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Boundary Behaviour of Eigenfunctions of the Laplace Operator on Trees.

ADAM KORÁNYI (*) - MASSIMO A. PICARDELLO (**)

1. – Introduction.

Let T be any tree, and let s, t be vertices of T. If s and t are neighbours, the ordered pair (s, t) is called an (*oriented*) *edge* issuing from s. A nearest-neighbour transition matrix is determined by assigning a positive number $p(s, t)$ to each edge (s, t). This matrix gives rise to a transition operator P on functions F on the vertices of T:

$$PF(s) = \sum p(s, t) F(t)$$

where the sum is taken over all edges issuing from s. The Laplace operator associated with P is defined by

$$\Delta F = PF - F\,.$$

The Laplace operator has been studied in detail in the case where the tree is homogeneous of degree $q+1$, and the transition matrix gives rise to a (nearest-neighbour) *isotropic* random walk: $p(s, t) = 1/(q+1)$ for every edge (s, t) [5, 2]. In this case, eigenfunctions of the Laplace operator (or, equivalently, of the transition operator P) can be represented as Poisson transforms of generalized functions on the «boundary» Ω of the tree [5], that is, its set of «ends». On a symmetric space, where a similar representation holds, the boundary behaviour of eigenfuctions of the Laplace-Beltrami operator can be described explicitly [6]. Indeed, let F be the Poisson

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transform of an integrable function defined on the Poisson boundary of a symmetric space. Then, if F is appropriately normalized, its asymptotic values along certain sets, introduced in [3] and called « admissible domains », coincide almost everywhere with the values of f. In the case when the symmetric space is the hyperbolic disc and the function F is harmonic, this result coincides with the classical Fatou theorem on nontangential convergence.

The purpose of this paper is to introduce the analogues of admissible domains for trees, and to use them to describe the boundary behaviour of eigenfunctions of Δ. Our results bear a close resemblance to those of [6] for symmetric spaces.

In section 2 we restrict attention to the homogeneous tree T_q with branching degree $q+1$, and the isotropic transition matrix. Fix a reference vertex o in T_q, and consider the random walk, starting at o, induced by the transition matrix. For each complex number γ, there exists, up to normalization, only one γ-eigenfunction of the transition operator which is radial, i.e., which depends only on the distance from o: these functions are called « spherical functions ». If μ is a complex measure on the boundary Ω of T_q, denote by μ_r the regular part of μ with respect to the Poisson measure ν on Ω. For each complex number z, consider the associated Poisson transform F_z of μ, and the corresponding spherical function φ_z. Let ϱ be the spectral radius of the transition operator P on $l^2(T_q)$. Then the main result of section 2 states that, for every complex eigenvalue of P outside the interval $(-\varrho, \varrho)$, the function F_z/φ_z converges asymptotically almost everywhere, along admissible sets, to the Radon-Nikodym derivative $d\mu_r/d\nu$.

Furthermore, for the eigenvalues $\pm\varrho$, we show that convergence to boundary values holds in a stronger sense than the usual admissible convergence. This is the analogue of a recent result of Sjögren for the hyperbolic disc [7].

The results of section 2 follow from sufficiently precise estimates for the Poisson kernel and the spherical functions, along the lines of [2]. The crucial part of the argument shows that the normalized Poisson transform of an integrable function f on the boundary Ω is bounded on an admissible domain by the Hardy-Littlewood maximal function at the vertex.

On the other hand, for non-homogeneous trees, sharp estimates of this type are not available. Moreover, a Poisson representation theorem is known only for not too small real eigenvalues and for *positive* eigenfunctions of the Laplace operator [1, thm. 2.1]. For these eigenfunctions a generalization of Fatou's convergence theorem was proved by Cartier [1]. Cartier's theorem concerns « radial » convergence to the boundary Ω, that is, convergence along geodesics in the tree. In section 3 we slightly improve this result to give admissible convergence instead of radial convergence. Our argument is independent from Cartier's, and is based upon the general Fatou-Naïm-Doob theorem (that is, the theory of fine convergence: see, for instance, [9]). A radial Fatou theorem for harmonic functions with respect to the isotropic Laplace operator on a homogeneous tree had been previously obtained in [8], in the framework of local fields.

