whitney extension theorems (cf. e.g. [7]) that if \( f \in \mathfrak{D}^\infty(L) \) and \( f \) is infinitely flat on \( \Sigma \) then it is possible to find a \( C^\infty \)-smooth extension \( F \) of \( f \) around \( \overline{D \cap U} \), vanishing on \( L \cap U \).

**2. - The semi-local case.**

Lemma 1.2 enables us to prove the following semi-local version of the main Theorem:

**PROPOSITION 2.1.** Let \( D \subset C^{n+1} \) be a bounded strictly pseudoconvex domain with \( C^\infty \)-smooth boundary and let \( g \in \mathcal{O}(D') \), where \( D \subset \subset D' \), such that, if
V = \{ g = 0 \}, then \( V \cap D = V \cap \overline{D} \neq \emptyset \); let \( x \in \overline{D} \) such that \( \partial g(x) \neq 0 \); then for every neighbourhood \( U \) of \( x \), there exists another neighbourhood \( W \) of \( x \) such that if \( f \in C^\infty(U) \) and \( f|_{U \cap D \cap V} \equiv 0 \) then for every pseudoconvex domain \( \hat{D} \) with \( C^\infty \)-smooth boundary such that \( D \subset \hat{D} \subset \subset \hat{D}' \) and \( D \cap W = \hat{D} \cap W \), we can find \( \lambda \in A^\infty(\hat{D}) \) such that \( \lambda|_D \in I^\infty(V) \), and \( a_1, \ldots, a_4 \in C^\infty(\hat{D}) \), in such a way that on \( \overline{W} \cap \overline{D} \) we have

\[
f = a_1 g + a_2 \overline{g} + a_3 \lambda + a_4 \overline{\lambda}.
\]

**Proof.**

1. We can assume \( x \in bD \cap V \) otherwise there is almost nothing to prove.

2. If \( V \) and \( bD \) are transversal at \( x \), we obtain the result with \( \lambda \equiv 0 \), using the well-known techniques for the regularly separated case.

3. If \( V \) and \( bD \) are not transversal at \( x \), then we can choose complex coordinates near \( x \) in such a way that \( z_{n+1} = g \) (and so we can identify near \( x, V \) with \( L = \{ z_{n+1} = 0 \} = T_z \overline{bD} \); performing the c.i.c.c. as in Lemma 1.1, again we can assume \( k > 0 \) and construct \( S, \Sigma, Z \) as in Lemma 1.2 b), in a neighbourhood \( W' \subset U \) of \( O \).

4. Let \( f \in C^\infty(U) \) such that \( f|_{U \cap D \cap V} \equiv 0 \); choose \( j \in \mathbb{Z}^+ \) in such a way that if \( \tilde{f} = f + jg \) then

\[
\frac{\partial \tilde{f}}{\partial z_{n+1}} - \frac{\partial \tilde{f}}{\partial \overline{z}_{n+1}} \neq 0
\]

in \( W' \); let \( M = \{ x \in W' | \tilde{f} = 0 \} \); then it is possible to find \( \varphi \in C^\infty(L, \mathbb{C}) \) such that \( \varphi|_{L \cap \overline{D}} \equiv 0 \) and

\[
M = \{ \varphi(z_1, \ldots, z_n) = z_{n+1} \} \cap W'
\]

then (cf. e.g. [7]) in \( W' \cap D \) we have

\[
\tilde{f} = a(\varphi - z_{n+1}) + b(\overline{\varphi - z_{n+1}}) \quad \text{for } a, b \in C^\infty(\hat{D});
\]

we want to factorize \( \varphi \).

