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Some Mean-Value Theorems for Exponential Sums.

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1. – Introduction.

Analytic number theory benefits from estimates for moments of exponential sums in a number of ways, typical examples being through the application of the circle method to the Waring problem, cf. [6] and Vinogradov's method [7] of estimating short exponential sums. An important application of the latter is to the Riemann zeta-function giving strong bounds for $\zeta(s)$ near the line $\sigma = 1$. Recently, the authors [1] established a new connection between mean-value theorems and upper bounds for individual exponential sums, which proves to be powerful for estimating $\zeta(s)$ on the critical line $\sigma = 1/2$; this paper deals with such mean-value theorems.

Our sums are generalized Weyl sums of type

$$S(\alpha, \beta, x; N) = \sum_{N < n \leqslant N_1} e(\alpha n^2 + \beta n + xf(n/N)) \tag{1.1}$$

where $1 \leqslant N < N_1 \leqslant 2N$ and $f(t)$ is a suitable smooth function. Our aim is to estimate the eighth power mean-value

$$\mathscr{I}_8(N, X) = \int_0^1 \int_0^1 \int_{-X}^{X} |S(\alpha, \beta, x; N)|^8 \, d\alpha \, d\beta \, dx \, .$$

Let $I_8(N, Y)$ be the number of solutions of the system

$$\sum_1^4 (n_j - n_{j+4}) = 0 \, , \tag{1.2}$$

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$$(1.3) \qquad \sum_{1}^{4} (n_j^2 - n_{j+4}^2) = 0\,,$$

$$(1.4) \qquad \left|\sum_{1}^{4} (f(n_j/N) - f(n_{j+4}/N))\right| \leqslant Y\,,$$

in integers n_j with $N < n_j \leqslant 2N$. It is clear that (cf. Lemma 2.3 of [1])

$$(1.5) \qquad I_8(N, Y) \ll X^{-1}\mathscr{I}_8(N, X) \ll I_8(N, Y)$$

with $XY = 1$, therefore the two problems of estimating $\mathscr{I}_8$ and I_8 are equivalent.

In this paper we restrict ourselves to investigating sums (1.1) with

$$(1.6) \qquad f(t) = t^\varkappa$$

for $\varkappa \neq 0, 1, 2$ and

$$(1.7) \qquad f(t) = \log t\,,$$

in the latter case we put $\varkappa = 0$. Our main result is

THEOREM. *Let $N \geqslant 1$ and X, $Y > 0$. We then have*

$$(1.8) \qquad \mathscr{I}_8(N, X) \ll (X + N)N^{4+\varepsilon}$$

and

$$(1.9) \qquad I_8(N, Y) \ll (1 + NY)N^{4+\varepsilon}$$

for any $\varepsilon > 0$, the constant implied in $\ll$ depending at most on ε and $\varkappa$.

It is easily seen that the order of magnitude of the bounds (1.8) and (1.9) is the best possible.

REMARKS. The fact that $f(t)$ is a monomial is important for the argument applied in the proof of Lemma 5. A modification is possible to treat few other cases but it has not been worked out in detail.

The bound (1.9) for $\varkappa = 3/2$ is used in [1] to prove that

$$\zeta(\tfrac{1}{2} + it) \ll t^{9/56+\varepsilon}\,, \qquad t \geqslant 1\,.$$

The main idea of this work was inspired by the series of papers of Hardy and Littlewood on Gauss sums $S(\alpha, \beta, 0; N)$, cf. [2] for example. Although their beautiful argument concerning the diophantine nature of the leading coefficient α (continued fraction expansion) is irrelevant to our method, the philosophy is the same. The fascinating point is that the Poisson summation is used a number of times to gain very little at each time. Since the Poisson summation is an involution, it is necessary to alternate the iteration steps by another operator which makes the whole process progressive. This operator is $T(\alpha) = \{1/4\alpha\}$. What really happens is illustrated symbolically in the following asymptotic formula:

$$\sum_{n\sim N} e(\alpha n^2 + g(n)) \sim \alpha^{-1/2} \sum_{n\sim\alpha N} e\left(-T(\alpha)n^2 + \tilde{g}\left(\frac{n}{4\alpha}\right)\right),$$

where g is considered as a perturbation and $\tilde{g}$ is a certain transform which changes g only slightly.