2. – The isotropic Laplace operator on a homogeneous tree.

In this section, we restrict attention to the homogeneous tree T_q with branching degree $q+1$, $q\geq 2$. Most of the preliminaries are as in [2], and notations will be, as much as possible, the same as in that reference with only minor modifications. Most of these notations also make sense (and will be tacitly adopted) for non-homogeneous trees. In particular, Ω denotes the boundary of the tree, that is, the set of infinite geodesics, starting at the reference vertex o, and $N(\omega, \omega')$ denotes the number of edges in common between $\omega, \omega' \in \Omega$. We can identify each vertex x of T_q with the finite simple path connecting o with x; then $N(x, \omega)$ denotes the number of edges in common between the finite geodesic x and the infinite geodesic ω. The distance $d(o, x)$ is denoted by $|x|$, and the vertex with length $n \leq |x|$ in the geodesic connecting o and x is denoted by x_n. Similarly, ω_n denotes the vertex with length n in the geodesic $\omega \in \Omega$. Finally, we endow Ω with a compact topology as follows.

As in [2], we introduce the sets $E_x = \{\omega\colon N(x, \omega) = |x|\}$. Then an open basis at $\omega \in \Omega$ consists of all the sets E_{ω_n}, $n \in \mathbb{N}$. The Poisson measure ν on Ω, that is, the hitting distribution on Ω of the random walk, starting at o, defined by the transition operator, is given by

$$\nu(E_x) = (q-1)^{-1} q^{1-n} \quad \text{if } |x| = n\,,$$

and the corresponding *Poisson kernel* is

$$K(x, \omega) = q^{2N(x,\omega)-|x|}$$

[2]. In particular

$$K(x, \omega) = q^{2j-n} \quad \text{if } \omega \in E_{x_j} - E_{x_{j+1}} \text{ and } |x| = n$$

(here $E_{x_0} = \Omega$, $E_{x_{n+1}} = \emptyset$).

We are now ready to define admissible sets in the tree (this definition is also valid for non-homogeneous trees). Admissible sets in a tree are the analogue of euclidean cones (that is, hyperbolic cylinders) in the upper half-spaces, with vertex at the boundary.

DEFINITION. For every integer $\alpha \geqslant 0$, and $\omega \in \Omega$

$$\Gamma_\alpha(\omega) = \{x\colon d(x, \omega_m) \leqslant \alpha \text{ for some } m \geqslant 0\}.$$

We say that a function F on the tree *converges admissibly* to l at ω if, for every $\alpha \geqslant 0$, $\lim_{|x|\to\infty,\, x\in\Gamma_\alpha(\omega)} F(x) = l$.

REMARK. The notion of admissible convergence is independent of the choice of reference vertex o. Indeed, denote by $\Gamma'_\alpha(\omega)$ the admissible domains with respect to a different reference vertex o'. Then, since there is a unique simple path connecting o and o', it follows that, if $|x|$ is large enough, $x \in \Gamma_\alpha(\omega)$ if and only if $x \in \Gamma'_\alpha(\omega)$. $\square$

Given two complex-valued functions f, g we write $f \approx g$ if $|f/g|$ is bounded above and below.