We need two preliminary lemmas; first of all let

\[
\mathcal{E} = \{ \sigma \in C^\infty(\mathbb{R}^+, \mathbb{R}^+) | \text{ for every } k \in \mathbb{Z}^+, \sigma^{(k)}(x) = 0, \sigma'(x) > 0 \text{ if } x > 0 \}
\]

then we have:

**Lemma 2.2** Given \( \varphi \in C^\infty(L, \mathbb{C}) \) such that \( \varphi|_{L \cap \overline{D}} \equiv 0 \), it is possible to find \( \tilde{\varphi} \in C^\infty(L, \mathbb{R}) \) such that \( \{ \tilde{\varphi} = 0 \} = L \cap \overline{D} \) and \( \sigma \in \mathcal{E} \) in such a way that

\[
\sigma(\tilde{\varphi}(x)) \geq |\varphi(x)|
\]
PROOF. For any $\varepsilon > 0$, let $K_\varepsilon = \{ z \in L : \operatorname{dist}(z, L \cap D) \leq \varepsilon \}$ and let $\lambda(\varepsilon) = \sup |\varphi(z)|$ thus we have: $\lambda(\varepsilon) \searrow 0$ if $\varepsilon \searrow 0$ and $\lambda(\varepsilon) = o(\varepsilon^k)$ for every $k \in \mathbb{Z}^+$; so it is possible to find $\hat{\lambda}, \hat{\mu} \in \mathcal{E}$ such that:

i) $\hat{\lambda} > \lambda$,

ii) $\hat{\lambda} = o(\hat{\mu}^k)$ for every $k \in \mathbb{Z}^+$ and so $\hat{\lambda} = \sigma \circ \hat{\mu}$ for $\sigma \in \mathcal{E}$.

Let now $\rho \in C^\infty(L \setminus \overline{D})$ such that for $z \in L \setminus \overline{D}$

$$\operatorname{dist}(z, L \cap \overline{D}) \leq \rho(z) \leq 2 \operatorname{dist}(z, L \cap \overline{D})$$

and set

$$\hat{\varphi}(z) = \begin{cases} 
\hat{\mu}(\rho(z)) & \text{on } L \setminus \overline{D} \\
0 & \text{on } L \cap \overline{D}
\end{cases}$$

thus $\hat{\varphi} \in C^\infty(L, \mathbb{R})$, $\{ \hat{\varphi} = 0 \} = L \cap \overline{D}$ and

$$\sigma(\hat{\varphi}(z)) = \sigma \circ \hat{\mu}(\rho(z)) \geq \sigma \circ \hat{\mu}(\operatorname{dist}(z, L \cap \overline{D}))$$

$$= \hat{\lambda}(\operatorname{dist}(z, L \cap \overline{D})) \geq \lambda(\operatorname{dist}(z, L \cap \overline{D})) \geq |\varphi(z)|.$$  

**Lemma 2.3.** Let $a \in C^\infty(L, \mathbb{C})$ such that $a|_{L \cap D} \equiv 0$; set $A(z_1, \ldots, z_n, z_{n+1}) = a(z_1, \ldots, z_n)$; then the following facts are equivalent:

i) $a(z) = o(|\lambda(z)|^k)$ for $z \rightarrow L \cap \overline{D} \cap W'$ and every $k \in \mathbb{Z}^+$

ii) $A|_{\overline{D} \cap W'}$ admits a $C^\infty$-smooth extension around $\overline{D} \cap W'$ vanishing on $L \cap W'$.

**Proof.** i) $\Rightarrow$ ii) we claim that, if $\alpha = (\alpha_1, \ldots, \alpha_n, 1, \ldots, 1) \in (\mathbb{Z}^+)^{2n+2}$, setting

$$f_\alpha(x) = \begin{cases} 
0 & \text{if } \alpha_{n+1} + \alpha_n > 0 \\
D^\alpha A(x) & \text{if } x \in \overline{D} \cap W' \\
0 & \text{if } L \setminus \overline{D} \cap W'
\end{cases}$$

then the $(f_\alpha)_{\alpha \in (\mathbb{Z}^+)^{2n+2}}$ are, under assumption i), Whitney data on $(\overline{D} \cap L) \cap W'$ i.e. for any $\alpha \in (\mathbb{Z}^+)^{2n+2}$, any $m \in \mathbb{Z}^+$

$$f_\alpha(x) = \sum_{|\beta| \leq m} \frac{1}{\beta!} f_{\alpha + \beta}(y)(x - y)^\beta + o(|x - y|^m)$$

uniformly for $|x - y| \rightarrow 0$; in fact:

1) if $x, y \in \overline{D} \cap W'$ or $x, y \in L \cap W'$, we have nothing to prove;

2) if $x \in \overline{D} \cap W' \setminus L$, $y \in L \cap W'$, from i) it follows that, for any $\alpha \in (\mathbb{Z}^+)^{2n+2}$ such that $\alpha_{n+1} + \alpha_n = 0$ and any $m \in \mathbb{Z}^+$, setting $x = (z, z_{n+1})$, we have:

$$f_\alpha(x) = D^\alpha a(z) = o(|\lambda(z)|^m)$$
and \(|h(z)| \leq c(|z_{n+1}| + |z - y|) \leq c'|x - y|;\)

3) if \(x \in L \cap W', y \in D \cap W' \setminus L, y = (z, z_{n+1})\) then for any \(\alpha \in (\mathbb{Z}^+)^{2n+2}, m \in \mathbb{Z}^+\)

\[ f_\alpha(x) - \sum_{|\beta| \leq m} \frac{1}{\beta!} D_{\alpha+\beta} A(y)(x - y)^\beta = -D^\alpha a(x) + o(|x - y|^m) = o(|x - y|^m) \]

and so ii) follows from Whitney extension theorems (cf. e.g. [7]).

ii)\(\Rightarrow\)i) let \(F\) be the extension in assumption ii); if \(z \in L \cap W'\), let \(x = (z, h(z)), y = (z, 0)\); if \(\alpha = (\alpha_1, \ldots, \alpha_n, 0, \alpha_{n+1}, \ldots, \alpha_{2n+2}, 0) \in (\mathbb{Z}^+)^{2n+2}\) then we have:

\[ D^\alpha a(x) = D^\alpha F(z) = \sum_{|\beta| \leq m} \frac{1}{\beta!} D_{\alpha+\beta} F(y)(x - y)^\beta + o(|x - y|^m) = o(|x - y|^m), \]

Going back to the proof of Proposition 2.1, using Lemma 2.2, we can find \(\phi \in C^\infty(L, \mathbb{R})\) and \(\sigma \in \mathcal{E}\) such that \(\{\phi = 0\} = L \cap D\) and \(\sigma(\phi(z)) \geq |\phi(z)|\).

We can find also \(\omega, q, \alpha \in \mathcal{E}\) such that

\[ \omega \circ q \circ \alpha = \sigma \]

and so setting \(s = \alpha \circ \phi\) we obtain

\[ \varphi(z) = o(|q(s)(z)|^k) \]

for \(z \to L \cap \overline{D} \cap W'\) and every \(k \in \mathbb{Z}^+\); since \(\varphi \equiv 0\) when \(h(z) \leq 0\), we have also

\[ \varphi(z) = o(|h(z) + q(s)(z)|^k) \]

for \(z \to L \cap \overline{D} \cap W'\) and every \(k \in \mathbb{Z}^+\).

Let now \(F: \mathbb{C}^n_+ \to \mathbb{C}^n_+\) defined by

\[ \begin{cases} w_j = z_j & 1 \leq j \leq n \\ w_{n+1} = q(s)(z_1, \ldots, z_n) + z_{n+1} \end{cases} \]

and \(G = F^{-1}: \mathbb{C}^n_+ \to \mathbb{C}^n_+\)

\[ \begin{cases} z_j = w_j & 1 \leq j \leq n \\ z_{n+1} = w_{n+1} - q(s)(w_1, \ldots, w_n) \end{cases} \]

be \(C^\infty\)-smooth changes of coordinates: then

\[ F(D \cap W') = \{ \text{Re } w_{n+1} > r'(w_1, \ldots, w_n, \text{Im } w_{n+1}) \} \]
where
\[ r'(w_1, \ldots, w_n, \text{Im } w_{n+1}) = r(w_1, \ldots, w_n, \text{Im } w_{n+1}) + q(s)(w_1, \ldots, w_n) \]
and so
\[ h'(w_1, \ldots, w_n) = h(w_1, \ldots, w_n) + q(s)(w_1, \ldots, w_n). \]