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2. – Auxiliary results.

In this and the next section we assume that

$$X = N\,. \tag{2.2}$$

LEMMA 1. *Let* $N^{-1} < \Delta \leqslant 1$. *We have*

$$\frac{1}{\Delta N}\int_0^{\Delta}\int_0^1\int_{-N}^{N}|S(\alpha, \beta, x; N)|^8\, d\alpha\, d\beta\, dx \ll N^5(\log N)^2\,. \tag{2.3}$$

PROOF. By Lemma 2.3 of [1], or directly, the left-hand side of (2.3) is bounded by the number of solutions of the system

$$\begin{cases} \sum_1^4 (n_j - n_{j+4}) & \ll 1\,, \\ \sum_1^4 (n_j^2 - n_{j+4}^2) & \ll \Delta^{-1} \ll N\,, \\ \sum_1^4 \big(f(n_j/N) - f(n_{j+4}/N)\big) & \ll N^{-1}\,, \end{cases} \tag{2.4}$$

in integers n_j with $N < n_j \leqslant 2N$. If we replace n_j in (2.4) by a real number in the unit interval around n_j the system (2.4) remains unchanged except for new constants involved in the symbol $\ll$. This makes it plain that the number of solutions of (2.4) is bounded by the Lebesgue measure of the corresponding set (with new implied constants) defined by extending n's to real numbers in the interval $[N-1, 2N+1]$.

Now we can replace $S(\alpha, \beta, x; N)$ in (2.3) by the corresponding integral

$$S_0(\alpha, \beta, x; N) = \int_{N-1}^{2N+1} e(\alpha t^2 + \beta t + xf(t/N))\, dt ,$$

and evaluate instead the eighth power moment of S_0.

In general, an integral over an interval of an exponential function $e(\lambda(t))$ with $|\lambda^{(r)}(t)| > \lambda_r$ throughout this interval, is bounded by $O(\lambda_r^{-1/r})$ (for this result with $r = 1, 2$ see Lemmas 4.3 and 4.4 of [5]). In our case, since $\varkappa \neq 1$, from the above principle with $r = 3$ one gets

$$S_0(\alpha, \beta, x; N) \ll |x|^{-1/3} N .$$

Suppose now that $\alpha < c_1|x|N^{-2}$, where $c_1 = c_1(\varkappa)$ is sufficiently small; then the principle with $r = 2$ shows that

$$S_0(\alpha, \beta, x; N) \ll |x|^{-1/2} N ,$$

if instead $\alpha > c_2|x|N^{-2}$, where $c_2 = c_2(\varkappa)$ is sufficiently large, we obtain

$$S_0(\alpha, \beta, x; N) \ll \alpha^{-1/2} ;$$

finally $S_0(\alpha, \beta, x; N) \ll N$ is trivial. In any case, we find

$$S_0(\alpha, \beta, x; N) \ll (\alpha|x|)^{-1/6} N^{2/3} . \tag{2.5}$$

We also have

$$\int_{-1}^{1} |S_0(\alpha, \beta, x; N)|^2\, d\beta \ll N . \tag{2.6}$$

Combining (2.5) and (2.6) we conclude that the left-hand side of (2.3) is

$$\begin{aligned} &\ll \int_0^{N^{-1}} \int_{-1}^{1} \int_{-N}^{N} |S_0(\alpha, \beta, x; N)|^8\, d\alpha\, d\beta\, dx , \\ &\ll N^5 + N^5 \int_{N^{-4}}^{N^{-1}} \int_{N^{-4}}^{N} (\alpha x)^{-1}\, d\alpha\, dx \ll N^5 (\log N)^2 . \quad \blacksquare \end{aligned}$$

Similar arguments will prove

LEMMA 2. *We have*

$$\int_0^1 \int_{-N}^{N} |S(\alpha, \beta, x; N)|^4 \, d\beta \, dx \ll N^3 (\log N)^2 . \tag{2.7} \qquad \blacksquare$$

LEMMA 3. *For* $N^{-2/3} < \Delta < 1$ *we have*

$$\frac{1}{\Delta N} \int_{\Delta/2}^{\Delta} \int_0^1 \int_{-N}^{N} |S(\alpha, \beta, x; N)|^6 \, d\alpha \, d\beta \, dx \ll (N \log N)^3 .$$