LEMMA 1. *Let $\omega^0 \in \Omega$ and $x \in \Gamma_\alpha(\omega^0)$ for some $\alpha \geqslant 0$, $|x| = n$. Then $K(x, \omega) \approx K(\omega_n^0, \omega)$, with bounds depending only on α. In particular*

$$K(x, \omega) \approx q^{2j-n} \quad \textit{for } \omega \in E_{\omega_j^0} - E_{\omega_{j+1}},\ 0 \leqslant j \leqslant n\,.$$

PROOF. It is enough to show that $|N(x, \omega) - N(\omega_n^0, \omega)| \leqslant \alpha$. For this, let $k = N(x, \omega^0) \geqslant n - \alpha$. Then, if $\omega_k = x_k$, both $N(x, \omega)$ and $N(\omega_n^0, \omega)$ are not less than k, and not larger than n, and the inequality follows. On the other hand, if $\omega_k \neq x_k$, then $N(x, \omega) = N(\omega_n^0, \omega)$. $\square$

It has been proved in [5] that all eigenfunctions of P can be represented as *Poisson transforms*, that is, integrals of the form $\mathcal{K}_z f = \int_\Omega K(x, \omega)^z f(\omega)\, d\nu(\omega)$, where f is a finitely additive measure (that is, a martingale) on Ω, and $z \in \mathbf{C}$. The corresponding eigenvalue is

$$\gamma(z) = (q+1)^{-1}(q^z + q^{1-z})\,. \tag{1}$$

By the symmetry properties of (1), we can restrict attention to complex numbers z such that $\operatorname{Re} z > \frac{1}{2}$ and $-\pi/\log q < \operatorname{Im} z \leqslant \pi/\log q$, or $\operatorname{Re} z = \frac{1}{2}$ and $0 \leqslant \operatorname{Im} z \leqslant \pi/\log q$. The latter set of parameters corresponds to eigenvalues $\gamma(z)$ in the l^2-spectrum of P.

Explicit expressions are known for the Poisson transforms of the constant function 1 on Ω. Indeed, there exists a constant a_z, depending on z, such that if $|x| = n$, the «spherical function» $\varphi_z = \mathcal{K}_z 1$ has the properties [2, ch. 3]:

$$(2)\quad \begin{cases} \varphi_z(x) \approx a_z q^{(z-1)n} & \text{if } \operatorname{Re} z > \frac{1}{2}, \\ \varphi_z(x) = \left(1 + \dfrac{q-1}{q+1} n\right) q^{-zn} & \text{if } z = \frac{1}{2} + im\pi/\log q,\ m = 0, 1, \\ \varphi_z(x) = \operatorname{Re}(c_z q^{-zn}) & \text{if } \operatorname{Re} z = \frac{1}{2},\ 0 < \operatorname{Im} z < \pi/\log q. \end{cases}$$

We are interested in the asymptotic behaviour of Poisson transforms normalized by the corresponding spherical functions. However, (2) shows that, for $z = \frac{1}{2} + it$, $0 < t < \pi/\log q$, the spherical function φ_z oscillates, and therefore dividing by it cannot be expected to lead to results about asymptotic convergence. For the other values of z we introduce the normalized Poisson kernel $K_z(x, \omega) = K^z(x, \omega)/\varphi_z(x)$.

Lemma 1 and the estimates (2) yield

LEMMA 2. *Let* $x \in \Gamma_\alpha(\omega^0)$, $n = |x|$. *Then, for* $\omega \in E_{\omega^0_j} - E_{\omega^0_{j+1}}$, $0 \leqslant j \leqslant n$,

$$K_z(x, \omega) \approx q^{n+2(j-n)z} \quad \text{if } \operatorname{Re} z > \tfrac{1}{2}$$

$$K_z(x, \omega) \approx q^j/(1+n) \quad \text{if } z = \tfrac{1}{2} + ik\pi/\log q,\ k = 0, 1$$

with bounds depending only on z *and* α.

Denote by $\mathcal{K}$ the group of all isometries of T_q which fix o.