Setting
\[ \Phi(w_1, \ldots, w_n, w_{n+1}) = \varphi(w_1, \ldots, w_n), \]
using (#) and Lemma 2.3, we obtain that \( \Phi|_{\overline{D \cap \overline{W}^\prime}} \) admits an extension which is \( C^\infty \)-smooth around \( \overline{D \cap \overline{W}^\prime} \) and vanishes on \( M = \{w_{n+1} = 0\} \) and so \( \Phi|_{D \cap \overline{W}^\prime} \) admits an extension which is \( C^\infty \)-smooth around \( D \cap \overline{W}^\prime \) and vanishes on
\[ (G(M) = \{q(s)(z_1, \ldots, z_n) + z_{n+1} = 0\}) \cap W^\prime; \]

since \( \Phi \) is \( n + 1 \)-flat on \( L \cap D \cap W' \), this implies (cf. [4]) that it is possible to find \( c \in C^\infty(D) \) such that on \( D \cap \overline{W}^\prime \) we have
\[ \varphi(z) = c(z, z_{n+1})(q(s)(z_1, \ldots, z_n) + z_{n+1}). \]

We want to factorize \( q(s) \).

5. Let \( W \subset B_{n+1}(0, \varepsilon/2) \subset B_{n+1}(0, \varepsilon) \subset \overline{W}^\prime \) be a neighbourhood of \( O \) and let \( \chi \in C^\infty_0(W' \cap L), \chi \equiv 1 \) on \( W \cap L \); set \( \tilde{s} = \chi \cdot s. \) Since \( S \) is totally real we can find (cf. [5]) \( \tilde{s} \in C^\infty(L, \mathbb{C}) \) such that

1) \( \tilde{s}|_{S \cap \overline{W}^\prime} = \tilde{s}|_{S \cap \overline{W}^\prime} \)
2) \( \overline{\partial s}|_{S \cap \overline{W}^\prime} = 0 \) up to infinite order
3) \( \text{supp} \tilde{s} \subset \text{supp} \tilde{s} \)

let \( \beta \in C^\infty_0(\mathbb{C}) \) such that \( \text{supp} \beta \subset B(0, \varepsilon), \beta \equiv 1 \) on \( B(0, \varepsilon/2) \): thus setting
\[ \tilde{s}(z_1, \ldots, z_{n+1}) = \beta(z_{n+1})\tilde{s}(z_1, \ldots, z_n) \]
we have that \( \overline{\partial \tilde{s}} \), as element of \( C^\infty_{(0,1)}(\overline{D \cap \overline{W}}) \), is infinitely flat on \( \Sigma \) and since \( Z = L \cup \Sigma \) and \( D \), are, by Lemma 1.2 b), regularly separated at \( O \), then the data
\[ \{ D^a \overline{\partial \tilde{s}} \text{ on } \overline{D \cap \overline{W}}, 0 \text{ on } \overline{Z \cap \overline{W}} \] as Whitney data coinciding on the intersection, are Whitney data on \( (\overline{D \cup Z}) \cap \overline{W} \) (cf. e.g. [7]) i.e. \( \overline{\partial \tilde{s}}|_{D \cap \overline{W}^\prime} \) admits an extension \( C^\infty \)-smooth around \( \overline{D \cap \overline{W}} \) vanishing on \( L \cap W \), and so
\[ \alpha = \frac{\partial \tilde{s}}{z_{n+1}} \in C^\infty_{(0,1)}(\overline{D \cap \overline{W}}); \]
since, for a suitable $\epsilon$, $\operatorname{supp} \tilde{s} \subset W'$, we have
\[ \alpha = \frac{\tilde{\partial} s}{g} \in C_{(0,1)}(\tilde{D}) \]
for any domain $\tilde{D}$ as in the statement of Proposition 2.1; thus, following [6], it is possible to find $u \in C^\infty(\tilde{D})$ such that $\tilde{\partial} u = \alpha$ on $\tilde{D}$ and
\[ \lambda = gu - \tilde{s} \in A^\infty(\tilde{D}), \quad \lambda|_{\tilde{D}} \in I^\infty(V). \]