We postpone the proof of Lemma 3 to the next section. Let $\Delta_0 = c(\varkappa) N^{-1}$, where $c(\varkappa)$ is a sufficiently large constant. From the upper bound

$$S(\alpha, \beta, x; N) \ll \alpha^{1/2} N \ll \Delta^{1/2} N ,$$

valid if $\alpha > \Delta_0$, see [5], Theorem 5.9, p. 90, and by Lemmas 1, 2, 3 one infers

COROLLARY. *We have*

$$\int_0^{N^{-1/2}} \int_0^1 \int_{-N}^{N} |S(\alpha, \beta, x; N)|^8 \, d\alpha \, d\beta \, dx \ll N^5 (\log N)^4 . \tag{2.8} \qquad \blacksquare$$

3. – An application of Poisson's formula.

In the particular range $N^{-2/3} < \alpha \leqslant 1$ we transform $S(\alpha, \beta, x; N)$ into a similar but shorter sum by means of Poisson summation. As in Theorem 4.9 of [5], we obtain

$$\sum_{N < n \leqslant 2N} e(g(n)) = \exp\left[\frac{\pi i}{4}\right] \sum_{g'(N) < m < g'(2N)} |g''(t_m)|^{-1/2} e(g(t_m) - m t_m) + E(N)$$

with $g(t) = \alpha t^2 + \beta t + x f(t/N)$ in mind, where t_m is the solution of $g'(t_m) = m$ and $E(N)$ is considered as an error term. Theorem 4.9 of [5] gives

$$E(N) \ll \lambda_2^{-1/2} + \log(2 + \lambda_2 N) + (\lambda_2 \lambda_3)^{1/5} N ,$$

while a slight modification of the argument leads to

$$E(N) \ll \lambda_2^{-1/2} + \log(1 + \lambda_2 N) + \lambda_3^{1/3} N.$$

In our case the latter says

$$E(N) \ll \alpha^{-1/2} + \log N + N^{1/3} \ll N^{1/3},$$

which is far better than we need.

Perhaps it is worthwhile at this point to emphasize that the problem of evaluating such error terms is best approached throughout the use of general asymptotic expansions, which arise by a higher order stationary phase technique. In other words, these remainders can be approximated by oscillatory sums of the same general nature as the main term, to which one may apply more sophisticated techniques in case far better estimates are needed. We draw the attention of the number theoretists to the paper [4] by L. Hörmander which proves the existence of asymptotic expansions under very general conditions. For the one dimensional case one should mention that the old result of I. M. Vinogradov [7] is amply sufficient for most applications, see also [3].

Notice that $g'(N) = 2\alpha N + O(1)$ and $g'(2N) = 4\alpha N + O(1)$, thus we can change the range of summation into

$$2\alpha N < m < 4\alpha N \tag{3.1}$$

at no cost. Next we compute

$$g''(t) = 2\alpha + O(N^{-1}),$$

thus we can replace the factor $|g''(t_m)|^{-1/2}$ by $(2\alpha)^{-1/2}$ at no cost. It remains to evaluate $g(t_m) - mt_m$. This requires a greater precision. We abbreviate

$$g_m(t) = g(t) - mt, \qquad F(t) = f(t/N), \qquad F_j = F^{(j)}_{(m/2\alpha)},$$

thus

$$g'_m(t_m) = 2\alpha t_m + (\beta - m) + xF'(t_m) = 0,$$

$$t_m = \frac{m-\beta}{2\alpha} - \frac{x}{2\alpha} F'(t_m) = \frac{m-\beta}{2\alpha} + O\left(\frac{1}{\alpha}\right).$$

and

$$g_m(t_m) = -\frac{(m-\beta)^2}{4\alpha} + xF(t_m) + \frac{1}{4\alpha}(xF'(t_m))^2.$$

We shall evaluate $F(t_m)$ and $F'(t_m)$ with a tolerable precision by a method of successive approximations. For notational simplicity we denote by $h(m)$ a function (nor necessarily the same in each occurrence) such that

$$m^j h^{(j)}(m) \ll 1\,, \quad j = 0, 1, 2\,, \tag{3.2}$$

the constant implied in $\ll$ being absolute.