COROLLARY 1. *Assume* $\operatorname{Re} z > \frac{1}{2}$ *or* $z = \frac{1}{2} + im\pi/\log q$, $m = 0, 1$. *Then*

i) *for every* k *in* $\mathcal{K}$ *and for every* x, ω,

$$K_z(x, \omega) = K_z(kx, k\omega);$$

ii) *for every* x,

$$\int_\Omega K_z(x, \omega)\, d\nu(\omega) = 1;$$

iii) *there exists a constant* M_z *such that, for every* x,

$$\int_\Omega |K_z(x, \omega)|\, d\nu(\omega) \leqslant M_z;$$

iv) *for every integer* j *and for every* ω^0,

$$\lim_n \sup_{\omega\in\Omega-E_{\omega_j^0}} |K_z(\omega_n^0, \omega)| = 0 .$$

PROOF. (i) is obvious, because $N(x, \omega)$ is invariant under $\mathcal{K}$, and (ii) is nothing else but the definition of φ_z. (iii) follows from Lemma 2 and the fact that $\nu(E_y) \approx q^{-|y|}$ with bounds independent of y. Finally, (iv) is an immediate consequence of Lemma 2. □

By standard arguments (see [10, chapter 17, thms. 1.20, 1.23]) the estimates in corollary 1 yield:

PROPOSITION 1. *Assume* $\operatorname{Re} z > \frac{1}{2}$ *or* $z = \frac{1}{2} + im\pi/\log q$, $m = 0, 1$. *Let* f *be defined on* Ω, *and denote by* F *its normalized Poisson transform*:

$$F(x) = \int_\Omega K_z(x, \omega) f(\omega)\, d\nu(\omega) = (\mathcal{K}_z f)(x)/\varphi_z(x) .$$

Then $\lim F(\omega_n) = f(\omega)$

i) *uniformly if* f *is continuous*;

ii) *in* $L^p(\Omega)$ *if* $f \in L^p(\Omega)$ $(1 \leqslant p < \infty)$;

iii) *in the weak*-topology of* $L^\infty(\Omega)$ *if* $f \in L^\infty(\Omega)$;

iv) *in the weak*-topology of* $M(\Omega)$ *if* f *is a regular signed measure.*

For the main result of this section, we need a final tool: the Hardy-Littlewood maximal theorem. If f is integrable function on Ω, its ***Hardy-Littlewood maximal function*** is defined as

$$Mf(\omega) = \sup_n \frac{1}{\nu(E_{\omega_n})} \int_{E_{\omega_n}} |f| .$$

The operator $f \to Mf$ is called the *maximal operator*.

LEMMA 3. *The maximal operator is weak type* $(1, 1)$ *and strong type* (p, p) *for* $1 < p \leqslant \infty$.

PROOF. Denote by $\mathcal{M}$ the stabilizer of ω^0 in the group $\mathcal{K}$ of all isometries of T_q fixing o. Then we introduce a «gauge» on Ω by the rule $|\omega| = q^{-N(\omega, \omega^0)}$. It is readily seen that this is a gauge for $(\mathcal{K}, \mathcal{M})$ in the sense of [4, definition 1.1]. Indeed, we only need to check condition (iii) of [4, definition 1.1]: this follows by observing that, if ω^1, ω^2 satisfy $N(\omega^0, \omega^1) < N(\omega^0, \omega^2)$, then $N(\omega^0, \omega^1) = N(\omega^1, \omega^2)$. Then the lemma follows from [4, Corollary, pg. 580]. □

We can now prove the nontangential convergence theorem.

THEOREM 1. *Let z be as in Proposition 1, and let μ be a measure on Ω. Denote by $F(x) = \int K_z(x, \omega)\, d\mu(\omega) = \mathcal{K}_z\mu/\varphi_z$ the normalized Poisson transform of μ, and by μ_r the regular part of μ with respect to the Poisson measure ν. Then F has admissible limits a.e. on Ω, equal to $d\mu_r/d\nu(\omega)$.*

PROOF. Standard methods [10, chapter 17] reduce the proof to the case of absolutely continuous μ, that is, to Poisson transforms of L^1-functions. Since every L^1-function can be approximated in L^1 by continuous functions, by Proposition 1.i. we only need to show that $\nu\{\omega\colon \sup |F(x)| > \lambda$ for $x \in \Gamma_\alpha(\omega)\} < \|f\|_1/\lambda$.