6. Extend now $q$ to $C_\varsigma$ in the obvious way: $q(\varsigma) = q(|\varsigma|)$; then we have
\[ q(\varsigma + \eta) = q(\varsigma) + \hat{a} \eta + \tilde{b} \eta \quad \text{for} \quad \hat{a}, \tilde{b} \in C^\infty(\varsigma); \]
we obtain on $W \cap D$
\[ s = s - \tilde{s} + \hat{s} = s - \tilde{s} + gu - \lambda \]
and
\[ q(s) = q(s - \tilde{s}) + \hat{a} \cdot (gu - \lambda) + \tilde{b} (gu - \lambda) \]
where $q(s - \tilde{s})$ as element of $C^\infty(\tilde{D} \cap \tilde{W})$ is infinitely flat on $\Sigma$ and, by the same argument as before,
\[ q(s - \tilde{s}) = d \cdot g \quad \text{for} \quad d \in C^\infty(\tilde{D}); \]
thus we have on $W \cap D$
\[ q(s) = d \cdot g + \hat{a} \cdot (gu - \lambda) + \tilde{b} \cdot (gu - \lambda) \]
\[ \varphi = c \cdot [(d + \hat{a} u + 1) \cdot g + \tilde{b} \tilde{g} - \hat{a} \lambda - \tilde{b} \lambda] \]
and, putting everything together, we obtain finally:
\[ f = a_1 g + a_2 \tilde{g} + a_3 \lambda + a_4 \lambda \]
with $a_1, a_2, a_3, a_4 \in C^\infty(\tilde{D})$.

REMARK 2.4. In general it is not possible to simplify the representation of a $C^\infty$-smooth function by means of holomorphic functions, given in Proposition 2.1, i.e., given $f \in \mathcal{S}^\infty(V)$, in general it is not possible to find a single $\lambda \in I^\infty(V)$ such that, at least locally
\[ f = a\lambda + b \lambda \quad \text{for} \quad a, b \in C^\infty(\tilde{D}). \]
In fact, let $V = L = \{z_{n+1} = 0\}$ and $f \in \mathcal{S}^\infty(L)$ such that:
\[ \begin{align*}
\text{i) } & \left| \frac{\partial f}{\partial z_{n+1}} \right| - \left| \frac{\partial f}{\partial \overline{z}_{n+1}} \right| \neq 0 \\
\text{ii) } & \{f = 0\} \cap D \supset L \cap D \end{align*} \]
(and this is possible whenever \( L \) has an infinite order of contact with \( bD \) along some real direction); if \( f = a\lambda + b\bar{\lambda} \) with \( \lambda \in I^\infty(L) \) and \( a, b \in C^\infty(\overline{D}) \), from i) we obtain
\[
(|a|^2 - |b|^2) \left| \frac{\partial \lambda}{\partial z_{n+1}} \right|^2 \neq 0
\]
and
\[
\lambda = (\bar{a}f - b\bar{f})(|a|^2 - |b|^2)^{-1};
\]
thus \( \{ \lambda = 0 \} \) is a complex submanifold of \( D \) containing \( \{ f = 0 \} \); contradiction.

3. - The general case.