By Taylor's expansion we obtain

$$F\left(\frac{m-\beta}{2\alpha}\right) = F_0 - \frac{\beta}{2\alpha}F_1 + \frac{\beta^2}{8\alpha^2}F_2 + \alpha^{-3}N^{-3}h(m)$$

and

$$\begin{aligned} F\left(\frac{m-\beta}{2\alpha}\right) &= F\left(t_m + \frac{x}{2\alpha}F'(t_m)\right) \\ &= F(t_m) + \frac{x}{2\alpha}(F'(t_m))^2\left(1 + \frac{x}{4\alpha}F_2\right) + \alpha^{-3}N^{-3}h(m)\,. \end{aligned}$$

Hence we express $F(t_m)$ in terms of $F_0, F_1, F_2, F'(t_m)$ and get

$$\begin{aligned} g_m(t_m) = &- \frac{(m-\beta)^2}{4\alpha} + xF_0 - \frac{\beta x}{2\alpha}F_1 + \frac{\beta^2 x}{8\alpha^2}F_2 \\ &- \frac{1}{4\alpha}(xF'(t_m))^2\left(1 - \frac{x}{2\alpha}F_2\right) + \alpha^{-3}N^{-2}h(m)\,. \end{aligned}$$

Seeking for $F'(t_m)$ we proceed as follows

$$\begin{aligned} F'(t_m) &= F'\left(\frac{m-\beta}{2\alpha} - \frac{x}{2\alpha}F'(t_m)\right) \\ &= F'\left(\frac{m-\beta}{2\alpha}\right) - \frac{x}{2\alpha}F'(t_m)F''\left(\frac{m-\beta}{2\alpha}\right) + \alpha^{-2}N^{-3}h(m) \\ &= F_1 - \frac{\beta}{2\alpha}F_2 - \frac{x}{2\alpha}F'(t_m)F_2 + \alpha^{-2}N^{-3}h(m)\,, \end{aligned}$$

whence, using the fact that $(x/2\alpha)F_2$ is small, we find

$$\begin{aligned} F'(t_m) &= \left(1 + \frac{x}{2\alpha}F_2\right)^{-1}\left(F_1 - \frac{\beta}{2\alpha}F_2\right) + \alpha^{-2}N^{-3}h(m) \\ &= F_1 - \frac{\beta}{2\alpha}F_2 - \frac{x}{2\alpha}F_1F_2 + \alpha^{-2}N^{-3}h(m)\,. \end{aligned}$$

This yields up to a remainder of type $\alpha^{-3}N^{-2}h(m)$:

$$g_m(t_m) = -\frac{(m-\beta)^2}{4\alpha} + \frac{x}{2\alpha}\left(2\alpha F_0 - \beta F_1 + \frac{\beta^2}{4\alpha}F_2\right)$$
$$-\left(\frac{x}{2\alpha}\right)^2(\alpha F_1 - \beta F_2)F_1 + \frac{3}{8}\frac{x^3}{\alpha^2}F_1^2F_2 + \alpha^{-3}N^{-2}h(m)\,.$$

We set $\alpha_1 = (4\alpha)^{-1}$, $\beta_1 = (2\alpha)^{-1}\beta$, $x_1 = (2\alpha)^{-\varkappa}x$ and use $F_j = (2\alpha)^{j-\varkappa}F^{(j)}(m)$ to find our final approximation

$$\text{(3.3)}\quad g_m(t_m) = -\alpha_1 m^2 + \beta_1 m + x_1 F(m) - p(\alpha_1, \beta_1, x_1; m) + q(\alpha_1, \beta_1, x_1; m) + \alpha^{-3}N^{-2}h(m)$$

where

$$p(\alpha_1, \beta_1, x_1; m) = \frac{1}{4\alpha_1}[x_1F'(m) + \beta_1]^2$$

and

$$q(\alpha_1, \beta_1, x_1; m) = \frac{x_1F'''(m)}{8\alpha_1^2}[4\beta_1^2 + 2\beta_1 x_1 F'(m) + 3(x_1F'(m))^2]\,.$$