Because of Lemma 3, this follows if we prove that, for f in $L^1(\Omega)$,

$$\sup_{x \in \Gamma_\alpha(\Omega)} |F(x)| < C(\alpha, z) Mf(\omega)\,. \tag{2}$$

We distinguish two cases.

Case 1. $\operatorname{Re} z > \frac{1}{2}$. For x in $\Gamma_\alpha(\omega^0)$, $|x| = n$, Lemmas 1 and 2 yield:

$$|F(x)| \leqslant \sum_{j=0}^{n-1} \int\limits_{E_{\omega_j^0} - E_{\omega_{j+1}^0}} |K_z(x, \omega) f(\omega)|\, d\nu(\omega) + \int\limits_{E_{\omega_n^0}} |K_z(x, \omega) f(\omega) f(\omega)|\, d\nu$$
$$\leqslant C \sum_{j=0}^{n} q^{(n-j)(1-2\operatorname{Re} z)} Mf(\omega_0) < (C/1 - q^{1-2\operatorname{Re} z}) Mf(\omega_0)$$

We have used the estimate $\nu(E_v) \approx q^{-|v|}$.

Case 2. $z = \frac{1}{2} + ik\pi/\log q$, $k = 0, 1$. Again by Lemma 2, as above,

$$|F(x)| \leqslant C \sum_{j=0}^{n} q^j (1+n)^{-1} \int\limits_{E_{\omega_j^0}} |f(\omega)|\, d\nu(\omega) \leqslant C \sum_{j=0}^{n} Mf(\omega_0)/1 + n = CMf(\omega^0)\,.$$

In both cases, (2) is proved, and the theorem follows. □

We recall that the case $z = \frac{1}{2} + ik\pi/\log q$ corresponds to the two eigenvalues $\pm\varrho$, where ϱ is the spectral radius of P on $l^2(T_q)$. For these critical eigenvalues, we can prove a stronger convergence theorem, along the lines of [7]. Let us consider *enlarged admissible domains*

$$\Gamma_\alpha^+(\omega^0) = \{x\colon d(x, \omega_m^0) \leqslant \alpha + \log_q m \text{ for some } m\}$$

and the corresponding « strengthened admissible convergence ».

THEOREM 2. *If* $z = \frac{1}{2} + ik\pi/\log q$, $k = 0, 1$, *then, with notations as in Theorem* 1, F *converges a.e. to* $d\mu_r/d\nu$ *in the strengthened admissible sense.*

PROOF. Let $x \in \Gamma_\alpha^+(\omega^0)$, $|x| = n$, and $m = n - [\log_q n] - \alpha$. We can assume $m > 0$. Observe that $N(x, \omega^0) \geqslant m$. Moreover, $x \in \Gamma_{\alpha + \log_q n}(\omega^0)$: thus Lemma 2 yields (write $E = E_{\omega_m^0}$):

$$\int_E |K_z(x, \omega) f(\omega)|\, d\nu \leqslant C q^n (1 + n)^{-1} q^{-n} Mf(\omega^0) \leqslant C(\alpha, z) Mf(\omega^0),$$

again as $\nu(E) \approx q^{-m}$.

On the other hand, if $\omega \in \Omega - E$, then $\omega_m \neq \omega_m^0$, hence $N(\omega_m^0, \omega) < m \leqslant N(x, \omega^0)$. Thus $K_z(x, \omega) = K_z(\omega^0, \omega)$, and the argument of Theorem 1 applies again to give

$$\int_{\Omega - E} |K_z(x, \omega) f(\omega)|\, d\nu < C'(\alpha, z) \cdot Mf(\omega^0).$$

By combining these estimates, we obtain a constant $C(\alpha, z)$ such that $|F(x)| < C(\alpha, z) Mf(\omega^0)$ for every $x \in \Gamma_\alpha^+(\omega^0)$, and the theorem follows. □

REMARK. The end of the proof of Theorem 2 exploits the geometry of the tree in the same way as the argument of Lemma 1. This argument is equivalent to observing that the metric on Ω induced by the gauge, $d(\omega, \omega') = q^{-N(\omega, \omega')}$, satisfies the ultrametric inequality $d(\omega, \omega') \leqslant \max\{d(\omega, \omega''), d(\omega'', \omega')\}$.