Our next step is to extend Proposition 2.1 to the case of arbitrary codimension.
Consider first the case \( V \) is a linear submanifold; in this direction, we have the following

**Lemma 3.1.** Let \( D \subset \mathbb{C}^{n+1} \) be a bounded strictly pseudoconvex domain with \( C^\infty \)-smooth boundary and let \( V = \{ z_{k+1} = \ldots = z_{n+1} = 0 \} \); assume
\[
\overline{D} \cap V = D \cap V \neq \emptyset;
\]
let \( x \in \overline{D} \); then for every neighbourhood \( U \) of \( x \), there exists another neighbourhood \( W \) of \( x \) such that, if \( f \in C^\infty(U) \) and \( f|_{\overline{D} \cap V} = 0 \), then it is possible to find \( \lambda \in I^\infty(V) \) and \( a, b, a_{k+1}, \ldots, a_{n+1}, b_{k+1}, \ldots, b_{n+1} \in C^\infty(\overline{D}) \) in such a way that on \( W \cap \overline{D} \) we have
\[
f = \sum_{j=k+1}^{n+1} (a_j z_j + b_j \bar{z}_j) + a\lambda + b\bar{\lambda}.
\]

**Proof.** 1. We can assume \( x \in bD \cap V \), \( V \) and \( bD \) are not transversal at \( x \) and therefore, e.g. \( T^x_x bD = L = \{ z_{n+1} = 0 \} \).

2. Let \( M = \{ z_{k+1} = \ldots = z_n = 0 \} \); thus \( bD \) and \( M \) are transversal at \( x \) and therefore in a neighbourhood \( W \subset U \) of \( x \); thus we can find another strictly pseudoconvex domain \( \tilde{D} \supset D \) such that \( D \cap W = \tilde{D} \cap W \) and \( M \) and \( b\tilde{D} \) are transversal everywhere, so \( \tilde{D}^{(1)} = M \cap \tilde{D} \) is a strictly pseudoconvex \((k + 1)\)-dimensional domain with \( C^\infty \)-smooth boundary.

Let \( f \in C^\infty(\overline{U}) \) such that \( f|_{\overline{D} \cup \overline{V}} = 0 \); since \( V \) is 1-codimensional in \( \tilde{D}^{(1)} \), applying proposition 2.1. to \( \tilde{D}^{(1)} \) and \( f|_{\overline{U} \cap M} \), we can find \( a_{n+1}, b_{n+1}, a, b \in C^\infty(\overline{D}), \mu \in A^\infty(\overline{D}^{(1)}), \mu|_{\overline{D} \cup \overline{V}} \equiv 0 \) such that, on \( \overline{D}^{(1)} \cap W \)
\[
f = a_{n+1} z_{n+1} + b_{n+1} \bar{z}_{n+1} + a\mu + b\bar{\mu}.
\]
Now, since $M$ and $b\overline{D}$ are transversal, by [4] (Lemma 2 ii)), it is possible to find $\lambda \in A^\infty(\overline{D})$ such that $\lambda|_{\partial D} = \mu$, so if

$$F = a_{n+1}z_{n+1} + b_{n+1}\overline{z}_{n+1} + a\lambda + b\overline{\lambda}$$

we have $(F - f)_{|\overline{D} \cap W') = 0$ and again on $\overline{D} \cap W$$$
F - f = \sum_{j=1}^{n} (a_jz_j + b_j\overline{z}_j)
$$
for $a_j, b_j \in C^\infty(\overline{D}), 1 \leq j \leq n$, so the proof of Lemma 3.1 is complete.