Here the variation of p and q when α_1 and β_1 change by less than 1 is a function $h(m)$ of type (3.2), therefore

$$g_m(t_m) = -\alpha_1 m^2 + \beta_1 m + x_1 F(m) - p(a, b, x_1; m) + q(a, b, x_1; m) + h(m)$$

for any a, b with $\alpha_1 < a \leqslant \alpha_1 + 1$ and $\beta_1 < b \leqslant \beta_1 + 1$. Fix Δ with $N^{-2/3} < \Delta < 1$ and let α range over $[\Delta/2, \Delta]$. Define $M = \Delta N$, so by (3.1)

$$\text{(3.4)}\qquad M < m < 4M\,.$$

By Lemmas 2.1 and 2.2 of [1] we can remove the factor $e(h(m))$ in $e(g_m(t_m))$ and change the range of summation (3.1) into (3.4) getting

$$\text{(3.5)}\quad S(\alpha, \beta, x; N) \ll N^{1/2} + \Delta^{-1/2}\int_0^1 \min\{N, \xi^{-1}\}$$
$$\Big|\sum_{M<m<4M} e(-\alpha_1 m^2 + (\beta_1 + \xi)m + x_1F(m) - p(a, b, x_1; m) + q(a, b, x_1; m))\Big|\,d\xi\,.$$

for any a, b with $\alpha_1 < a \leqslant \alpha_1 + 1$ and $\beta_1 < b \leqslant \beta_1 + 1$.

PROOF OF LEMMA 3. It easily follows from (3.5) by Hölder's inequality and an application of the general mean-value theorem

$$\int_0^1\int_0^1\Big|\sum_{M<m\leqslant 4M} c_m e(\gamma m^2+\delta m)\Big|^6 d\gamma\, d\delta \ll M^3(\log 2M)^2$$

valid for any complex numbers c_m with $|c_m|\leqslant 1$.

After having completed the proof of Lemma 3 and, with it, the proof of its Corollary, we may now suppose that $\Delta\geqslant N^{-1/2}$. This simplifies substantially (3.5), as

$$S(\alpha,\beta,x;N)\ll N^{1/2}+\Delta^{-1/2}\int_0^1 \min\{N,\xi^{-1}\}$$
$$\Big|\sum_{M<m<4M} e(-\alpha_1 m^2+(\beta_1+\xi)m+x_1F(m)-p(a,b,x_1;m))\Big|\,d\xi$$

because $q(\alpha_1,\beta_1,x_1;m)$ is a function $h(m)$ of type (3.2). By Hölder's inequality

$$|S(\alpha,\beta,x;N)|^8\ll N^4+(\log N)^7\Delta^{-4}\int_0^1\min\{N,\xi^{-1}\}\Big|\sum_m\Big|^8 d\xi\,.$$

For notational simplicity define

$$R(\Delta,N)=\int_{\Delta/2}^{\Delta}\int_0^1\int_{-N}^{N}|S(\alpha,\beta,x;N)|^8\,d\alpha\,d\beta\,dx\,.$$

We integrate the last inequality over a in $[\alpha_1,\alpha_1+1]$, over b in $[\beta_1,\beta_1+1]$, over α in $[\Delta/2,\Delta]$, over β in $[0,1]$ and over x in $[-N,N]$ and get

$$R(\Delta,N)\ll N^5+(\log N)^8\Delta^{-1}\int_0^{\Delta^{-1}}\int_{(4\Delta)^{-1}}^{\Delta^{-1}}\int_0^1\int_0^1\int_{-N}^{N}$$
$$\cdot\Big|\sum_{M<m<2M} e\Big(\gamma m^2+\delta m+xf\Big(\frac{m}{M}\Big)-\frac{1}{a}\Big(\frac{2x}{M}f'\Big(\frac{m}{M}\Big)\Big)^2-\frac{2bx}{aM}f'\Big(\frac{m}{M}\Big)\Big)\Big|^8 db\,da\,d\delta\,d\gamma\,dx$$

where $M=\Delta N$ or $M=2\,\Delta N$. Now we make the change of variables

$a = x^2u^{-1}$ and $b = |x|u^{-1}v$ and we find

$$R(\Delta, N) \ll N^5 + (\log N)^8 \Delta^{-4} \int_0^N x^{-3} \int_0^1 \int_0^1 \int_0^{4\Delta x^2} \int_0^{4x}$$

$$\cdot \left| \sum_{M<m<2M} e\left(\gamma m^2 + \delta m + xf\left(\frac{m}{M}\right) - u\left(\frac{2}{M}f'\left(\frac{m}{M}\right)\right)^2 - v\,\frac{2}{M}f'\left(\frac{m}{M}\right)\right)\right|^8 dx\,d\gamma\,d\delta\,du dv.$$