3. – Non-homogeneous trees.

In this section, T is a non-homogeneous tree. Following [1], we consider a transition operator P such that, for every vertex s, the « transition coefficients » $p(s, t)$ are positive and nearest-neighbour, but do not satisfy, in general, the « markovian condition » $\sum_t p(s, t) = 1$.

Denote by Δ the corresponding Laplace operator, and by Δ_m the markovian Laplace operator on a homogeneous tree considered in the previous section. Note that the eigenfunctions g of Δ_m corresponding to a given eigenvalue $\gamma > -1$ satisfy $\Delta g = 0$ for an appropriate non-markovian Laplace operator Δ: nevertheless, in this section we follow the terminology of [1], and refer to the solutions of $\Delta g = 0$ (or $\Delta g \geqslant 0$) as « harmonic » (respectively, superharmonic) functions. A path c in T is a finite or infinite collection of adjoining edges (c_i, c_{i+1}) in T: we write $c = (c_0, c_1, \ldots, c_n, \ldots)$.

In the markovian case, the transition operator P gives rise to a random walk X_n on the vertices of T. In the general case, for every infinite path c, we write $X_n(c) = c_n$ [1, section 3.1].

For every finite path $c = (c_0, \ldots, c_n)$, define $p(c) = \prod_{i=1}^{n} p(c_{i-1}, c_i)$, and consider the *Green kernel* $G(x, y) = \sum p(c)$, where the sum is taken over all paths joining the vertices x and y. Following [1], *we assume that* $G(x, y)$ *is finite* for every x, y (in the markovian case, this implies that the random walk X_n is transient).

Let $\omega \in \Omega$, and let $K(x, \omega) = \lim_n G(x, \omega_n)/G(o, \omega_n)$: the functions $K_\omega(x) = K(x, \omega)$ are the minimal harmonic functions on T, normalized by $K_\omega(o) = 1$. Every positive harmonic function h has a unique Poisson representation $h = \int K_\omega \, d\mu_h$, where μ_h is a positive Borel measure on Ω [1, thm. 2.1]. In fact, Ω gives rise to the Martin compactification of T (a basis for the topology of $T \cup \Omega$ is given by the sets $E_x \cup \{y\colon E_y \subset E_x\}$).

Denote by W be the set of infinite paths and by W' the subset of paths c which have a limit (i.e., for which there exists ω in Ω such that $c_n \to \omega$ in the above topology). By [1, Corollary 3.1], for every x in T and ω in Ω there is a probability measure ν_x^ω on W, with support in the set of paths starting at x and tending to ω, which has the following property: for every finite path c joining x and y,

$$\nu_x^\omega(W_c) = p(c) K(y, \omega)$$

where W_c is the set of infinite paths whose starting segment is c.

We are now ready to state the nontangential convergence theorem for the non-homogeneous case. Our theorem is an extension of [1, thm. 3.3]. The argument relies upon the following fact, which is a consequence of the Fatou-Naïm-Doob theorem [9] (but can also be proved easily, by using Corollary 3.1 b, Theorem 3.1 and Theorem 3.2 of [1]): *for every positive harmonic* h *and positive superharmonic* g, *the limit* $\lim_n g/h(X_n)$ *exists* ν_x^ω*-almost surely for* μ_h*-almost every* ω.

In order to connect the next statement with Theorem 1, suppose that the tree is homogeneous and the transition operator is as in section 2, and observe that, if $h \equiv 1$, then its representing measure μ_h is ν.