We have now the following

**Proposition 3.2.** Let $D, V, g_1, \ldots, g_k$ as in the main Theorem and assume $g_j \in \mathcal{O}(D')$ $1 \leq j \leq k$, where $D' \supset \overline{D}$; then, for every neighbourhood $U$ of $x$ there exists another neighbourhood $W$ of $x$ such that for every function $f \in C^\infty(U)$ such that $f|_{D \cap U} = 0$, it is possible to find $\lambda \in I^\infty(V)$ and $a, b, a_1, \ldots, a_k, b_1, \ldots, b_k \in C^\infty(D)$ in such a way that in $\overline{W} \cap D$ we have

$$f = \sum_{j=1}^{k} (a_jg_j + b_j\overline{g}_j) + a\lambda + b\overline{\lambda}.
$$

**Proof.** 1. As usual, we can assume $x \in V \cap bD$; let $G : D' \rightarrow \mathbb{C}^k$ be the holomorphic map given by $G(z) = (g_1(z), \ldots, g_k(z))$ and let $\Gamma$ be its graph.

2. Let $f \in C^\infty(\overline{U})$ such that $f|_{D \cap U} = 0$; since $(g_1, \ldots, g_k)$ is a complete defining system for $V$, we can find (cf. [4], Lemma 5) a neighbourhood $A$ of $x$ complex coordinates $v_1, \ldots, v_q$, $q = n + 1 + k$, in such a way that

$$A \cap C^{n+1} = \{v_{n+2} = \cdots = v_q = 0\}
$$

$$A \cap \Gamma = \{v_{n+2-d} = \cdots = v_{n+1-d+k} = 0\}
$$

where $d = n + 1 - \dim_{\mathbb{C}} V \leq k$, thus, since $\Gamma \cap D' = V$,

$$V \cap A = \{v_{n+2-d} = \cdots = v_q = 0\}.
$$

3. Let now $W \subset W' \subset U$ be two neighbourhoods of $x$ in $\mathbb{C}^{n+1}$ such that $A \cap C^{n+1} \supset W'$ and let $\rho = C^\infty(W')$ such that $\rho \equiv 1$ on $W$; set $\tilde{f} = \rho f$; setting

$$\tilde{F}(v_1, \ldots, v_q) = \tilde{f}(v_1, \ldots, v_{n+1}) \text{ for } (v_1, \ldots, v_q) \in [(W' \cap D) \times \mathbb{C}^k] \cap A
$$

we obtain $\tilde{F}|_{\Gamma \cap [(W' \cap D) \times \mathbb{C}^k] \cap A} = 0$ so we can construct in $D' \times \mathbb{C}^k$ a strictly pseudoconvex domain $B$ with $C^\infty$-smooth boundary such that

i) $B \cap (D' \times \{0\}) = D$

ii) $B \cap A \subset [(W' \cap D) \times \mathbb{C}^k] \cap A$
and we can extend $\tilde{F}$ to an element $F$ of $C^\infty(\mathbb{B})$ in such a way that $F |_{\Gamma \cap B} \equiv 0$ and $F |_{D \cap W} = f$.

4. Now $\Gamma \cap B$ is holomorphically equivalent to a plane section, thus, using Lemma 3.1., we can find a neighbourhood $\tilde{W}$ of $x$ in $\mathbb{C}^{n+1} \times \mathbb{C}^k$, $\Lambda \in \mathcal{A}^\infty(\mathbb{B})$ such that $\Lambda |_{\Gamma \cap B} \equiv 0$, $\tilde{a}, \tilde{b}, \tilde{a}_1, \ldots, \tilde{a}_k, \tilde{b}_1, \ldots, \tilde{b}_k \in C^\infty(\mathbb{B})$ in such a way that on $\mathbb{B} \cap \tilde{W}$

$$F = \sum_{j=1}^{k} [a_j \cdot (g_j - w_j) + b_j \cdot (\overline{g_j} - \overline{w_j})] + \tilde{a}\Lambda + \tilde{b}\overline{\Lambda}$$

and therefore, setting

$$a_j = \tilde{a}_j |_{\mathbb{B}}, \quad b_j = \tilde{b}_j |_{\mathbb{B}}, \quad 1 \leq j \leq k,$$

$$a = \tilde{a} |_{\mathbb{B}}, \quad b = \tilde{b} |_{\mathbb{B}}, \quad \lambda = \Lambda |_{\mathbb{B}} \in \mathcal{I}^\infty(V),$$

we obtain precisely

$$f = \sum_{j=1}^{k} (a_j g_j + b_j \overline{g_j}) + a\lambda + b\overline{\lambda}.$$ 