Next, we divide the range of integration in x into $\ll \log N$ subintervals of type $(X/2, X)$ together with a last interval $(0, N^{-4})$, and we obtain

$$R(\Delta, N) \ll N^5 + (\log N)^9 \Delta^{-4} X^{-3} \int_{-X}^{-X} \int_0^1 \int_0^1 \int_{-4\Delta X^2}^{-4\Delta X^2} \int_{-4X}^{-4X} \left|\sum_m\right|^8 dx\,d\gamma\,d\delta\,du\,dv$$

for some X with $N^{-4} < X \leqslant N$.

The expression we have obtained for $R(\Delta, N)$ bears some striking resemblance to the original integral $\mathscr{I}_8(N, X)$, except for the two additional terms $u((2/M)f'(m/M))^2$ and $v(2/M)f'(m/M)$ and corresponding integration over u and v. The main point, as in the Hardy and Littlewood argument, is that the range of summation is over $m \sim \Delta N$, which is somewhat smaller than the original range. The next lemma shows how to remove the terms involving u, v and prepares the way for the final inductive argument.

LEMMA 4. *Let $X_1, \ldots, X_r > 0$ and $c(y_1, \ldots, y_r)$ be complex numbers bounded by $d(y_1, \ldots, y_r)$ in absolute value. We then have*

$$\int_{-X_1}^{X_1} \cdots \int_{-X_r}^{X_r} \left|\sum_{\mathfrak{y}} c(\mathfrak{y})\, e(\mathfrak{x}\cdot\mathfrak{y})\right|^2 dx_1 \ldots dx_r \leqslant 2^r \int_{-2X_1}^{2X_1} \cdots \int_{-2X_r}^{2X_r} \left|\sum_{\mathfrak{y}} d(\mathfrak{y})\, e(\mathfrak{x}\cdot\mathfrak{y})\right|^2 dx_1 \ldots dx_r\,.$$

PROOF. It follows from

$$\int_{-1}^{1} (1 - |x|)\, e(xy)\, dx = \left(\frac{\sin \pi y}{\pi y}\right)^2 \geqslant 0\,. \quad \blacksquare \tag{3.5}$$

By Lemma 4 we finally conclude a kind of recurrence formula

LEMMA 5. *If $N^{-1/2} < \Delta \leqslant 1/4$ we have*

$$R(\Delta, N) \ll N^5 + (\log N)^9 \Delta^{-3} \mathscr{I}_8(\Delta N, N)\,. \quad \blacksquare$$

4. – Proof of Theorem.

Lemma 5, with $\Delta = N^{-\varepsilon}$, can be used in an inductive argument to estimate $\mathscr{I}_8(N, N)$ in terms of $\mathscr{I}_8(N', N')$ with $N' \ll N^{1-\varepsilon}$. Repeating this procedure $[1/\varepsilon]$ times, and estimating the final integral trivially, we obtain our theorem by letting $\varepsilon \to 0$. In a sense, we may say that the proof of our theorem requires the application of Poisson's summation formula an arbitrarily large number of times.

In practice, we present this argument as follows. Let η_0 be the infimum of the set of those real numbers $\eta > 4$ for which

$$\mathscr{I}_8(N, X) \ll (X + N)N^{\eta} \tag{4.1}$$

for all $N \geqslant 1$ and $X > 0$, the constant implied in $\ll$ depending at most on η and $\varkappa$. We wish to show that $\eta_0 = 4$.

We have

$$\mathscr{I}_8(N, X) \leqslant \mathscr{I}_8(N, X_1) \ll \frac{X_1}{\Delta X_0} \int_0^1 \int_0^{\Delta} \int_{-X_0}^{X_0} |S(\alpha, \beta, x;\, N)|^8 \, d\alpha \, d\beta \, dx$$

provided $0 < \Delta < 1$, $0 < X \leqslant X_1$ and $0 < X_0 \leqslant X_1$: Hence taking $X_0 = N$ and $X_1 = X + X_0$ we get

$$\mathscr{I}_8(N, X) \ll (X + N)(\Delta N)^{-1} \int_0^1 \int_0^{\Delta} \int_{-N}^{N} |S(\alpha, \beta, x;\, N)|^8 \, d\beta \, d\alpha \, dx\,. \tag{4.2}$$