THEOREM 3. *Assume that there exists $\delta > 0$ such that $p(x, y) \geqslant \delta$ for all neighbours x, y in T, and that the Green kernel is finite. Let h be a positive harmonic function with representing measure μ_h, and let g be a positive superharmonic function. Then g/h has admissible limits μ_h-almost everywhere on Ω.*

PROOF. By the remark preceding the statement, it suffices to show that g/h has admissible limit at ω for all ω such that $\lim_n g/h(X_n)$ exists ν_0^ω-almost surely. By contradiction, let $l = \lim_n g/h(X_n)$ and suppose that there exists $\varepsilon > 0$, an integer α and a sequence $\{x_k\}$ such that $x_k \in \Gamma_\alpha(\omega)$, $|x_k| \to \infty$ and $|g/h(x_k) - l| \geqslant \varepsilon$ for all k. Then we claim that there exists $\eta > 0$ such that, for every $x \in \Gamma_\alpha(\omega)$, $\nu_0^\omega[X_n = x$ for some $n] \geqslant \eta$. Once the claim is proved, the theorem follows: indeed, the process X_n meets infinitely many of the x_k's with ν-probability $\geqslant \eta$, which contradicts the assumption $\lim g/h(X_n) = l$.

To prove this claim, let $F^\omega(o, x) = \nu_0^\omega[X_k = x$ for some $k]$. As in [1, chapter 2] let $F(o, x) = \sum_c p(c)$, where the sum is taken over all paths from o to x *reaching* x *only at the end* (in the markovian case, $F(o, x)$ is the probability of hitting x starting at o). By definition of ν_0^ω, $F^\omega(o, x) = F(o, x)K(x, \omega)$. We have to prove $F^\omega(o, x) \geqslant \eta > 0$ for all x in $\Gamma_\alpha(\omega)$. By definition of $K(x, \omega)$, this amounts to showing that, for large n,

$$F(o, x)\,G(x, \omega_n)/G(o, \omega_n) \geqslant \eta\ .$$

By [1, prop. 2.5], the left hand side equals $F(o, x)F(x, \omega_n)/F(o, \omega_n)$. As $x \in \Gamma_\alpha(\omega)$, there exists an integer j such that $d(x, \omega_j) \leqslant \alpha$. Without loss of generality, we can assume $\delta < 1$. Then, by [1, coroll. 2.3], the above expression equals $F(o, \omega_j)F(\omega_j, x)F(x, \omega_j)F(\omega_j, \omega_n)/F(o, \omega_j)F(\omega_j, \omega_n) = F(\omega_j, x)F(x, \omega_j) \geqslant \delta^{2d(x,\omega_j)} \geqslant \delta^{2\alpha}$, and the claim is proved. □

REMARK. A more general but less precise statement holds without the assumption that the transition coefficients be bounded below. Indeed, the theorem holds with the same proof if we define $\Gamma_\alpha(\omega) = \{x\colon F(o, x)K(x, b) \leqslant \alpha^{-1}\}$. On the other hand, a less general but more explicit statement is obtained by assuming $0 < \delta \leqslant p(x, y) \leqslant \eta < \frac{1}{2}$ for all x, y: this condition automatically guarantees the finiteness (even the uniform boundedness) of the Green kernel.

In the case of a homogeneous tree with isotropic transition probabilities, it is interesting to consider the overlapping between Theorems 1 and 3. As noticed above, multiplication of the transition matrix by a positive constant gives rise to a dilation of its eigenvalues. By this token, Theorem 3 handles all *positive* eigenfunctions of the Laplace operator whose eigenvalues satisfy $\gamma > -1$. On the other hand, it follows immediately by [2, ch. 3] that the existence of a positive eigenfunction is equivalent to $\gamma \geqslant \varrho - 1$, where, as before, ϱ denotes the spectral radius of P in l^2. In other words, Theorem 3 can be used to deal with a proper subset of the set of eigenvalues considered in Theorem 1.

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