We are now in the position to prove our main Theorem: using Proposition 3.2, we can construct an open cover $\mathcal{U} = (W^{(h)})_{1 \leq h \leq m}$ of $\mathbb{D}$ in such a way that, for every $f \in \mathcal{C}^\infty(V)$ one can find $\lambda_1, \ldots, \lambda_m \in \mathcal{I}^\infty(V)$, $a_1^{(h)}, \ldots, a_k^{(h)}, b_1^{(h)}, \ldots, b_k^{(h)}$, $c^{(h)}, d^{(h)} \in C^\infty(D)$ $1 \leq h \leq m$ such that on $D \cap W^{(h)}$

$$f = \sum_{j=1}^{k} (a_j^{(h)} g_j + b_j^{(h)} \overline{g_j}) + c^{(h)}\lambda_h + d^{(h)}\overline{\lambda_h}.$$ 

Let $\mathcal{A}$ be the sheaf on $\mathbb{D}$ of germs of functions $C^\infty$-smooth up to $bD$ and let

$$\mathcal{B} = (g_1, \ldots, g_k, \overline{g}_1, \ldots, \overline{g}_k, \lambda_1, \ldots, \lambda_m, \overline{\lambda}_1, \ldots, \overline{\lambda}_m) \mathcal{A}$$

thus $f \in H^0(\mathbb{D}, \mathcal{B})$.

Consider the exact sequence of sheaves

$$O \rightarrow \mathcal{R} \rightarrow \mathcal{A}^{\otimes(2k+m)} \rightarrow \mathcal{B} \rightarrow O$$

where:

$$\mu(a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_m, d_1, \ldots, d_m) = \sum_{j=1}^{k} (a_j g_j + b_j \overline{g_j}) + \sum_{h=1}^{m} (c_h \lambda_h + d_h \overline{\lambda_h})$$

and $\mathcal{R}$ is the sheaf of relations $C^\infty$-smooth up to $bD$ between $g_1, \ldots, g_k$, $\overline{g}_1, \ldots, \overline{g}_k, \lambda_1, \ldots, \lambda_m, \overline{\lambda}_1, \ldots, \overline{\lambda}_m$; since $\mathcal{R}$ is a fine sheaf, passing to the
cohomology sequence, we obtain:

\[ O \rightarrow H^\alpha(\mathcal{D}, \mathbb{R}) \rightarrow [H^\alpha(\mathcal{D}, \mathcal{A})]^{\otimes (k+m)} \rightarrow H^\alpha(\mathcal{D}, \mathcal{B}) \rightarrow O \]

is exact and this concludes the proof of the main Theorem.

From the main Theorem we can deduce the following (cf. also [2]).

**COROLLARY 3.3.** Let \( D, V, g_1, \ldots, g_k \) as in the main Theorem; then the following statements are equivalent:

i) \( D \) and \( V \) are regularly separated;

ii) \( g_1, \ldots, g_k \) generate \( \mathcal{I}^\infty(V) \) over \( \mathcal{A}^\infty(D) \).

**Proof.** i)\( \Rightarrow \)ii): see [1] and [4].

ii)\( \Rightarrow \)i) if \( g_1, \ldots, g_k \) generate \( \mathcal{I}^\infty(V) \) over \( \mathcal{A}^\infty(D) \), from the main Theorem it follows that \( g_1, \ldots, g_k, \overline{g}_1, \ldots, \overline{g}_k \) generate \( \mathcal{S}^\infty(V) \) over \( \mathcal{C}^\infty(D) \), so (see introduction) \( D \) and \( V \) are regularly separated.

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