By (4.1) and by Lemma 5 we have

$$N^{-1} \int_0^1 \int_{\frac{1}{2}\Delta_1}^{\Delta_1} \int_{-N}^{N} |S(\alpha, \beta, x;\, N)|^8 \, d\beta \, d\alpha \, dx \ll \Delta_1^{\eta-3} N^{\eta} \tag{4.3}$$

for any $N^{-1/2} < \Delta_1 < 1/4$. By (4.2), (4.3) and (2.8) we conclude that

$$\mathscr{I}_8(N, X) \ll (\Delta^{-1} N^4 + \Delta^{\eta-4} N^{\eta})(X + N)(\log N)^{10}$$

for any $\eta > \eta_0$ and Δ with $N^{-1/2} < \Delta < 1/4$, the constant implied in $\ll$ depending on $\varkappa$ and η alone. We choose $\Delta = N^{-(\eta-4)/(\eta-3)}$ and we find

$$4 \leqslant \eta_0 < \frac{5\eta - 16}{\eta - 3}$$

for any $\eta > \eta_0$, which ends the proof. ■

5. – Remarks.

Our method can be applied to estimate similar integrals of type

$$\mathscr{I}_6(N, X) = \int_{-X}^{X}\int_{0}^{1}\Big|\sum_{1}^{N} e(\alpha n^2 + xf(n))\Big|^6 dx\, d\alpha . \tag{5.1}$$

In the special case of $f(n) = \log n$ we can show that

$$\int_{0}^{T}\int_{0}^{1}\Big|\sum_{1}^{N} e(\alpha n^2) n^{it}\Big|^6 dt\, d\alpha \ll (T+N)N^{3+\varepsilon} . \tag{5.2}$$

Perhaps, it is interesting to mention that (5.2) contains our theorem for $\varkappa = 3$. In other words, (5.2) implies

$$P\int_{0}^{P^{-1}}\int_{0}^{1}\int_{0}^{1}\Big|\sum_{1}^{N} e(\alpha n^2 + \beta n + \gamma n^3)\Big|^8 d\gamma\, d\beta\, d\alpha \ll (N^2 + P)N^{2+\varepsilon} , \tag{5.3}$$

or, what is equivalent, that the number of solutions of the system

$$\sum_{1}^{4} (n_j - n_{j+4}) = 0 , \tag{5.4}$$

$$\sum_{1}^{4} (n_j^2 - n_{j+4}^2) = 0 , \tag{5.5}$$

$$\Big|\sum_{1}^{4} (n_j^3 - n_{j+4}^3)\Big| \leqslant P \tag{5.6}$$

in integers n_j with $|n_j| \leqslant N$ is

$$\nu(N, P) \ll (N^2 + P)N^{2+\varepsilon}. \tag{5.7}$$

In order to see this consider the Hermite matrices

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad H_8 = H_2 \otimes H_2 \otimes H_2 \otimes H_2 .$$

They are non-singular orthogonal. We have

$$\det H_8 = (\det H_2)^4 = 2^4 .$$

By the linear change of variables

$$\mathfrak{m} = H_8 \mathfrak{n} , \tag{5.8}$$

the system (5.4)-(5.6) reduces to one equation

$$\sum_1^3 (m_j^2 - m_{j+3}^2) = 0 \tag{5.9}$$

and one inequality

$$|m_1 m_2 m_3 - m_4 m_5 m_6| \leqslant Z \tag{5.10}$$

in integers m_j with $|m_j| \leqslant 2^8 N$ where $Z = 8P$. Of two remaining variables m_7, m_8 one is determined uniquely as a linear combination of $m_1, \ldots, m_6$ and the other ranges freely over the interval $[-2^8 N, 2^8 N]$. Therefore, letting $I_6(N, Z)$ be the number of solutions of (5.9)-(5.10) in integers m_j with $|m_j| \leqslant N$ we have shown that

$$\nu(N, P) \ll N I_6(2^8 N, 8P) . \tag{5.11}$$

On the other hand (5.2) implies, by Hölder's inequality,

$$I_6(N, Z) \ll (N^2 + Z) N^{1+\varepsilon} . \tag{5.12}$$

By (5.11) with (5.12) one gets (5.7) completing the proof. ∎

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